ESSENTIAL MATHEMATICS FOR ECONOMICS AND BUSINESS

SECOND EDITION

TERESA BRADLEY
AND PAUL PATTON

REVISED BY TERESA BRADLEY

Essential Mathematics for Economics and Business

Second Edition

This page intentionally left blank

Essential Mathematics for Economics and Business

Teresa Bradley
Limerick Institute of Technology

Paul Patton
Limerick Senior College

Second Edition Revised by
Teresa Bradley

JOHN WILEY & SONS, LTD

Second Edition
Copyright © 2002 by John Wiley & Sons Ltd,
Baffins Lane, Chichester,
West Sussex PO19 1UD, England

National 01243 779777
International (+44) 1243 779777
e-mail (for orders and customer service enquiries): cs-books@wiley.co.uk
Visit our Home Page on http://www.wiley.co.uk
or http://www.wiley.com

First edition printed in 1998, 1999

All Rights Reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except under the terms of the Copyright, Designs and Patents Act 1988 or under the terms of a licence issued by the Copyright Licensing Agency, 90 Tottenham Court Road, London, W1P 9HE, UK, without the permission in writing of the Publisher.

Teresa Bradley and Paul Patton have asserted their rights under the Copyright, Designs and Patents Act 1988, to be identified as the authors of this work.

Other Wiley Editorial Offices

John Wiley & Sons, Inc., 605 Third Avenue,
New York, NY 10158-0012, USA

WILEY-VCH Verlag GmbH, Pappelallee 3,
D-69469 Weinheim, Germany

John Wiley & Sons, Australia, Ltd, 33 Park Road, Milton,
Queensland 4064, Australia

John Wiley & Sons (Asia) Pte Ltd, 2 Clementi Loop #02-01,
Jin Xing Distripark, Singapore 129809

John Wiley & Sons (Canada) Ltd, 22 Worcester Road,
Rexdale, Ontario M9W 1L1, Canada

British Library Cataloguing in Publication Data

A catalogue record for this book is available from the British Library

ISBN 0-470-84466-3

Typeset in 9.5/11.5pt Times by C.K.M. Typesetting, Salisbury, Wiltshire
Printed and bound in Great Britain by Biddles Ltd, Guildford and King's Lynn
This book is printed on acid-free paper responsibly manufactured from sustainable forestation, for which at least two trees are planted for each one used for paper production.

To my parents, Lily and Richard   T.B.

To my mother and the memory of my father   P.P.

This page intentionally left blank

ebooksdownloadrace.blogspot.in

Contents

Introduction xi

Introduction to the first edition xiii

1 Mathematical preliminaries 1
- 1.1 Arithmetic operations 2
- 1.2 Fractions 4
- 1.3 Solving equations 7
- 1.4 Currency conversions 9
- 1.5 Simple inequalities 14
- 1.6 Calculating percentages 17
- 1.7 Using the calculator 19
- 1.8 Introducing Excel 20

2 The straight line and applications 29
- 2.1 The straight line 30
- 2.2 Mathematical modelling 47
- 2.3 Applications: demand, supply, cost, revenue 51
- 2.4 More mathematics on the straight line 66
- 2.5 Translations of linear functions 71
- 2.6 Elasticity of demand, supply and income 72
- 2.7 Budget and cost constraints 80
- 2.8 Excel for linear functions 81
- 2.9 Summary 86

3 Simultaneous equations 89
- 3.1 Solving simultaneous linear equations 89
- 3.2 Equilibrium and break-even 98

ebooksdownloadrace.blogspot.in

3.3 Consumer and producer surplus 114
3.4 The national income model and the *IS-LM* model 118
3.5 Excel for simultaneous linear equations 123
3.6 Summary 127
Appendix 129

4 Non-linear functions and applications 131
4.1 Quadratic, cubic and other polynomial functions 132
4.2 Exponential functions 151
4.3 Logarithmic functions 165
4.4 Hyperbolic functions of the form $a/(bx + c)$ 177
4.5 Excel for non-linear functions 183
4.6 Summary 185

5 Financial mathematics 188
5.1 Arithmetic and geometric sequences and series 189
5.2 Simple interest, compound interest and annual percentage rates 195
5.3 Depreciation 205
5.4 Net present value and internal rate of return 207
5.5 Annuities, debt repayments, sinking funds 213
5.6 The relationship between interest rates and the price of bonds 224
5.7 Excel for financial mathematics 227
5.8 Summary 229
Appendix 231

6 Differentiation and applications 234
6.1 Slope of a curve and differentiation 235
6.2 Applications of differentiation, marginal functions, average functions 245
6.3 Optimisation for functions of one variable 260
6.4 Economic applications of maximum and minimum points 276
6.5 Curvature and other applications 291
6.6 Further differentiation and applications 306
6.7 Elasticity and the derivative 319
6.8 Summary 328

7 Functions of several variables 332
7.1 Partial differentiation 332
7.2 Applications of partial differentiation 349
7.3 Unconstrained optimisation 369
7.4 Constrained optimisation and Lagrange multipliers 378
7.5 Summary 390

8 Integration and applications 394
8.1 Integration as the reverse of differentiation 394
8.2 The power rule for integration 396
8.3 Integration of the natural exponential function 401
8.4 Integration by algebraic substitution 402
8.5 The definite integral and the area under a curve 407

8.6 Consumer and producer surplus 414
8.7 First-order differential equations and applications 422
8.8 Differential equations for limited and unlimited growth 433
8.9 Summary 438

9 Linear algebra and applications 441
9.1 Linear programming 441
9.2 Matrices 452
9.3 Solution of equations: elimination methods 462
9.4 Determinants 468
9.5 The inverse matrix and input/output analysis 481
9.6 Excel for linear algebra 495
9.7 Summary 498

10 Difference equations 503
10.1 Introduction to difference equations 503
10.2 Solution of difference equations (first-order) 505
10.3 Applications of difference equations (first-order) 517
10.4 Summary 526

Solutions to progress exercises 528

Bibliography 621

List of worked examples 622

Index 628

This page intentionally left blank

Introduction

This book is intended for students who are studying mathematics as a subject on economics, business or management courses. The text assumes minimal mathematical background but demonstrates the usefulness and relevance of basic mathematics in economics and business.

To assist students, particularly those who are apprehensive about mathematics, the mathematical methods are set out and explained step by step then illustrated in worked examples. Economic and business applications of these methods follow immediately, again illustrated in worked examples. It is essential that students attempt the progress exercises at the end of each section in order to consolidate and retain the ideas and methods introduced and to test and enhance understanding. Detailed solutions are given at the back of the text.

In this edition several sections within chapters have been rewritten in a clearer, more accessible style, for example, the straight line, logs, differentiation, optimisation and integration. Chapters 4 and 8 were reorganised, introducing e^x earlier, therefore including it as an integral part of the rules for indices, logs, integration and applications. New topics have been introduced: currency conversions in Chapter 1; annuities, debt repayment, sinking funds in Chapter 5; integration by substitution (for functions of linear functions) in Chapter 8; elimination methods (Gaussian and Gauss–Jordan) applied to the solution of systems of equations and the inverse matrix in Chapter 9. A short section on the use of Excel in linear algebra is also included in Chapter 9.

In an attempt to produce a slimmer volume, some of the less frequently used sections (on the mathematics, but more especially on applications) have been moved to the web site which has been developed to accompany this book at www.wiley.co.uk/bradley2ed. This material is referenced by WWW within the text, so that it may be retrieved quickly and easily from the web site as required. Also available on the web site:

- Integration by substitution and parts with progress exercises and solutions (Chapter 8).
- PowerPoint slides: these slides present introductory material and a resumé of certain topics from selected chapters.

- Instructors manual: this is available, via password, to instructors who adopt the book as the main course text. This is an updated version of the manual that was formerly available in hardcopy form. Details of content are given in the introduction to the first edition of the text.

Acknowledgements

I thank all those whose help was invaluable in producing the second edition: Bart Mooney, Gabriel Daly, Chris Naughton. I thank those who reviewed the first edition and made very helpful suggestions for the second edition; these suggestions were adopted whenever possible: Dr Reza Arabsheibani, University of Wales; Dr Donal Hurley, University College Cork; Professor John O'Donoghue, Dr Eugene Gath and Dr Michael Hayes, University of Limerick; J. Colin Glass, University of Ulster; Hillary Lamaison, Brunel University; Dr Alexander Lee, University of Western Sydney; Helen Chadda, Limerick Institute of Technology; Anca Porojan, University of Bradford; R.J. Stroeker, Erasmus University; Dr Stephen Weissenhofer, University of Western Sydney. I thank John McGuinness, who made suggestions for the design of the cover. And I thank Steve Hardman, publishing editor and Sarah Booth of John Wiley & Sons Ltd, Chichester.

Teresa Bradley

A web site accompanies this book:

www.wiley.co.uk/bradley2ed

Introduction to the First Edition

Mathematics in Economics and Business Studies

Many students who pursue the study of economics and business studies are surprised and perturbed when they discover that mathematics is a core subject on first-year courses. However, a certain level of skill and understanding of the basic mathematical methods is required if the study of economics or business is to be pursued beyond the descriptive level. Students should be reassured that it is not necessary to become a mathematician to use mathematical techniques and methods effectively.

An Approach to Learning to Use Mathematics

Learning to use mathematics could be compared to learning to drive. In either case, the quote from the Chinese philosopher Lao Tse is appropriate:

> You read and you forget;
> you see and you remember;
> you do and you learn.

At the outset the learner-driver is presented with a bewildering set of rules and tasks, some of which must be performed simultaneously, some sequentially. There are sound, sensible reasons for each of these rules, as learners will discover on their first outing on a public road. Mastering driving skills and gaining a sense of how to control the car only comes about by following closely the routines demonstrated by the instructor, then practising them over and over again, sometimes patiently, sometimes not! In the end, the new driver will be able to handle a car easily and effortlessly, as if it were second nature. With these newly acquired skills life is enhanced with previously unavailable choices. The new driver (with a car!) can choose where to go, who to go with, what route to take, what time is convenient, etc. And so it is with maths.

Worked Examples

At an introductory level, you will not become proficient at using mathematics by simply 'reading' a mathematics book, cover to cover. A better approach is to read just one section at a time, then put pen to paper and follow the methods which are demonstrated in the Worked Examples, line by line. An index of Worked Examples is provided at the end of the book for easy reference.

Graphs

Can you imagine someone who has never seen a car before, attempting to understand what the controls, gears, steering, etc. look like from a verbal description? Understanding would be enhanced enormously by the provision of some well-labelled sketches and diagrams. Visualising mathematical functions/equations is not easy, especially for beginners. When the function is graphed, much of the vagueness and abstractness is removed. In fact, many of the properties of the function are revealed. In Worked Examples, whenever appropriate, graphs are plotted over an interval by calculating a table of points, then drawing the graph. Unfortunately, in many economic applications, this process is often a very time-consuming exercise and errors are all too easily made in calculations. In this text, the use of spreadsheets has been introduced to expedite the process.

Use of Spreadsheets (Excel or Others)

The text refers to Excel, since it is available in Microsoft Office, but in fact, any spreadsheet serves the same purpose. Learning Excel is not difficult and outline steps are given in the text. However, no attempt is made to teach Excel, so beginners are advised to refer to introductory texts on the subject. The use of Excel is discussed at the ends of Chapters 1 to 5, with appropriate Worked Examples. Progress Exercises in which Excel would be particularly useful are set at the ends of Chapters 2 and 3. In the remainder of the text, Excel may be used whenever the student chooses, for graphs, tables, etc. With practice, the student should discover that a spreadsheet is an invaluable tool: Tables of points are calculated and good accurate graphs are drawn in minutes.

☺ **Remember:** the text can be used in its entirety without touching on the Excel sections, but, when used, Excel will prove invaluable for graph plotting and in certain calculations.

Progress Exercises and Test Exercises

The Progress Exercises provide the vitally important practice on the mathematics and on the economic/business applications covered in the preceding section. Answers to all problems, with outline solutions to some of the longer ones, are given at the end of the text. However, the Progress Exercises alone are not sufficient. In more realistic applications (not to mention exam papers) the student is required to draw on a variety of mathematical techniques. Questions based on the entire chapter are set in the Test Exercises at the end of each chapter. Answers/solutions to the Test Exercises are given in the Lecturer's Manual only.

You Are Not Expected to be a Mathematician!

It is important to point out that this text is introductory, hence the approach adopted is not mathematically rigorous. The mathematical functions used in Worked Examples and in applications are those which are normally encountered at this level. However, the student is alerted to the unusual and exceptional cases. For example, the student is made aware of the fact that equations do not always have solutions, that functions and/or their derivatives may not exist at every point and this gives rise to problems in graph sketching and optimisation, etc. This situation is similar to that faced by the vast majority of car drivers. Many excellent drivers have only a vague idea about how the engine works, but, nonetheless quickly recognise when normal conditions break down. So they consult manuals or seek advice from the experts. For reference, a short Bibliography of mathematical texts is given at the end of the book.

Instructor's Manual

This manual by Teresa Bradley is available from the publisher to instructors who adopt this book as the main course text. For each chapter in the text the manual contains:

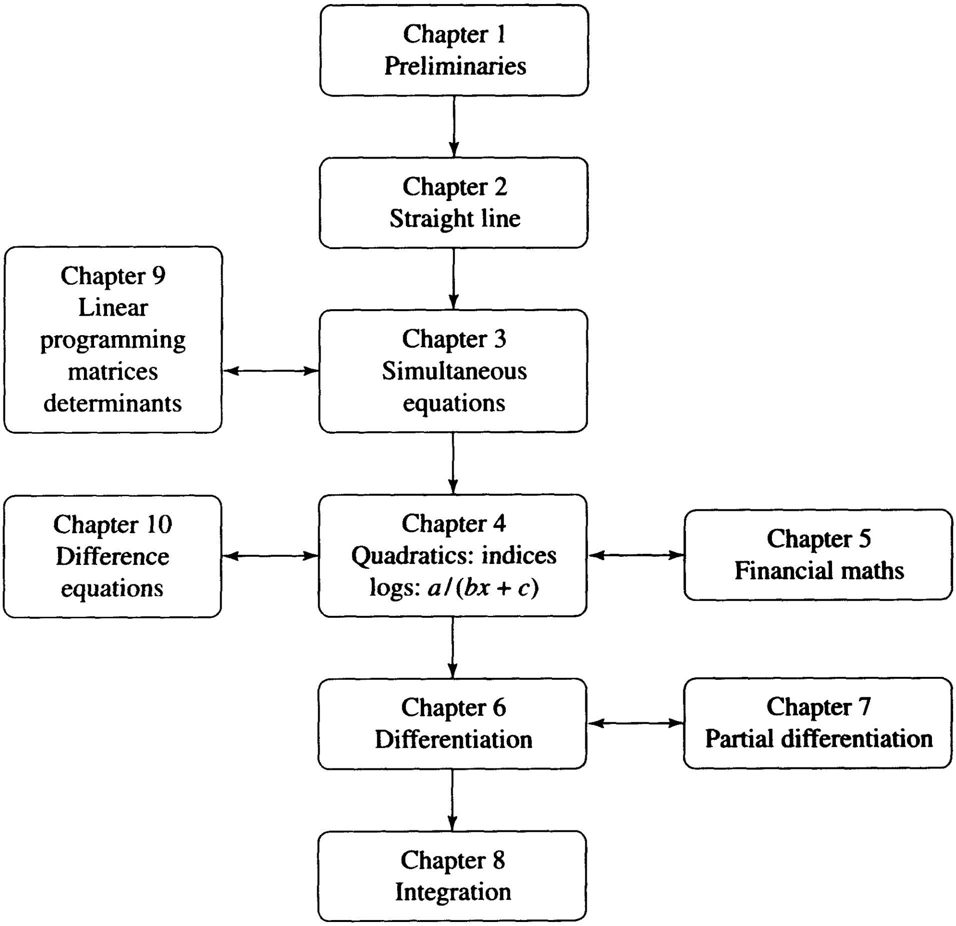
- Introduction: general comments on the contents of each chapter
- Details of PowerPoint presentations on disk, if applicable
- Reference to mathematical software: Calmat; Maple
- General comments on areas of difficulty experienced by students
- Answers/solutions to the Test Exercises
- Additional problems, with answers/solutions
- Worked Examples on applications of translations are given in Chapter 2
- Sample Examination Papers: stage 1 papers examine material from the text at a basic level: stage II papers at a more advanced level

A disk with Powerpoint presentations on selected topics from the text
Transparency masters on selected figures from the text may be printed from the PowerPoint presentations.

Structure of the Text

Mathematics is a hierarchical subject. The core topics in most foundation mathematics courses (linear functions, non-linear functions, differentiation and integration) are covered in

Chapters 1, 2, 3, 4, 6 and 8. These provide the basis for the usual economic applications. (Note: an introductory course may be pursued by covering the earlier sections within these chapters.) Some flexibility is possible in deciding when to introduce further material based on these core areas, such as linear algebra, difference equations, etc. The prerequisite chapters are indicated by the chart below.


Chapter 1
Preliminaries
Chapter 2
Straight line
Chapter 9
Linear programming matrices determinants
Chapter 3
Simultaneous equations
Chapter 10
Difference equations
Chapter 4
Quadratics: indices logs: $a/(bx+c)$
Chapter 5
Financial maths
Chapter 6
Differentiation
Chapter 7
Partial differentiation
Chapter 8
Integration

For example, for those who wish to continue with more mathematics and applications of the basic simultaneous linear equations in Chapter 3, go straight to Chapter 9. The financial maths in Chapter 5 requires a knowledge of indices and logs covered in Chapter 4. Difference equations (Chapter 10) could be introduced on completion of the section on indices in Chapter 4. The material in Chapter 7 follows immediately from Chapter 6. However, many introductory mathematics courses may omit this chapter completely, unless required for related subjects on the course.

Acknowledgements

We would like to thank a number of friends and colleagues:

Chris Naughton and Dave O'Neill for the painstaking hours spent reviewing various chapters of the text and who contributed many invaluable ideas and suggestions.

Dr Jan Podivinsky (Southampton University), Hilary Lamaison (Brunel University), Julian Wells (Southbank University), Dr Matthias Lutz (Sussex University), Lars Wahlgren (Lund University), Mark Baimbridge (University of Bradford), Professor Hassan Molana (University of Dundee) and any anonymous reviewers who offered useful suggestions for improvements, many of which we have incorporated into this text.

Alan Barry for his technical support and advice at all stages throughout the project. Orla Gavigan for her patience and assistance in the final preparation of the manuscript.

Steve Hardman, publishing editor and his assistant See Hanson, and Mary Seddon, marketing manager and the production team at John Wiley.

Finally, thanks to Harry and Joe and our families for their support.

We bear full responsibility for all errors and omissions.

T.B. and *P.P.*

This page intentionally left blank

CHAPTER 1

Mathematical Preliminaries

At the end of this chapter you should be able to:

- Perform basic arithmetic operations and simplify algebraic expressions
- Perform basic arithmetic operations with fractions
- Solve equations in one unknown, including equations involving fractions
- Understand the meaning of no solution and infinitely many solutions
- Currency conversions
- Solve simple inequalities
- Calculate percentages

In addition, you will be introduced to the calculator and a spreadsheet.

❐ Some mathematical preliminaries

Brackets in mathematics are used for grouping and clarity.

Brackets may also be used to indicate multiplication.

Brackets are used in functions to declare the independent variable (see later).

Powers: positive whole numbers such as 2^3, which means $2 \times 2 \times 2 = 8$:

$$(\text{anything})^3 = (\text{anything}) \times (\text{anything}) \times (\text{anything})$$

$$(x)^3 = x \times x \times x$$

$$(x + 4)^5 = (x + 4)(x + 4)(x + 4)(x + 4)(x + 4)$$

Note

Brackets: $(A)(B)$ or $A \times B$ or AB
all indicate A multiplied by B.

Variables and letters: When we don't know the value of a quantity, we give that quantity a symbol, such as x. We may then make general statements about the unknown quantity, x, for example 'For the next 15 weeks, if I save £x per week I shall have £4000 to spend on a holiday'. This statement may be expressed as a mathematical equation:

$$15 \times \text{weekly savings} = 4000$$

$$15 \times x = 4000 \quad \text{or} \quad 15x = 4000$$

Now that the statement has been reduced to a mathematical equation, we may solve the equation for the unknown, x:

$$15x = 4000$$

$$\frac{15x}{15} = \frac{4000}{15} \quad \text{divide both sides of the equation by 15}$$

$$x = 266.67$$

Square roots: the square root of a number is the reverse of squaring:

$$(2)^2 = 4 \rightarrow \sqrt{4} = 2$$

$$(2.5)^2 = 6.25 \rightarrow \sqrt{6.25} = 2.5$$

1.1 Arithmetic Operations

❒ Addition and subtraction

Adding: If all the signs are the same, simply add all the terms and give the answer with the common overall sign.

Subtracting: When subtracting any two numbers or two similar terms, give the answer with the sign of the largest number or term.

If terms are identical, for example all x-terms, all xy-terms, all x^2-terms, then they may be added or subtracted as shown in the following examples:

Add/subtract with numbers, mostly		**Add/subtract with variable terms**
$5+8+3=16$	similarity →	$5x+8x+3x=16x$
$5+8+3+y=16+y$	similarity →	(i) $5x+8x+3x+y=16x+y$
The y-term is different, so it cannot be added to the others		(ii) $5xy+8xy+3xy+y=16xy+y$

		The y-term is different, so it cannot be added to the others
$7 - 10 = -3$	similarity →	(i) $7x - 10x = -3x$ (ii) $7x^2 - 10x^2 = -3x^2$
$7 - 10 - 10x = -3 - 10x$	similarity →	$7x^2 - 10x^2 - 10x = -3x^2 - 10x$
The x-term is different, so it cannot be subtracted from the others		The x-term is different, so it cannot be subtracted from the others

Worked Example 1.1

Addition and Subtraction

For each of the following, illustrate the rules for addition and subtraction:

(a) $2 + 3 + 2.5 = (2 + 3 + 2.5) = 7.5$

(b) $2x + 3x + 2.5x = (2 + 3 + 2.5)x = 7.5x$

(c) $-3xy - 2.2xy - 6xy = (-3 - 2.2 - 6)xy = -11.2xy$

(d) $8x + 6xy - 12x + 6 + 2xy = 8x - 12x + 6xy + 2xy + 6 = -4x + 8xy + 6$

(e) $3x^2 + 4x + 7 - 2x^2 - 8x + 2 = 3x^2 - 2x^2 + 4x - 8x + 7 + 2 = x^2 - 4x + 9$

❐ Multiplying and dividing

Multiplying or dividing two quantities with like signs gives an answer with a positive sign. Multiplying or dividing two quantities with different signs gives an answer with a negative sign.

Worked Example 1.2

Multiplication and Division

Each of the following examples illustrate the rules for multiplication.

(a) $5 \times 7 = 35$

(b) $-5 \times -7 = 35$

(c) $5 \times -7 = -35$

(d) $-5 \times 7 = -35$

(e) $7/5 = 1.4$

(f) $(-7)/(-5) = 1.4$

(g) $(-7)/5 = -1.4$

(h) $7/(-5) = -1.4$

☺ **Remember**

It is very useful to remember that a minus sign is a -1, so -5 is the same as -1×5

(i) $5(7) = 35$
(j) $(-5)(-7) = 35$
(k) $(-5)y = -5y$
(l) $(-x)(-y) = xy$

☺ **Remember**

$0 \times$ (any real number) $= 0$
$0 \div$ (any real number) $= 0$
But you cannot divide by 0

(m) $2(x + 2) = 2x + 4$ multiply each term inside the bracket by the term outside the bracket

(n) $(x + 4)(x + 2)$ multiply the second bracket by x, then multiply the second bracket by (+4) and add,

$= x(x + 2) + 4(x + 2)$
$= x^2 + 2x + 4x + 8$ multiply each bracket by the term outside it
$= x^2 + 6x + 8$ add or subtract similar terms, such as $2x + 4x = 6x$

(o) $(x + y)^2$
$= (x + y)(x + y)$ multiply the second bracket by x and then by y
$= x(x + y) + y(x + y)$ add the similar terms: $xy + yx = 2xy$
$= xx + xy + yx + yy$
$= x^2 + 2xy + y^2$

The following identities are important:

1. $(x + y)^2 = x^2 + 2xy + y^2$
2. $(x - y)^2 = x^2 - 2xy + y^2$
3. $(x + y)(x - y) = x^2 - y^2$

☺ **Remember:** Brackets are used for grouping terms together in maths for:

(i) Clarity
(ii) Indicating the order in which a series of operations should be carried out

1.2 Fractions

Terminology:

$$\text{fraction} = \frac{\text{numerator}}{\text{denominator}}$$

$\frac{3}{7}$
← 3 is called the numerator
← 7 is called the denominator

1.2.1 Add/subtract fractions: method

The method for adding or subtracting fractions is:

Step 1: Take a common denominator, that is, a number or term which is divisible by the denominator of each fraction to be added or subtracted. A safe bet is to use the product of all the individual denominators as the common denominator.

Step 2: For each fraction, divide each denominator into the common denominator, then multiply the answer by the numerator.
Step 3: Simplify your answer if possible.

WORKED EXAMPLE 1.3

ADD AND SUBTRACT FRACTIONS

Each of the following illustrates the rules for addition and subtraction of fractions.

Numerical example	**Same example, but with variables**
$\frac{1}{7}+\frac{2}{3}-\frac{4}{5}$ Step 1: The common denominator is (7)(3)(5) Step 2: $\frac{1}{7}+\frac{2}{3}-\frac{4}{5}$ $=\frac{1(3)(5)+2(7)(5)-4(7)(3)}{(7)(3)(5)}$ Step 3: $=\frac{15+70-84}{105}=\frac{1}{105}$	$\frac{x}{7}+\frac{2x}{3}-\frac{4x}{5}$ $=\frac{x(3)(5)+2x(7)(5)-4x(7)(3)}{(7)(3)(5)}$ $=\frac{15x+70x-84x}{105}$ $=\frac{x}{105}$
$\frac{1}{7}+\frac{2}{3}$ Step 1: The common denominator is (7)(3) Step 2: $\frac{1}{7}+\frac{2}{3}=\frac{1(3)+2(7)}{(7)(3)}$ Step 3: $=\frac{3+14}{21}=\frac{17}{21}$	$\frac{1}{x+4}+\frac{2}{x}=\frac{1(x)+2(x+4)}{(x+4)(x)}$ $=\frac{x+2x+8}{x^2+4x}$ $=\frac{3x+8}{x^2+4x}$

1.2.2 Multiplying fractions

In multiplication, write out the fractions, multiply the numbers across the top lines and multiply the numbers across the bottom lines.
Note: Write whole numbers as fractions by putting them over 1.
Terminology: RHS means right-hand side and LHS means left-hand side.

WORKED EXAMPLE 1.4

MULTIPLYING FRACTIONS

(a) $\left(\frac{2}{3}\right)\left(\frac{5}{7}\right)=\frac{(2)(5)}{(3)(7)}=\frac{10}{21}$

(b) $\left(-\frac{2}{3}\right)\left(\frac{7}{5}\right)=\frac{(-2)(7)}{(3)(5)}=\frac{-14}{15}$

(c) $3\times\frac{2}{5}=\left(\frac{3}{1}\right)\left(\frac{2}{5}\right)=\frac{(3)(2)}{(1)(5)}=\frac{6}{5}=1\frac{1}{5}$

The same rules apply for fractions involving variables, x, y, etc.

(d) $\left(\frac{3}{x}\right)\frac{(x+3)}{(x-5)}=\frac{3(x+3)}{x(x-5)}=\frac{3x+9}{x^2-5x}$

1.2.3 Dividing fractions

General rule:

Dividing by a fraction is the same as multiplying by the fraction inverted

WORKED EXAMPLE 1.5

DIVISION WITH FRACTIONS

The following examples illustrate how division with fractions operates.

(a) $\frac{\left(\frac{2}{3}\right)}{\left(\frac{5}{11}\right)}=\left(\frac{2}{3}\right)\left(\frac{11}{5}\right)=\frac{22}{15}$

(b) $\frac{5}{\left(\frac{3}{4}\right)}=5\times\frac{4}{3}=\frac{5}{1}\times\frac{4}{3}=\frac{20}{3}=6\frac{2}{3}$

(c) $\frac{\left(\frac{7}{3}\right)}{8}=\frac{\left(\frac{7}{3}\right)}{\frac{8}{1}}=\frac{7}{3}\times\frac{1}{8}=\frac{7}{24}$

(d) $$\frac{\left(\dfrac{2x}{x+y}\right)}{\left(\dfrac{3x}{2(x-y)}\right)} = \frac{2x}{x+y}\,\frac{2(x-y)}{3x}$$

$$= \frac{4x(x-y)}{3x(x+y)} = \frac{4(x-y)}{3(x+y)}$$

Note: The same rules apply to all fractions, whether the fractions consist of numbers or variables.

Progress Exercises 1.1 Revision on Basics

Show, step by step, how the expression on the left-hand side simplifies to that on the right.

1. $2x + 3x + 5(2x - 3) = 15(x - 1)$

2. $4x^2 + 7x + 2x(4x - 5) = 3x(4x - 1)$

3. $2x(y + 2) - 2y(x + 2) = 4(x - y)$

4. $(x + 2)(x - 4) - 2(x - 4) = x(x - 4)$

5. $(x + 2)(y - 2) + (x - 3)(y + 2)$
$= 2xy - y - 10$

6. $(x + 2)^2 + (x - 2)^2 = 2(x^2 + 4)$

7. $(x + 2)^2 - (x - 2)^2 = 8x$

8. $(x + 2)^2 - x(x + 2) = 2(x + 2)$

9. $\dfrac{1}{3} + \dfrac{3}{5} + \dfrac{5}{7} = 1\dfrac{68}{105}$

10. $\dfrac{x}{2} - \dfrac{x}{3} = \dfrac{x}{6}$

11. $\dfrac{\left(\dfrac{2}{3}\right)}{\left(\dfrac{1}{5}\right)} = \dfrac{10}{3}$

12. $\dfrac{\left(\dfrac{2}{7}\right)}{3} = \dfrac{2}{21}$

13. $2\left(\dfrac{2}{x} - \dfrac{x}{2}\right) = \dfrac{4 - x^2}{x} = \dfrac{4}{x} - x$

14. $\dfrac{-12}{P}\left(\dfrac{3P}{2} + \dfrac{P}{2}\right) = -24$

15. $\dfrac{\left(\dfrac{3}{x}\right)}{x + 3} = \dfrac{3}{x(x + 3)}$

16. $\dfrac{\left(\dfrac{5Q}{P + 2}\right)}{\left(\dfrac{1}{(P + 2)}\right)} = 5Q$

1.3 Solving Equations

The solution of an equation is simply the value or values of the unknown(s) for which the left-hand side (LHS) of the equation is equal to the right-hand side (RHS).

For example, the equation, $x + 4 = 10$, has the solution $x = 6$. We say $x = 6$ 'satisfies' the equation. We say this equation has a **unique solution**.

Not all equations have solutions. In fact, equations may have no solutions at all or may have infinitely many solutions. Each of these situations is demonstrated in the following examples.

Case 1: **Unique solutions** An example of this is given above: $x + 4 = 10$ etc.

Case 2: **Infinitely many solutions** The equation, $x + y = 10$ has solutions $(x = 5,\ y = 5)$; $(x = 4,\ y = 6)$; $(x = 3,\ y = 7)$, etc. In fact, this equation has infinitely many solutions or pairs of values (x, y) which satisfy the formula, $x + y = 10$.

Case 3: **No solution** The equation, $0(x) = 5$ has no solution. There is simply no value of x which can be multiplied by 0 to give 5.

☐ Methods for solving equations

Solving equations can involve a variety of techniques, many of which will be covered later.

WORKED EXAMPLE 1.6

SOLVING EQUATIONS

(a) Given the equation $x + 3 = 2x - 6 + 5x$, solve for x.
(b) Given the equation $(x + 3)(x - 6) = 0$, solve for x.
(c) Given the equation $(x + 3)(x - 3) = 0$, solve for x.

Solution

(a)

$$\begin{aligned} x + 3 &= 2x - 6 + 5x && \\ x + 3 &= 7x - 6 && \text{adding the } x \text{ terms on the RHS} \\ x + 3 + 6 &= 7x && \text{bringing over } -6 \text{ to the other side} \\ x + 9 &= 7x && \\ 9 &= 7x - x && \text{bringing } x \text{ over to the other side} \\ 9 &= 6x && \\ \frac{9}{6} &= x && \text{dividing both sides by 6} \\ 1.5 &= x && \end{aligned}$$

(b) $(x + 3)(x - 6) = 0$

The LHS of this equation consists of the product of two terms $(x + 3)$ and $(x - 6)$. A product is equal to 0, if either or both terms in the product is 0.
Here there are two solutions: $x = -3$ and $x = 6$.
Each of these solutions can be confirmed by checking that they satisfy the original equation.
(c) This is similar to part (b), a product on the LHS, zero on the right, hence the equation $(x - 3)(x + 3) = 0$ has two solutions, $x = 3$ from the first bracket and $x = -3$ from the second bracket.

Alternatively, on multiplying out the brackets, this equation simplifies as follows:

$$
\begin{aligned}
(x-3)(x+3) &= 0 \\
x(x+3) - 3(x+3) &= 0 \\
x(x) + x(3) - 3(x) - 3(x) &= 0 \\
x^2 + 3x - 3x - 0 &= 0 \\
x^2 - 9 &= 0 \\
x^2 &= 9 \qquad \text{simplified equation} \\
x &= \pm\sqrt{9} \\
x &= 3 \quad \text{or} \quad x = -3
\end{aligned}
$$

The reason for two solutions is that x may be positive or negative; both satisfy the simplified equation $x^2 = 9$.
If x is positive, then $x = 3 \rightarrow x^2 = 9$, so satisfying the equation.
If x is negative, then $x = -3 \rightarrow x^2 = (-3)^2 = 9$, also satisfying the equation.
Therefore, in solving equations of the form $x^2 =$ number there are always two solutions:

$$\boxed{x = +\sqrt{\text{number}} \quad \text{and} \quad x = -\sqrt{\text{number}}}$$

1.4 Currency Conversions

You may have browsed the internet in order to purchase books, software, music, etc. Frequently prices will be quoted in some currency other than your own. With some simple maths and knowledge of the current rate of exchange, you should have no difficulty converting the price to your own currency. The following worked examples use the Euro exchange rates in Table 1.1. This table equates each of the currencies listed to 1 Euro on a given day in August 2001.

Table 1.1 Euro exchange rates

Currency	**Rate**	**Currency**	**Rate**
British pound	0.6187	Canadian dollar	1.3460
US dollar	0.8770	Australian dollar	1.6988
Japanese yen	108.5400	Polish zloty	3.7143
Danish krone	7.4424	Hungarian forint	247.5000
Swedish krona	9.1550	Hong Kong dollar	6.8399
Swiss franc	1.5060	Singapore dollar	1.5607
Norwegian krone	7.9566		

WORKED EXAMPLE 1.7

CURRENCY CONVERSIONS

(a) A book is priced at US\$20. Calculate the price of the book in (i) Euro, (ii) British pounds and (iii) Australian dollars.
(b) How many British pounds are equivalent to (i) \$500 Australian and (ii) \$10 000 Singapore?

Solution

The calculations (correct to 4 decimal places) may be carried out as follows:
Step 1 State the appropriate rates from Table 1.1.
Step 2 Set up the equation: 1 unit of *given* currency = y units of *required* currency.
Step 3 Solve the equation: x units of *given* currency = $x \times$ (y units of *required* currency).

(a) (i) The price is *given* in US dollars, the *required* price (currency) is in euros, hence

Step 1 \$0.8770 US = 1 Euro from Table 1.1

Step 2 $\$1 \text{ US} = \frac{1}{0.8770}$ Euro dividing both sides by 0.8770 to get rate for \$1

Step 3 \$20 US = 20 Euro multiplying both sides by 20
= 22.8050 Euro

(ii)

Step 1 From Table 1.1 write down the exchange rates for 1 Euro with both British pounds and US dollars:

$$\left.\begin{array}{r}\$0.8770 = 1 \text{ Euro}\\ £0.6187 = 1 \text{ Euro}\end{array}\right\} \rightarrow \$0.8770 = £0.6187$$

\$0.8770 = £0.6187 since they are each equivalent to 1 Euro

The *given* price is in US dollars, the *required* price (currency) is British pounds

Step 2 $\$1 \text{ US} = £1 \times \frac{0.6187}{0.8770}$ Euros dividing both sides of the previous equation by 0.8770 to get rate for \$1

Step 3 $\$20 \text{US} = £20 \times \frac{0.6187}{0.8770}$ multiplying both sides by 20

$= £14.1095$

(iii)

Step 1 From Table 1.1 write down the exchange rates for 1 Euro with Australian dollars and US dollars:

$$\left.\begin{array}{r}\$0.8770 \text{ US} = 1 \text{ Euro}\\ £1.6988 \text{ Aus} = 1 \text{ Euro}\end{array}\right\} \rightarrow \$0.8770 \text{ US} = £1.6988 \text{ Aus}$$

\$0.8770 US = \$1.6988 Aus since they are each equivalent to 1 Euro

The *given* price is US dollars, the *required* price (currency) is Australian dollars

Step 2 $\$1\text{ US} = \$1 \times \dfrac{1.6988}{0.8770}\text{ Aus}$ dividing both sides by 0.8770

Step 3 $\$20\text{ US} = \$20 \times \dfrac{1.6988}{0.8770}\text{ Aus}$ multiplying both sides by 20

$= 38.7412\text{ Aus}$

(b) (i)

Step 1 From Table 1.1 write down the exchange rates for 1 Euro with Australian dollars and British pounds:

$$\left.\begin{aligned} £0.6187 &= 1\text{ Euro} \\ \$1.6988\text{ Aus} &= 1\text{ Euro} \end{aligned}\right\} \rightarrow £0.6187 = \$1.6988\text{ Aus}$$

Here we are *given* $500 Aus and we *require* its equivalent in British pounds $1.6988 Aus = £0.6187 since they are each equivalent to 1 Euro

Step 2 $\$1\text{ Aus} = £1 \times \dfrac{0.6187}{1.6988}$ dividing both sides by 1.6988 to get rate for $1 Aus

Step 3 $\$500\text{ Aus} = £20 \times \dfrac{0.6187}{1.6988}$ multiplying both sides by 500

$= £182.0991$

(ii)

Step 1 From Table 1.1 write down the exchange rates for 1 Euro with Singapore dollars and British pounds:

$$\left.\begin{aligned} £0.6187 &= 1\text{ Euro} \\ \$1.5607\text{ Singapore} &= 1\text{ Euro} \end{aligned}\right\} \rightarrow £0.6187 = \$1.5607\text{ Singapore}$$

$1.5607 Singapore = £0.6187 since they are each equivalent to 1 Euro

Step 2 $\$1\text{ Singapore} = £1 \times \dfrac{0.6187}{1.5607}$ dividing both sides by 1.5607

Step 3 $\$10\,000\text{ Singapore} = £10\,000 \times \dfrac{0.6187}{1.5607}$ multiplying both sides by 10 000

$= £3964.2468$

WORKED EXAMPLE 1.8

SOLVING A VARIETY OF EQUATION TYPES

In this worked example, try solving the following equations yourself. The answers are given below, followed by the detailed solutions.

1. $2x + 3 = 5x - 8$

2. $\dfrac{1}{x} + \dfrac{2}{x} = 5$

3. $(Q - 1)(Q + 7) = 0$

4. $x^2 + 4x - 6 = 2(2x + 5)$

5. $(x - y) = 4$

6. $x^3 - 2x = 0$

Now check your answers:

1. $x = 11/3$ **2.** $x = 0.6$ **3.** $Q = 1, Q = -7$ **4.** $x = 4, x = -4$

5. Infinitely many solutions for which $x = y + 4$

6. $x = 0, x = \sqrt{2}, x = -\sqrt{2}$.

Suggested solutions to Worked Example 1.8

1.

$$\begin{aligned} 2x + 3 &= 5x - 8 \\ 3 + 8 &= 5x - 2x \\ 11 &= 3x \\ 11/3 &= x \end{aligned}$$

2.

$$\begin{aligned} \frac{1}{x} + \frac{2}{x} &= 5 \\ \frac{(2+1)}{x} &= 5 \\ \frac{3}{x} &= \frac{5}{1} \\ \frac{x}{3} &= \frac{1}{5} && \text{inverting both sides} \\ x &= 3\left(\frac{1}{5}\right) && \text{multiplying both sides by 3} \\ x &= \frac{3}{5} = 0.6 \end{aligned}$$

3. $(Q - 1)(Q + 7) = 0$

Since the RHS = 0, we can now check for conditions which make the LHS = 0.
First, if $(Q - 1) = 0$, the LHS = 0, thus satisfying the equation.
Second, if $(Q + 7) = 0$, the LHS = 0, thus satisfying the equation.
Therefore: $Q = 1$ or $Q = -7$ are solutions.

4. $x^2 + 4x - 6 = 2(2x + 5)$

This time simplify first, by multiplying out the brackets and collecting similar terms:

$$\begin{aligned} x^2 + 4x - 6 &= 4x + 10 \\ x^2 + 4x - 6 - 4x - 10 &= 0 \\ x^2 - 16 &= 0 \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

5. $(x - y) = 4$: here we have one equation in two unknowns, so it is not possible to find a unique solution. The equation may be rearranged as

$$x = y + 4$$

This equation now states that for **any** given value of y (there are infinitely many values), x is equal to that value plus four. So, there are infinitely many solutions.

6. Note: This is not a quadratic.

For a quadratic, the highest power to which the variable is raised should be 2. However, x is in both terms, therefore, we can separate or factor x from each as follows:

$$x^3 - 2x = 0$$

$$x(x^2 - 2) = 0$$ since the RHS = 0, factor the left

$$\downarrow \quad \downarrow$$

$$x = 0, x^2 = 2$$ either or both factors equal to 0 will satisfy the equation

$$\downarrow \quad \downarrow$$

Solution: $x = 0, x = \pm\sqrt{2}$ there are three solutions

Progress Exercises 1.2 Use Basics to Solve Equations

Solve the following equations. Remember that equations may have no solution or infinitely many solutions.

1. $2x + 3x + 5(2x - 3) = 30$

2. $4x^2 + 7x - 2x(2x - 5) = 17$

3. $(x - 2)(x + 4) = 0$

4. $(x - 2)(x + 4) = 2x$

5. $(x - 2)(x + 4) = -8$

6. $x(x - 2)(x + 4) = 0$

7. $4x(x - 2)(x - 2) = 0$

8. $2x(y + 2) - 2y(x + 2) = 0$

9. $(x + 2)(y + 2) = 0$

10. $(x + 2)(y + 2) + (x - 3)(y + 2) = 0$

11. $(x - 2)(x + 4) - 2(x - 4) = 0$

12. $(x + 2)^2 + (x - 2)^2 = 0$

13. $(x + 2)^2 - (x - 2)^2 = 0$

14. $x(x^2 + 2) = 0$

15. $\frac{x}{3} - \frac{x}{2} = \frac{2}{3}$

16. $\frac{x}{3} = 2x$

17. $\frac{2}{x} - \frac{3}{2x} = 1$

18. $\frac{4x(x - 4)(x + 3.8)}{x^4 - 4x^3 + 7x^2 - 5x + 102} = 0$

19. $\left(\frac{-12P}{P}\right)\left(\frac{3P}{2} + \frac{P}{2}\right) = 480$

20. $\frac{2}{Q} = 0$

21. $\frac{\left(\frac{25}{x}\right)}{x} = 1$

22. $\frac{\left(\frac{5Q}{P + 2}\right)}{\left(\frac{1}{P + 2}\right)} = 20$

23. $10 - \frac{1}{10Q} = 0$

Refer to exchange rates given in Table 1.1 to answer questions 24 to 30. Give answers correct to 4 decimal places.

24. A bag is priced at 400 Canadian dollars. Calculate the price of the bag in (a) Euro and (b) yen.

25. How many Euro are equivalent to 800 US dollars?

26. A flight is priced at 1200 Swedish kronor. What is the equivalent price in Danish kroner?

27. Calculate a table of exchange rates for Hong Kong dollars for the first five currencies in Table 1.1.

28. A book is priced at 22 British pounds. Calculate the equivalent price in Hungarian forints.

29. Convert 500 Australian dollars to (a) US dollars and (b) Canadian dollars.

30. Assume you have 200 Euro. How much would you have left if you bought a book for 35 British pounds and a T-shirt for 420 Hong Kong dollars?

1.5 Simple Inequalities

An equation is an equality. It states that the expression on the LHS of the '=' sign is equal to the expression on the RHS. An inequality is a statement in which the expression on the LHS is either greater than (denoted by the symbol >) or less than (denoted by the symbol <) the expression on the RHS.

For example, $5 = 5$ or $5x = 5x$ are equations, while

$$5 > 3 \qquad 5x > 3x$$

are inequalities which read, '5 is greater than 3'; '$5x$ is greater than $3x$' (for any positive value of x).

Note: Inequalities may be read from left to right, as above, or the inequality may be read from right to left, in which case the above inequalities are

$5 > 3$ (5 is greater than 3) is the same as $3 < 5$ ('3 is less than 5')
$5x > 3x$ ('$5x$ is greater than $3x$') is the same as $3x < 5x$ ('$3x$ is less than $5x$')

❐ Inequality symbols

$>$	greater than	$<$	less than
$\geq$	greater than or equal to	$\leq$	less than or equal to

❐ The number line

The number line is a horizontal line on which every point represents a real number. The central point is zero, the numbers on the left are negative, numbers on the right are positive, as illustrated for selected numbers in Figure 1.1.

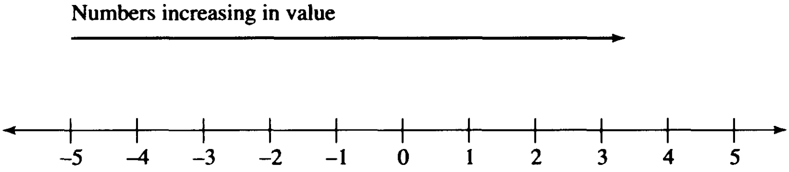


Figure 1.1 Number line, numbers increasing from left to right

Look carefully at the negative numbers; as the numbers increase in value they decrease in magnitude, for example -1 is a larger number than -2; -0.3 is a larger number than -0.5. Another way of looking at it is to say the numbers become less negative as they increase in value. (Like a bank account, you are better off when you owe £10 (−10) than when you owe £1000 (−1000).)

An inequality statement, such as $x > 2$, means all numbers greater than, but not including, 2. This statement is represented graphically as every point on the number line to the right of the number 2, as shown in Figure 1.2.

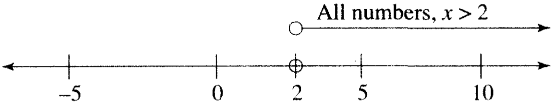


Figure 1.2 The inequality, $x > 2$

In economics, it is meaningful to talk about positive prices and quantities only, so in solving inequalities, we shall assume that the variable in the inequality (x above) is always positive.

❒ Manipulating inequalities

Inequalities may be treated as equations for many arithmetic operations. The inequality remains true when constants are added or subtracted to both sides of the inequality sign, or when both sides of the inequality are multiplied or divided by positive numbers or variables. For example, the equalities above are still true when 8 or −8 is added to both sides,

$5 + 8 > 3 + 8$, that is, $13 > 11$, similarity $5x + 8 > 3x + 8$

$5 - 8 > 3 - 8$, that is $-3 > -5$. Remember -5 is less than -3. See Figure 1.1.

However, when both sides of the inequality are multiplied or divided by negative numbers or variables, then the statement remains true only when the direction of the inequality changes: > becomes < and vice versa. For example, multiply both sides of the inequality, $5 > 3$ by -2:

$5(-2) > 3(-2)$ or $-10 > -6$ is not true ☹

$5(-2) < 3(-2)$, or $-10 < -6$ is true ☺

❒ Solving inequalities

The solution of an equation is a distinct value(s) for which the equation statement is true. For example, $x + 4 = 10$ is true when $x = 6$ only. On the other hand, the solution of an inequality is a range of values for which the inequality statement is true; for example, $x + 4 > 10$ is true when $x > 6$.

WORKED EXAMPLE 1.9

SOLVING SIMPLE INEQUALITIES

Find the range of values for which the following inequalities are true, assuming that $x > 0$. State the solution in words and indicate the solution on the number line.

(a) $10 < x - 12$ (b) $\dfrac{-75}{x} > 15$ (c) $2x - 6 \leq 12 - 4x$

Solution

(a) $10 < x - 12 \quad \rightarrow \quad 10 + 12 < x \quad \rightarrow \quad 22 < x$ (or $x > 22$).

The solution states: 22 is less than x or x is greater than 22. It is represented by all points on the number line to the right of, but not including, 22, as shown in Figure 1.3.

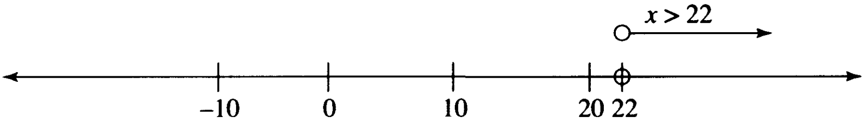


Figure 1.3 $x > 22$

(b) $\dfrac{-75}{x} > 15$

Multiply both sides of the inequality by x. Since $x > 0$, the direction of the inequality sign does not change.

$$-75 > 15x$$

$-5 > x$ dividing both sides by 15

The solution states: -5 is greater than x or x is less than -5. x cannot be less than -5 and greater than 0 at the same time. So there is no solution, as shown in Figure 1.4.

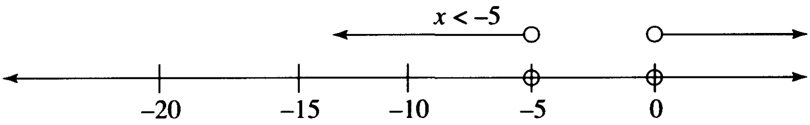


Figure 1.4 $x < -5$ and $x > 0$ is not possible

(c)

$$2x - 6 \leq 12 - 4x$$

$2x + 4x - 6 \leq 12$ add $4x$ to both sides

$6x \leq 12 + 6$ add 6 to both sides

$$6x \leq 18$$

$x \leq 3$ dividing both sides by 6

The solution is all values of x less than or equal to 3. This is represented by all points on the number line to the left of, but including, 3, as shown in Figure 1.5.

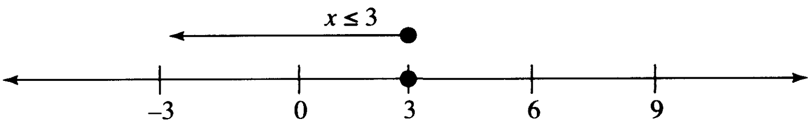


Figure 1.5 $x \leq 3$

☐ Intervals defined by inequality statements

When an application uses values within a certain range only, then inequality signs are often used to define this interval precisely. For example, suppose a tax is imposed on all incomes, (£Y), between £10 000 and £15 000 inclusive, we say the tax is imposed on all salaries within the interval, $10\,000 \leq Y \leq 15\,000$.

A certain bus fare applies to all children of ages (x) 4 but less than 16, we say the fare applies to those whose ages are in the interval, $4 \leq x < 16$. The age 4 is included, and all ages up to, but not including 16.

1.6 Calculating Percentages

- When we speak of 5% of a number we mean

$$\frac{5}{100} \times \text{number}$$

- When we say a number increases by x%, then the increase in the number

$$= \text{number} \times \frac{x}{100}$$

- The increased number

$$= \text{number} + \text{the increase} = \frac{\text{number}}{1} + \left(\frac{\text{number}}{1} \times \frac{x}{100}\right) = \text{number}\left(1 + \frac{x}{100}\right)$$

You should notice that, in calculations, percentages are always written as fractions, where the fraction is percentage quoted/100. These definitions and other calculations with percentages are best illustrated through worked examples.

Worked example 1.10

Calculations with percentages

(a) Calculate (i) 23% of 1534 (ii) 100% of 1534

(b) A salary of £55 240 is to be increased by 12%. Calculate (i) the increase, (ii) the new salary.

(c) In 1999, an apartment is valued at £63 400. This is 5% higher than the price paid for the apartment in 1997. Calculate the price paid in 1997.

Solution

(a) In calculations, quoted percentages are always written as the fraction, percentage quoted/100.

(i)

$$23\% \text{ of } 1534 = \frac{23}{100}\frac{1534}{1} = \frac{(23)(1534)}{100} = \frac{35\,282}{100} = 352.82$$

(ii)

$$100\% \text{ of } 1534 = \frac{100}{100}\,\frac{1534}{1} = \frac{(100)(1534)}{100} = 1534$$

So, 100% of (anything) = (anything), i.e.

$$\frac{100}{100} \times (\text{anything}) = (\text{anything}).$$

(b) (i)

$$12\% \text{ of } 55\,240 = \frac{12}{100}\,\frac{55\,240}{1} = \frac{12(5524)}{10} = \frac{66\,288}{10} = 6628.8 = \text{increase}$$

So the increase in salary is £6628.8

(ii) The new salary is £55 240 + £6628.8 = £61 868.8

(c) Let the 1997 price be the basic price, i.e. the 1997 price = 100% x basic price. The price in 1999 is 105% of the 1997 price, i.e. 1999 price = 105% x basic price. So £63 400 = 105% of the basic price and we want to find 100% of the basic price.

Method

First find 1% of the basic price by dividing by 105. Then multiply by 100 to calculate 100% of the basic price.

$$63\,400 = \frac{105}{100} \times \text{basic price}$$

$$\frac{63\,400}{105} = \frac{1}{100} \times \text{basic price}$$

$$\frac{63\,400}{105} \times \frac{100}{1} = \frac{100}{100} \times \text{basic price}$$

$$60\,380.95 = \text{basic price}$$

So, in 1997, the apartment cost £60 380.95.

Progress Exercises 1.3 Percentages and Inequalities

1. Graph the intervals given by the following inequalities on the number line:

   (a) $x > 2$ (b) $x < 25$ (c) $x > -4$ (d) $x \geq -1.5$ (e) $-4 \geq x$ (f) $60 < x$

2. Solve the following inequalities, stating the solution in words. Graph the inequality on the number line.

   (a) $x - 25 > 7$ (b) $5 < 2x + 15$ (c) $\frac{25}{x} < 10$ (d) $\frac{x}{2} + \frac{x}{3} \geq \frac{17}{6}$

   (e) $3x - 29 \leq 7x + 11$

3. Calculate (a) 12% of 5432.7; (b) 85% of 23.65; (c) 11.5% of 6.5.

4. A fast-food chain proposes to increase the basic hourly rate of pay by 14%. If the present rate is £5.65 per hour, calculate (a) the increase in the hourly rate, (b) the new hourly rate.
5. The 1998 price of a basic computer is 35% lower than the 1995 price. If the 1998 price is £910, calculate the price in 1995.
6. A company which produces printers proposes to increase its output by 6% each week. Calculate the company's projected output for the next three weeks (to the nearest whole printer) if the present output is 720 per week.
7. A company plans to phase out a particular model of car by reducing the output by 20% each week. If the present output is 400 cars per week, calculate the number of cars to be produced per week over the next six weeks.
8. The price of a new washing machine is £485. The price includes a value added tax (VAT) of 21% of the selling price (VAT is explained on page 28). Calculate the price without VAT.
9. A retailer sells a TV for £658. If the cost price was £480, calculate his profit as a percentage of the cost price. (Note: profit = selling price − cost price.)
10. A retailer sells a video recorder for £880. The price includes VAT at 21% and a profit of 34%. Calculate the cost price of the recorder.
11. A retailer buys TVs for £425. He must then pay VAT at 21%. What price should he charge if he is to make a profit of 25% of the cost price?
12. 154 students attend a maths lecture. If 22 students are absent, calculate the percentage of students absent.

1.7 Using the Calculator

You will require a scientific calculator for the remainder of the text. The keys which shall be required most frequently are:

Mathematical functions

add: [+] subtract: [−] multiply: [×] divide: [÷] change of sign [+/−]

squares: [x^2] square roots: [$\sqrt{}$] powers: [x^y] roots: [$x^{1/y}$]

logs to base 10: [log] antilog to base 10 [10^x]

logs to base e: [ln] antilog to base e: [e^x]

The memory keys: clear memory; put into memory; add to the contents of memory; subtract from the contents of memory.

The symbols representing the above functions may vary from one brand of calculator to another, so check the instruction booklet which is supplied with your calculator.

The calculator is an extremely useful aid to mathematical calculations, but it can only produce the correct answer if the data is keyed in correctly and in the correct order.

For example, 200(34/690) + 124 – 80/5 could be evaluated by keying in the figures in the following order:

200[×]34[÷]690[+]124[–]80[÷]5[=]117.855

Also check that the result produced by the calculator is about the right order of magnitude, for example 200(34/690) is about 10: 124 is 124: 80/5 is 16
So the result should be approximately 134 – 16.

Accuracy: rounding numbers correct to *x* decimal places

When you use the calculator you will frequently end up with a string of numbers after the decimal point. For example, (15)/7 = 2.1428571. For most purposes you do not require all these numbers. However, if some of the numbers are dropped, subsequent calculations are less accurate. To minimise this loss of accuracy, there are rules for 'rounding' numbers correct to a specified number of decimal places, as illustrated by the following example.

Consider (a) (15)/7 = 2.1428571, (b) 6/7 = 0.8571428. Assume that three numbers after the decimal point are required. To 'round' correct to three decimal places, denoted as 3D, inspect the number in the fourth decimal place:

- if the number in the fourth decimal place is less than 5, simply retain the first three numbers after the decimal place: (b) 6/7 = 0.8571428: use 0.857, when rounded correct to 3D.
- If the number in the fourth decimal place is 5 or greater, then increase the number in the third decimal place by 1, before dropping the remaining numbers: (a) (15)/7 = 2.1428571: use 2.143, when rounded correct to 3D.

To get some idea of the greater loss of accuracy by truncating (chopping off after a specified number of decimal places) rather than rounding to the same specified number of decimal places, consider the following:

$$\left(\frac{15}{7}\right)^{20} = (2.142)^{20} \quad = 4\,134\,179.507 \quad \text{error} = -33\,212.753, \text{ when truncated to 3D}$$

$$\left(\frac{15}{7}\right)^{20} = (2.143)^{20} \quad = 4\,172\,952.303 \quad \text{error} = 5\,560.043, \text{ when rounded to 3D}$$

1.8 Introducing Excel

A spreadsheet is another aid to mathematical and statistical problem solving. An electronic spreadsheet is a software package that accepts data in the rows and columns of its worksheet as shown in Figure 1.6.

One single location on the spreadsheet is called a 'cell'. Cells A4, C2, C7, C9, E2 and H8 are highlighted in Figure 1.6. A cell is referenced by the column letter followed by the row number, for example, cell C7, is located in column C, row 7. When the data is entered, the user may then perform calculations, such as:

- Sum the data in specified rows and/or columns of the spreadsheet.

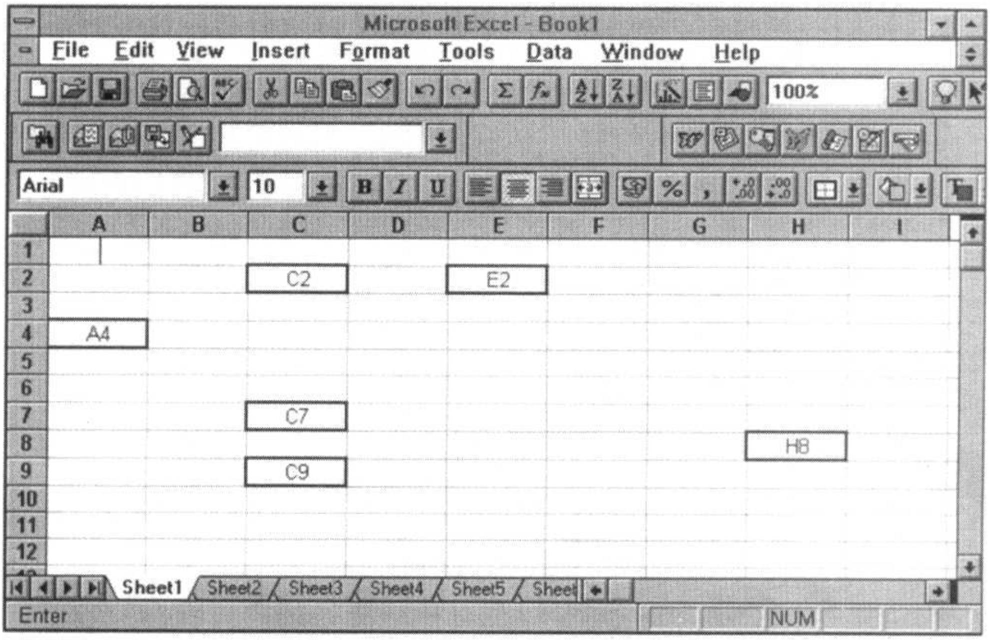

Microsoft Excel - Book1
File Edit View Insert Format Tools Data Window Help
100%
Arial
10
A B C D E F G H I
1 2 3 4 5 6 7 8 9 10 11 12
C2
E2
A4
C7
C9
H8
Sheet1 Sheet2 Sheet3 Sheet4 Sheet5 Shee
Enter
NUM

Figure 1.6 Cell reference on a spreadsheet

- Enter formula which calculates results from data in the spreadsheet.
- Plot graphs from data in the spreadsheet.

There is a variety of different spreadsheet packages available. We shall use Excel, which is available in Microsoft Office.

WORKED EXAMPLE 1.11

USING EXCEL TO PERFORM CALCULATIONS AND PLOT GRAPHS

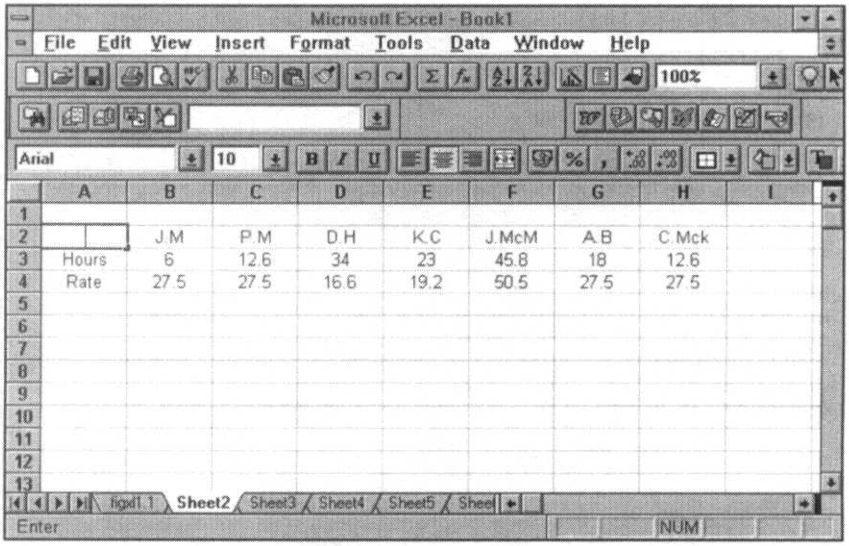
Part-time staff are paid on an hourly basis. The number of hours worked with the hourly rate of pay for seven staff are as follows:

Name	**J.M**	**P.M**	**D.H**	**K.C**	**J.McM**	**A.B**	**C.McK**
Hours	6	12.6	34	23	45.8	18	12.6
Rate	£27.5	£27.5	£16.6	£19.2	£50.5	£27.5	£27.5

(a) Enter the data onto a spreadsheet.

(b) Enter a formula to calculate the total pay for each member of staff.

(c) Plot a barchart, showing the total pay received by each member of staff.

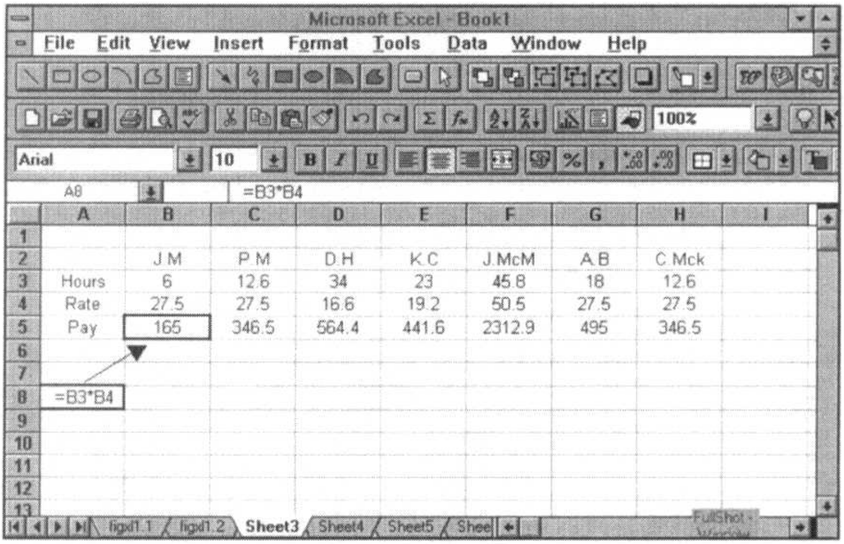

Microsoft Excel - Book1
File Edit View Insert Format Tools Data Window Help
100%
Arial
10
figxl1.1
Sheet2
Sheet3
Sheet4
Sheet5
Enter
NUM

	A	B	C	D	E	F	G	H
1								
2		J.M	P.M	D.H	K.C	J.McM	A.B	C.Mck
3	Hours	6	12.6	34	23	45.8	18	12.6
4	Rate	27.5	27.5	16.6	19.2	50.5	27.5	27.5

Figure 1.7 Data for Worked Example 1.11 entered on a spreadsheet

Solution

(a) Enter the data onto the spreadsheet as shown in Figure 1.7. You may start entering the data anywhere in the spreadsheet. In this example the data was entered starting at cell A2.


Microsoft Excel - Book1
File Edit View Insert Format Tools Data Window Help
100%
Arial
10
A8
=B3*B4
figxl1.1
figxl1.2
Sheet3
Sheet4
Sheet5
FullShot

	A	B	C	D	E	F	G	H
1								
2		J.M	P.M	D.H	K.C	J.McM	A.B	C.Mck
3	Hours	6	12.6	34	23	45.8	18	12.6
4	Rate	27.5	27.5	16.6	19.2	50.5	27.5	27.5
5	Pay	165	346.5	564.4	441.6	2312.9	495	346.5
8	=B3*B4							

Figure 1.8 Type the formul=3*B4 into cell B5 to calculate pa= 165 for JM. Copy the formula across row 5 to calculate pay for the remaining employees.

(b) Next, enter a formula to calculate total pay for each individual in the table.

$$\text{Total pay} = \text{number of hours} \times \text{hourly rate}$$

To indicate to the computer that a formula is being entered into a given cell, the first character typed should be, = , followed by the formula as follows:

- Position the cursor in cell B5 (so that hours worked, the hourly rate and total pay for J.M are all in the same column).
- Now, from the keyboard, enter the formula to calculate total pay by typing

$$= B3{*}B4$$

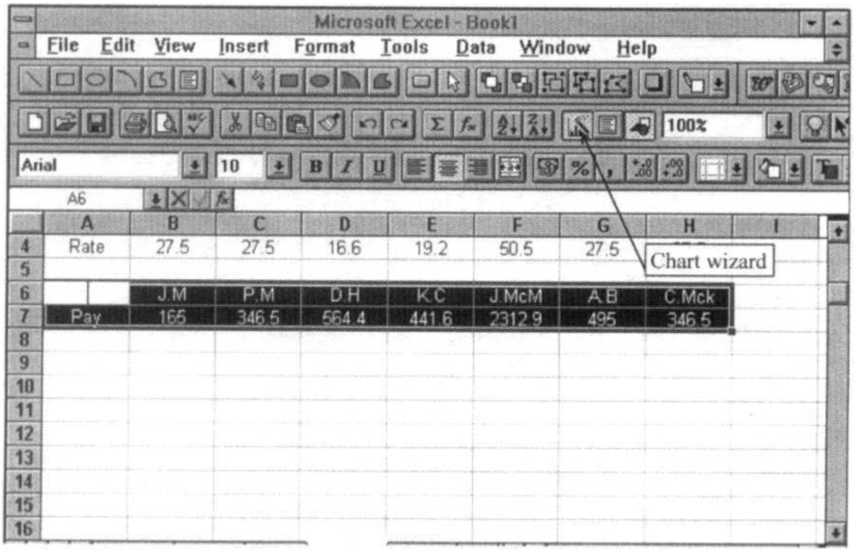
- Repeat this process for each cell in row 5: '= C3*C4': '= D3*D4' etc.

The results of the calculations are given in row 5, Figure 1.8.

Note: the advantage of entering the cell names instead of the actual values is that the result is automatically recalculated if the numbers in these cells are changed. Try it!

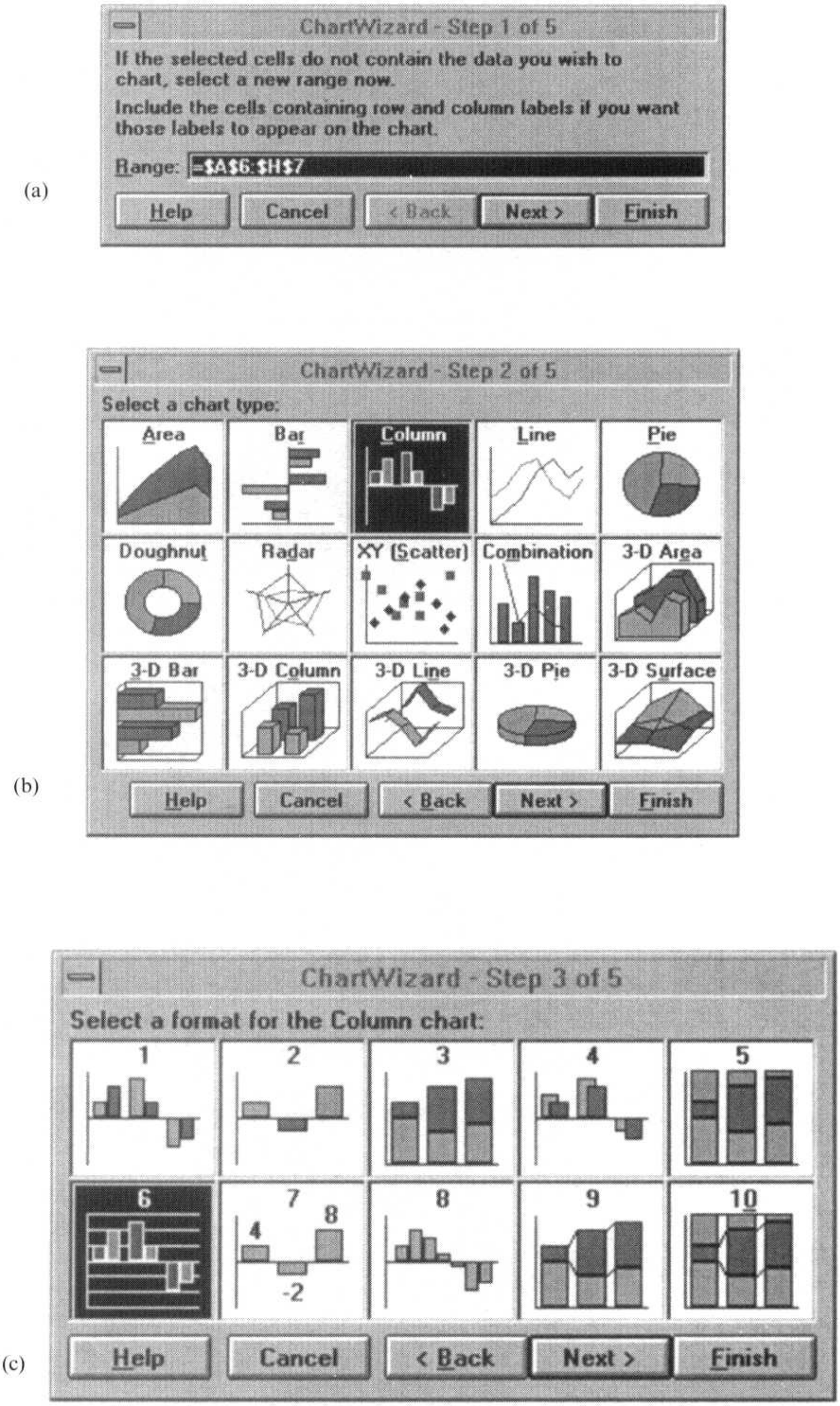
It is possible to copy the first formula in cell B5 across row 5, instead of keying the formula into each individual cell. This may be done by clicking on cell B5. It is now the active cell with a black box around it. Point to the bottom right-hand corner of the black box until a solid black cross appears, hold the left click on the mouse down and drag across cells C5 to H5. (Use the HELP button or consult some of the many reference books on Excel.)

(c) To simplify the graph sketching, copy or move the rows containing the data required for plotting together. In this example, it is required to plot total pay for each person, so the row of initials is moved from row 2 to row 6, and the row of total pay from row 5 to row 7 as shown in Figure 1.9. Now start to plot the graph.



	A	B	C	D	E	F	G	H
4	Rate	27.5	27.5	16.6	19.2	50.5	27.5	
5								
6		J.M	P.M	D.H	K.C	J.McM	A.B	C.Mck
7	Pay	165	346.5	564.4	441.6	2312.9	495	346.5

Figure 1.9 Highlight the data required for graph plotting then click on the chart wizard



ChartWizard - Step 1 of 5

If the selected cells do not contain the data you wish to chart, select a new range now.

Include the cells containing row and column labels if you want those labels to appear on the chart.

Range: =$A$6:$H$7

ChartWizard - Step 2 of 5

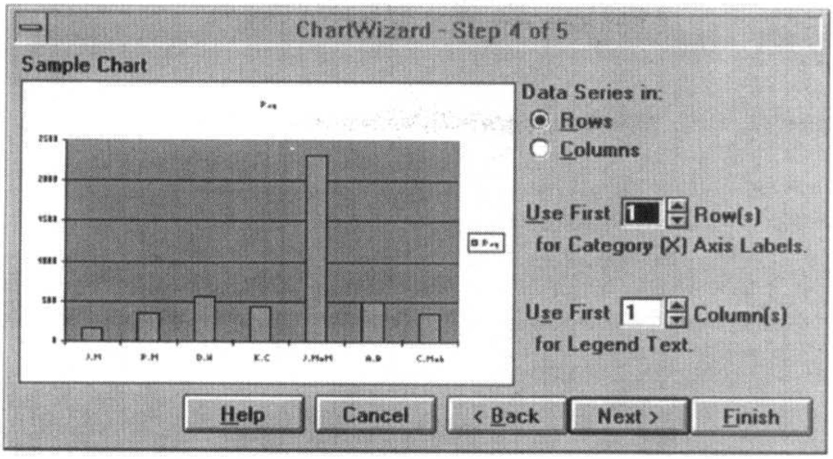
Select a chart type:

ChartWizard - Step 3 of 5

Select a format for the Column chart:

(a)
Help
Cancel
< Back
Next >
Finish
(b)
Area
Bar
Column
Line
Pie
Doughnut
Radar
XY (Scatter)
Combination
3-D Area
3-D Bar
3-D Column
3-D Line
3-D Pie
3-D Surface
Help
Cancel
< Back
Next >
Finish
(c)
1
2
3
4
5
6
7
4
8
-2
8
9
10
Help
Cancel
< Back
Next >
Finish

Figure 1.10 Excel menus for graph plotting

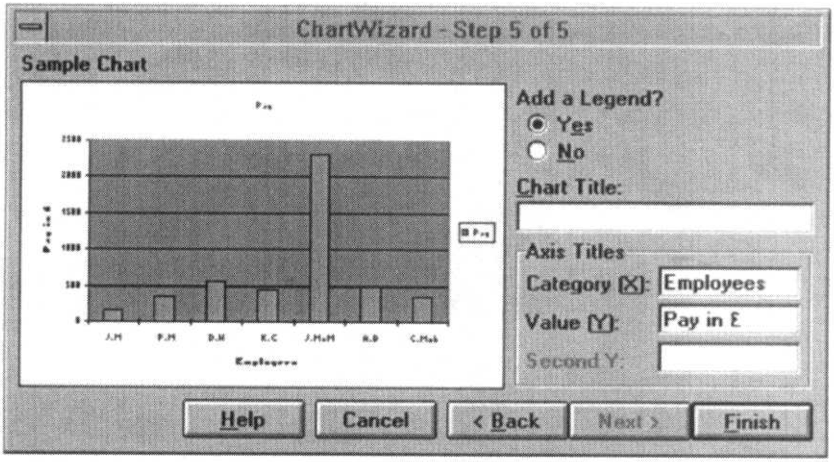

ChartWizard - Step 4 of 5
Sample Chart
Pay
2500
2000
1500
1000
500
0
Pay
Help
Cancel
< Back
Next >
Finish

Data Series in:
● Rows
○ Columns

Use First 1 Row(s) for Category (X) Axis Labels.

Use First 1 Column(s) for Legend Text.

(d)


ChartWizard - Step 5 of 5
Sample Chart
Pay
2500
2000
1500
1000
500
0
Pay in £
Employees
Pay
Help
Cancel
< Back
Next >
Finish

Add a Legend?
● Yes
○ No

Chart Title:

Axis Titles
Category (X): Employees
Value (Y): Pay in £
Second Y:

(e)

Figure 1.10 (*continued*)

To plot the graph, select the data required for the graph plotting. Then click on 'chart wizard' from the standard toolbar, as shown in Figure 1.9. A cross should now appear with the chart wizard symbol attached. Select any location on the spreadsheet for the final graph. This location and its size is not crucial since both may be changed when the graph is finished.

Excel will now guide you through the graph plotting by presenting a series of five menus as shown in Figure 1.10(a) to (e).

Step 1 of 5: Select the rows or columns of data on the spreadsheet for which the graph is required (if different from those selected already). In this example, rows 6 and 7 should already be selected, ranging from cell A6 to cell H7, as shown in Figure 1.10(a).

Step 2 of 5: Select the chart or graph type. A column bar chart is selected, Figure 1.10(b).

Step 3 of 5: Select the type of column chart. Type 6 was selected, though type 1 is also suitable, see Figure 1.10(c).

Step 4 of 5: Sample chart. This menu asks if you wish to use any row of data as names for the x-values. In this example, the first of the data rows (the initials) is to be used as the names for the x-values, while the first column represents the value of the pay for each employee.

Step 5 of 5: This menu allows you to enter a chart title and labels for axes. In Figure 1.10(e), 'Employees' was entered for the x-category, pay in £ for the y-values. Finish.

The finished graph now appears at the preselected location on the spreadsheet. However,the location may be changed and the chart edited. Consult HELP and reference manuals for further details.

Progress Exercises 1.4 Electronic Aids: the Calculator and Spreadsheets

For questions 1 and 2, evaluate the expressions using the calculator.

1. (a) $(23.6)\frac{23}{108} - 56 + \frac{90}{108}$ (b) $\sqrt{\frac{128}{20.4}}$ (c) $\sqrt{\frac{24.88}{92.5} + 256}$

2. (a) $\frac{23(1.90) - (-39)(1.8)}{12(1.55) + 18.6}$ (b) $\frac{23}{1.56} - \frac{14.57}{2.55}$ (c) $\sqrt{12 + \frac{89}{0.045}}$

3. The hours worked, expenses due and hourly rate of pay for a group of staff are given as:

Initials	J.C	J.M	S.T	R.G	P.M	K. McK
Hours	16	34	64	12.5	15.5	14.5
Rate	£12.6	£38.8	£102.5	£15.5	£32.5	£45
Expenses	£120	£55	0	£25	£12	£155

Use a spreadsheet to calculate:

(a) The total pay
(b) The net pay when a flat 20% tax is deducted from each employee
(c) The net pay, plus expenses

(d) Plot a bar chart and a pie chart of
  (i) Hours worked
  (ii) Net pay
  (iii) Expenses

Test Exercises 1

1. Simplify

(a) $2x(4) - 3(x + 1)$ (b) $(x - 1)(x + 2)$ (c) $\frac{1}{2x} - 4x$

2. Solve the equations

(a) $x(x + 1) = 0$ (b) $\frac{1}{x} + 2 = 5$ (c) $\frac{x - 1}{x^2 + 5x + 11} = 0$

3. Determine the range of values for which the following inequalities are true:

(a) $x + 2 > 10$ (b) $x < 2x - 12$ (c) $3x + 5 \geq 2x - 4$

4. The selling price of a laptop computer is £3750.
(a) If the price includes the retailer's profit of 20%, calculate the cost price of the computer.
(b) If the customer must pay 21% tax on the retail price, calculate the total paid by the customer.

5. Solve the equations

(a) $(x + 2)(x - 1) = 0$ (b) $(x + 2)(x - 1) = x^2$

6. Determine whether the expression on the LHS simplifies to the expression on the RHS.

(a) $\frac{x}{2} + \frac{5x}{4} = \frac{7x}{4}$ (b) $\frac{4x}{1/3} = 12x$

7. Theatre tickets are classified A (£35), B (£27), C (£17.50), D (£12.50). The numbers of each type sold are 180, 260, 450, 240, respectively.
(a) Set up a table to calculate the revenue from each type of ticket.
(b) Plot a bar chart of (i) the numbers of each class of ticket sold and (ii) the revenue received from each class of ticket.

8. A company's imports of tea are given by country of origin and weight: 400 tons from India, 580 tons from China, 250 tons from Sri Lanka, 120 tons from Burma.
(a) Plot the weight of tea imported on (i) a bar chart and (ii) a pie chart.
(b) Calculate the percentage of the company's imports from each country.

9. Solve the equations

(a) $20x - 3x^2 = 10(2x - 3)$ (b) $\frac{2}{x} = \frac{x}{2x} + 1$

10. Solve the equations

(a) $2x(x-2) = 2x(x+2)$ (b) $3\frac{x}{5} - 4\frac{x}{20} = 2$

11. The exchange rates for US dollars are 1.32 British pounds and 1.23 Euro.

(a) Calculate the equivalent price in (i) British pounds and (ii) Euro of a PC priced 850 US dollars.

(b) Convert 400 Euro to (i) US dollars and (ii) British pounds.

Value added tax

Value added tax (VAT) is a tax levied at each stage of the production and distribution of certain goods and services. VAT is called an 'ad valorem' tax, in that it is set at a fixed percentage of the value (or price) of the good or service.

CHAPTER

2

The Straight Line and Applications

At the end of this chapter you should understand the features of a straight line and its applications in economics and business which are covered in the following sections:

2.1 The straight line
2.2 Mathematical modelling
2.3 Applications: demand, supply, cost, revenue
2.4 More mathematics on the straight line
2.5 Translations of linear functions (web site)
2.6 Elasticity of demand, supply and income
2.7 Budget and cost constraints (web site)
2.8 Excel for linear functions
2.9 Summary

This chapter begins to establish a formal relationship between mathematics and its use in business and economics. Economics studies the relationship between different parts of an economic system; for example, the relationship between the price of a good and quantity demanded, between personal income and consumption, between production and costs. These relationships may be described by both qualitative and quantitative means. Qualitatively, the relationship between price and quantity demanded may be described by the statement 'as price increases, quantity demanded decreases'. This verbal statement, however, is not precise as it does not tell us by how much quantity demanded decreases when price increases. It is, therefore, necessary to establish a quantitative relationship between price and quantity

Table 2.1 Price and quantity observations

Price	Quantity demanded
$P = 10$	$Q = 58$
$P = 40$	$Q = 40$
$P = 80$	$Q = 23$
$P = 120$	$Q = 10$

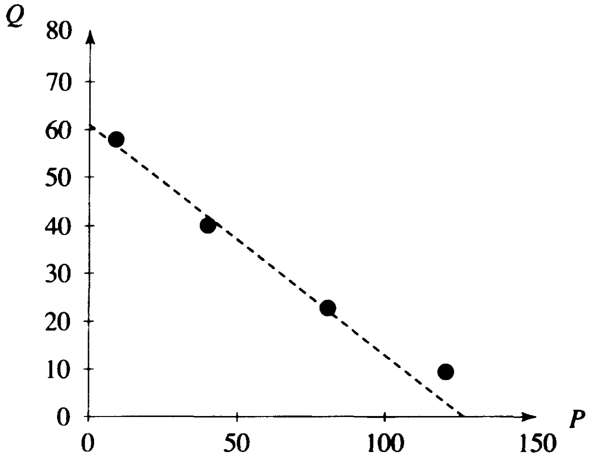


Figure 2.1 Graph of quantity demanded (Q) at set prices (P)

demanded. The first step could be to make several observations of the quantity demanded (Q) at different prices (P) (see Table 2.1). The results are plotted on a graph (see Figure 2.1).

From the graph, the mathematical form of the relationship, a formula relating Q to P, can usually be deduced. In the example given, the relationship is quite closely modelled by a straight line, or we say the relationship is linear. This formula, or equation, relating Q to P is called a mathematical model. Mathematical models are essential if a system is to be analysed and the exact relationships between economic quantities identified and understood. The simplest quantitative relationship is the straight line or linear function. Due to its simplicity, the linear function serves as an introductory mathematical model for a variety of business and economic applications. This chapter introduces some of these applications. Initially, however, it is necessary to study the straight line.

2.1 The Straight Line

At the end of this section you should be able to:

- Define the slope and intercept of a straight line and plot its graph
- Understand what 'the equation of a line' means
- Plot any straight line given the equation in the form $y = mx + c$ or $ax + by + d = 0$

2.1.1 The straight line: slope, intercept and graph

The straight line is one of the simplest mathematical functions: it is easy to manipulate and to graph. It is, therefore, essential to fully understand the straight line, what each part of the equation means and the effect of changes in each part on the graph of the line.

❐ Introductory background on graphs

Figure 2.2 shows the horizontal axis, x, and the vertical axis, y, which intersect at the origin where $x = 0$ and $y = 0$. Any point can be plotted on this graph if the coordinates, (x, y) are known; that is, the x-coordinate (with sign) is stated first and measured along the horizontal axis, followed by its y-coordinate (with sign) measured along the vertical axis. The points (2, 2), (1, 1), (2, −2), (− 4, −3), (−3, 2) and (0, 0) are plotted in Figure 2.2.

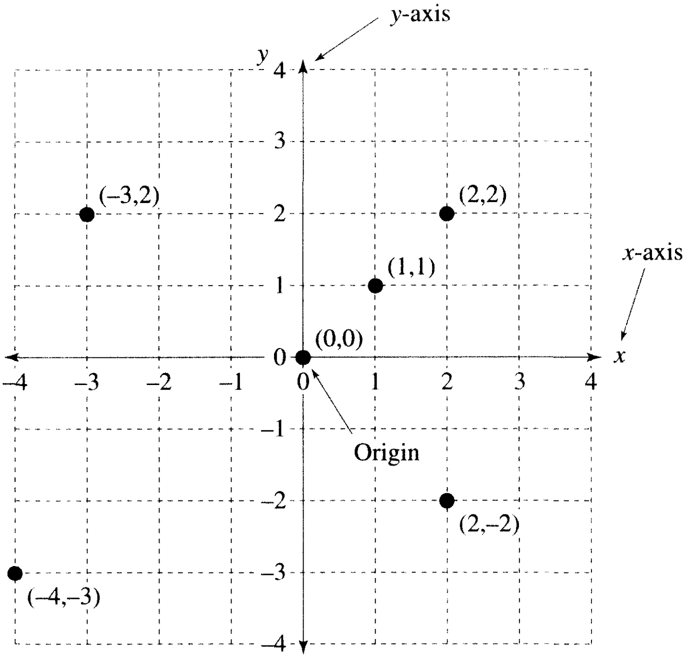


Figure 2.2 Plotting points on a graph

❐ How to define a straight line

A straight line may be defined by two properties:

- **Slope**, usually represented by the symbol m
- **Vertical intercept**, the point at which the line crosses the y-axis, usually represented by the symbol c.

Note: The vertical or y-intercept is often simply called 'the intercept'. This is the convention adopted throughout this text.

❒ Slope

The slope of a line is simply the '**slant**' of the line. The slope is negative if the line is falling from left to right and the slope is positive if the line is rising from left to right, (see Figure 2.3). In Figure 2.3, each line has a different slope and a different intercept. Note the horizontal line which has a zero slope and the vertical line which has an infinite slope.

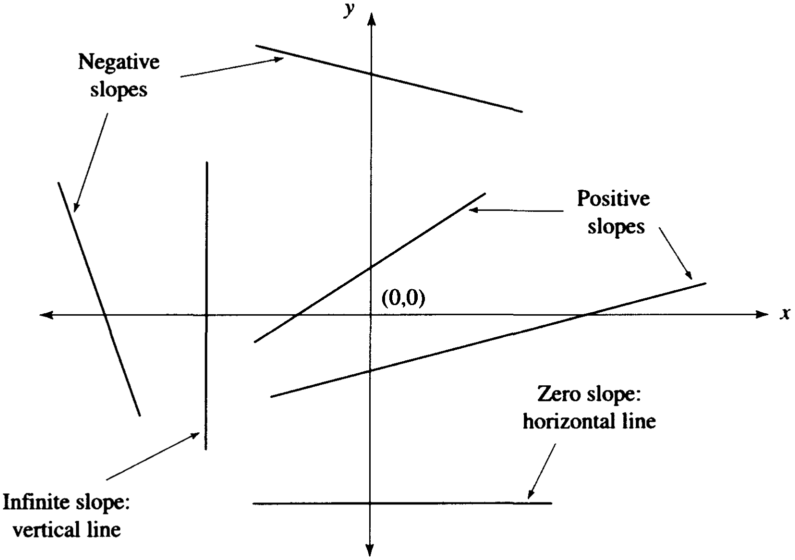


Figure 2.3 Lines with different slopes and different intercepts

❒ Measuring the slope of a line

Figure 2.4 shows how the slope of a line is measured.

- The line through the points A and B has a positive slope since it is rising from left to right. The slope is measured as follows:

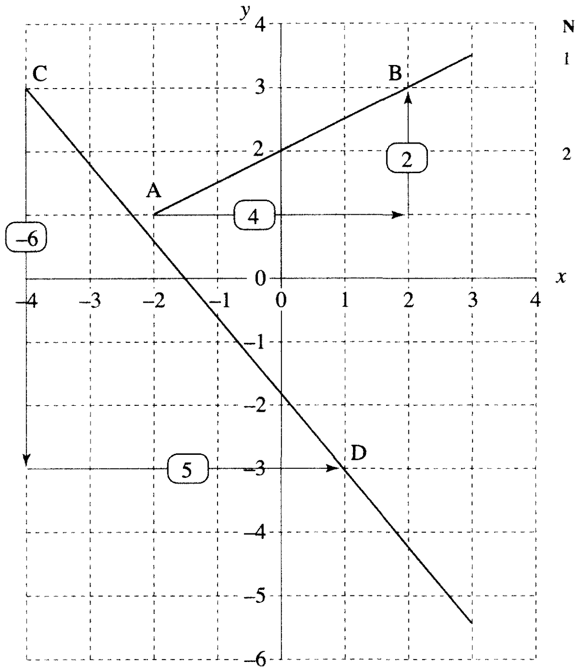
$$\text{Slope} = \frac{\text{change in height}}{\text{change in distance}} = \frac{2}{4} = \frac{1}{2} = \frac{0.5}{1} = 0.5$$

That is, height increases by 0.5 units when the horizontal distance increases by 1 unit.

- The line through the points C and D has a negative slope since the line is falling from left to right. A decrease in height is indicated by a minus sign. The slope is measured as follows:

$$\text{Slope} = \frac{\text{change in height}}{\text{change in distance}} = \frac{-6}{5} = \frac{-1.2}{1} = -1.2$$

That is, height decreases by 1.2 units when the horizontal distance increases 1 unit.



Notes

1 Height increases by 0.5 for every one unit increase in distance along the line AB

2 Height decreases by 1.2 for every one unit increase in horizontal distance along CD

Figure 2.4 Measuring slope

Note: The change in height and change in distance are generally referred to as Δy and Δx respectively where the symbol Δ means change.

The slope of a line may be described in several ways:

1. The change in height (Δy) divided by the corresponding increase in horizontal distance (Δx):
$$\text{slope} = \frac{\Delta y}{\Delta x}$$
2. The change in height per unit increase in horizontal distance
3. The number of units by which y changes when x increases by 1 unit

- Lines with the same slope, but different intercepts are different lines: they are parallel
- Lines with the same intercept, but different slopes are different lines
- Two lines are identical only if their slopes and intercepts are identical

WORKED EXAMPLE 2.1

PLOTTING LINES GIVEN SLOPE AND INTERCEPT

Plot the lines given the following information on slope and intercept:
(a) Slope = 1, intercept = 0 ($m = 1, c = 0$) (b) Slope = 1, intercept = 2 ($m = 1, c = 2$)

Solution

(a) Locate the vertical intercept, $y = 0$ on the y-axis. This gives the point (0, 0) which is the origin. Since slope is 1 and, slope $= \Delta y/\Delta x$, start at the origin (intercept) and move forward a horizontal distance of 1 unit and move up a vertical height of 1 unit; see Figure 2.5. This gives the point (1, 1). Join the points (0, 0) and (1, 1); continue the straight line until it is the required length. This line is often referred to as the 45° line: a line through the origin which makes an angle of 45° with the horizontal axis (use $\Delta x = 1$ for simplicity). Any change in the horizontal distance is accompanied by an equal change in the vertical height.

(b) Locate the vertical intercept, $y = 2$. At $y = 2$, set up a slope of 1, that is, move out a horizontal distance of 1 unit and move up a vertical height of 1 unit. This gives the point (1, 3). Join the points (0, 2) and (1, 3); continue the straight line until it is the required length. This line is illustrated in Figure 2.6.

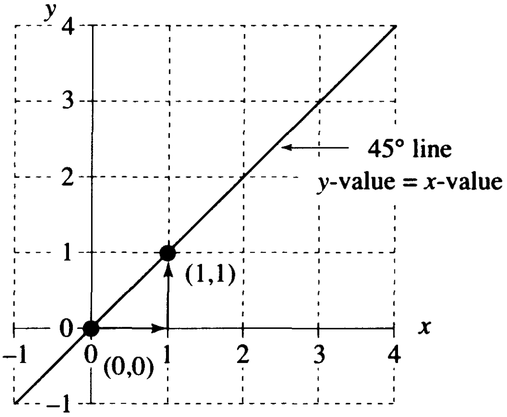


Figure 2.5 45° line through the origin

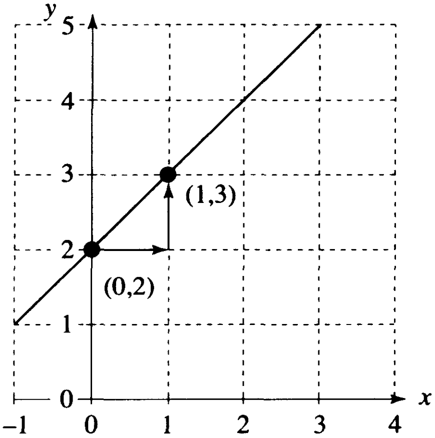


Figure 2.6 Line with slope = 1, intercept = 2

Progress Exercises 2.1 Points, Slopes and Intercepts for Linear Functions

1. Plot the points $(-2, 0), (0, 2), (2, 4), (4, 6)$, and $(5, 7)$. The plotted points lie on a straight line. Measure the slope and vertical intercept.
2. Plot the points $(-1, 7), (0, 6), (1, 5), (3, 3), (5, 1)$ and $(7, -1)$. The plotted points lie on a straight line. Measure the slope and vertical intercept.

Draw the following lines given in questions 3 to 6. Measure the point where each line cuts the horizontal axis.

3. Slope = 2, y-intercept = 0.
4. Slope = 1, y-intercept = −2.
5. Slope = −2, y-intercept = 0.
6. Slope = −1, y-intercept = 2.

2.1.2 The equation of a line

Worked Example 2.1 demonstrated how a straight line is determined by its slope and intercept. It is not, however, convenient for mathematical manipulations and other applications to work with a line described only in terms of slope and intercept. A line defined by an equation is more useful. The equation of a line is simply a formula which shows how the y-coordinate is related to the x-coordinate for every point on the line.

The formula which calculates the y-coordinate for any given x-value on the line is called

THE EQUATION OF THE LINE

Worked Example 2.2 will demonstrate that the equation of a line has the general form

$$y = mx + c \tag{2.1}$$

where m is the symbol representing slope and c is the symbol representing the intercept on the y-axis.

WORKED EXAMPLE 2.2

DETERMINE THE EQUATION OF A LINE GIVEN SLOPE AND INTERCEPT

(a) Demonstrate that the equation of the line with slope $m = 3$ and intercept $c = 1$ is $y = 1 + 3x$. Hence deduce that the equation of any line is $y = mx + c$.
(b) Find the equations of the lines with slope and intercept given in Worked Example 2.1:
(i) $m = 1, c = 0$
(ii) $m = 1, c = 2$

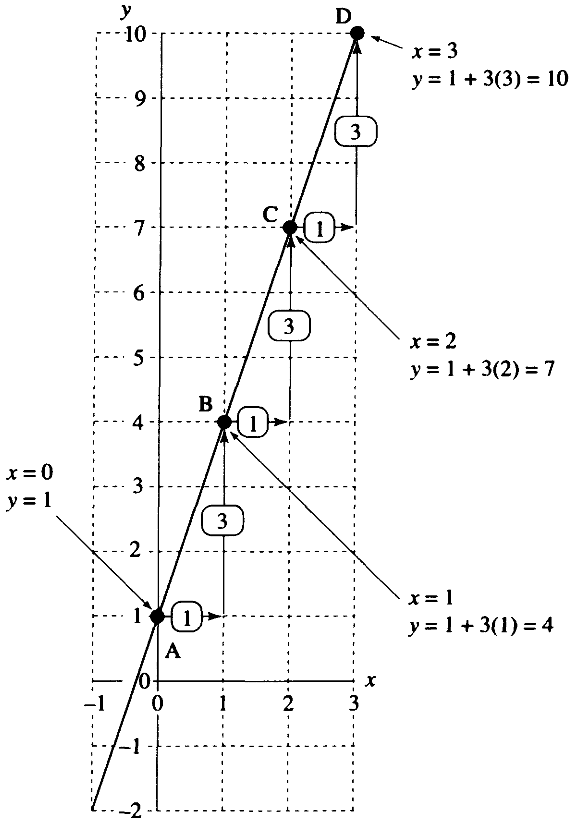


Figure 2.7 Deducing the equation of the line $y = 1 + 3x$

Solution

(a) See Figure 2.7. The line is drawn starting at the intercept $c = 1$ on the y-axis, that is at the point $x = 0$, $y = 1$.

Then, for each subsequent unit increase in x, y increases by 3 units since slope $m = 3$.

☺ **Remember**, slope is the change in y per unit increases in x.

Now observe the pattern relating the x and y values for selected points on the line.

x-value	y-value	Calculation of y-value
$x = 0$	$y = 1$	this is the intercept
$x = 1$	$y = 4$	$y = 1 + 3 \times$ (x-value)
$x = 2$	$y = 7$	$y = 1 + 3 \times$ (x-value)
$x = 3$	$y = 10$	$y = 1 + 3 \times$ (x-value)
any x-value	$y = 1 + 3x$	$y = 1 + 3 \times$ (x-value)

$$\boxed{y = 1 + 3x \text{ is the equation of the line}}$$

To generalise the above situation, see Figure 2.8. Figure 2.8 is simply Figure 2.7 redrawn with the intercept referred to as c, slope as m. Now observe the pattern relating the x and y values for selected points on the line.

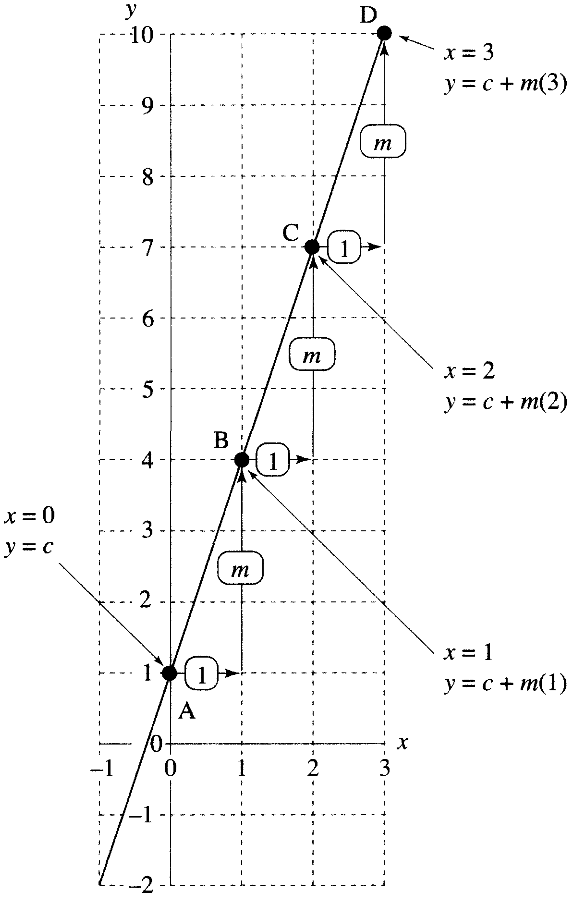


Figure 2.8 Deducing the equation of the line $y = mx + c$ with $m = 3$, $c = 1$

x-value	**y-value**	**Calculation of y-value**
$x = 0$	$y = c$	this is the intercept
$x = 1$	$y = c + m(1)$	$y = c + m \times$ (x-value)
$x = 2$	$y = c + m(2)$	$y = c + m \times$ (x-value)
$x = 3$	$y = c + m(3)$	$y = c + m \times$ (x-value)
any x-value	$y = c + m(x)$	$y = c + m \times$ (x-value)

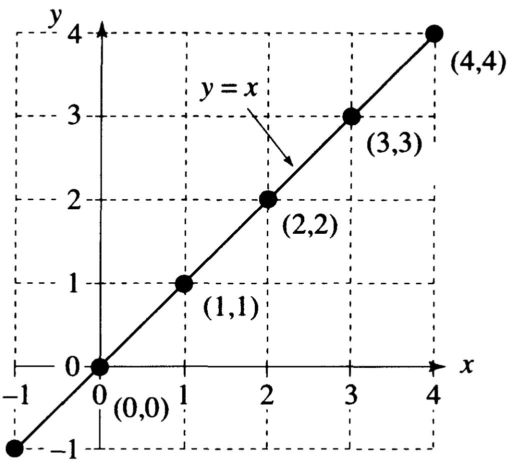


Figure 2.9 Graph of the line $y = x$

For each value of x, the y-coordinate is calculated by the formula $y = mx + c$:

> $y = c + mx$ or $y = mx + c$ is the equation of the line

Hence, if the slope and intercept of a line are given, the equation of that line may be obtained by substituting the given values of m and c into the general formula $y = mx + c$.

(b) (i) Figure 2.9 is simply Figure 2.5 redrawn with additional points labelled. The equation of the line may be determined by substituting $m = 1$, $c = 0$ into the general formula $y = mx + c$:

$$\begin{aligned} y &= 1x + 0 \\ y &= x \end{aligned} \tag{2.2}$$

This is the equation of the line.

☺ **Remember**, $y = x$ is the equation of the 45° line through the origin.

Observe that, for each point on the line, the y-coordinate is equal to the x-coordinate.

The equation of the line is $y = x$.

(ii) The equation of the line with $m = 1$, $c = 2$ is determined by substituting $m = 1$, $c = 2$ into the general formula:

$$\begin{aligned} y &= mx + c \\ y &= 1x + 2 \\ y &= x + 2 \end{aligned} \tag{2.3}$$

See Figure 2.10 for the graph of this line.

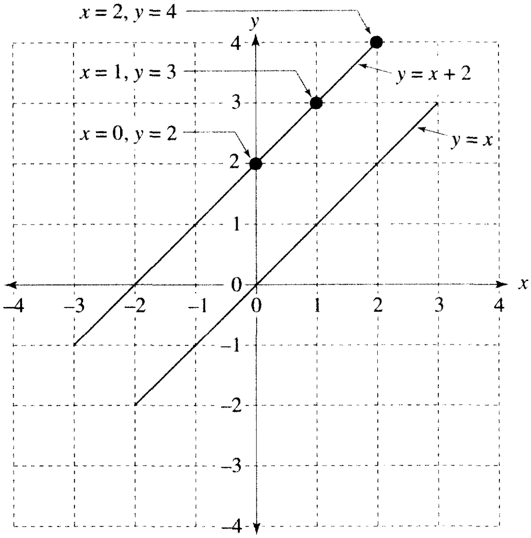


Figure 2.10 Comparing the lines $y = x$ and $y = x + 2$

Notes

(i) Observe that for each point on the line $y = x + 2$ the value of the y-coordinate is equal to the value of the x-coordinate plus 2, hence its equation.

(ii) The lines $y = x$ and $y = x + 2$ have the same slope, $m = 1$, with different intercepts; they are parallel lines.

❐ Horizontal intercepts

The above analysis has been concerned with the slope of a line and the vertical intercept only insofar as these two properties may be used to define a straight line. However, there are times when the calculation of the horizontal intercept is required, that is, the point where the line crosses the horizontal axis. This calculation uses the fact that every point on the horizontal axis has a y-coordinate $y = 0$. Since y is known, substitute $y = 0$ into the equation of the line, $y = mx + c$, and solve for x: this value of x ($x = -c/m$) is the horizontal intercept. See the following worked example.

WORKED EXAMPLE 2.3

CALCULATION OF HORIZONTAL AND VERTICAL INTERCEPTS

Calculate the horizontal and vertical intercepts for each of the following lines:

(i) $y = mx + c$ (ii) $y = x + 2$ (iii) $y = 1 + 3x$.

Method	**(i) $y = mx + c$**	**(ii) $y = x + 2$ (Figure 2.10)**	**(iii) $y = 1 + 3x$ (Figure 2.7)**
Equation of line	$y = mx + c$	$y = x + 2$	$y = 1 + 3x$
(i) Horizontal or x-intercept			
Line cuts x-axis at $y = 0 \rightarrow$	$0 = mx + c$	$0 = x + 2$	$0 = 1 + 3x$
Solve for $x \rightarrow$	$-c = mx$	$-2 = x$	$-1 = 3x$
	$-c/m = x$		$-1/3 = x$
This is the horizontal intercept	$x = -\frac{c}{m}, y = 0$	$x = -2, y = 0$	$x = -\frac{1}{3}, y = 0$
(ii) Vertical or y-intercept*	$y = mx + c$	$y = x + 2$	$y = 1 + 3x$
Line cuts y-axis at $x = 0 \rightarrow$	$y = m(0) + c$	$y = 0 + 2$	$y = 1 + 3(0)$
Substitute $x = 0$, solve for y	$y = c$	$y = 2$	$y = 1$
This is the vertical intercept	$x = 0, y = c$	$x = 0, y = 2$	$x = 0, y = 1$

*Alternatively, read off the vertical intercept, $y = c$, from the equation $y = mx + c$

Note: In the rest of the text, the horizontal intercept may be denoted by $x(0)$; this simply means the 'value of x when $y = 0$' or x (when $y = 0$), abbreviated to $x(0)$. Similarly, the vertical intercept, which is the value of y when x is zero, may be denoted by $y(0)$.

☐ Equation of horizontal and vertical lines

Figure 2.11 illustrates that the equation of a horizontal line simply takes the value of the coordinate where the line crosses the y-axis, and the equation of a vertical line simply takes the value of the coordinate where the line crosses the x-axis.

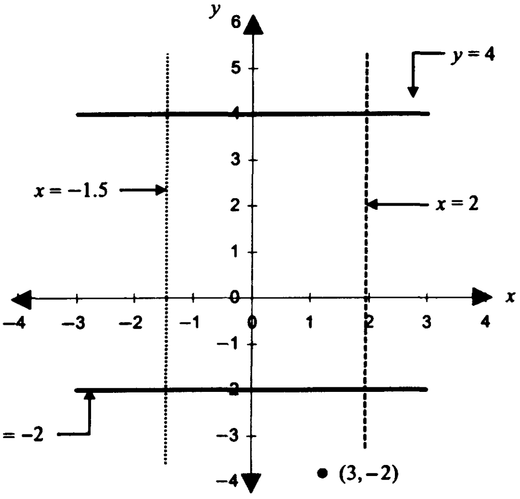


Figure 2.11 Equation of horizontal and vertical lines

❐ **For any straight line**

The equation is simply the formula which relates the x- and y-coordinates for every point on the line (or curve, as we shall see later).

y = slope (change in y per unit increase in x × (units moved from $x = 0$) + intercept
$y = m \times x + c$

2.1.3 To graph a straight line from its equation

❐ **To graph a straight line from its equation $y = mx + c$**

- Earlier in this chapter lines were plotted, starting at the intercept (first point), then plotting further points using the fact that the slope is the change in y per unit increase in x.
- In this section a straight line is graphed from its equation by calculating at least two points then drawing a line through them. To calculate the required points, use either of the following methods.

Method A

Calculate a table of points (at least two points).

Step 1: Rearrange the equation into the form $y = mx + c$, if it is not already in this form.

Step 2: Choose any x-values (choose convenient values of x, such as $x = 0, x = 10, x = -2$); calculate the corresponding y-values, using the rearranged formula.

Step 3: Plot the points and draw the line through these points.

Method B

In economics, graphs are frequently required in the region where x-values and y-values are positive. To plot a line in this region, calculate the horizontal and vertical intercepts.

Step 1: Rearrange the equation into the form $y = mx + c$ if necessary.

Step 2: The x-intercept is at $(x = -c/m, y = 0)$; the y-intercept is at $(x = 0, y = c)$.

Step 3: Draw the line through the points.

Note: While a minimum of two points are required to plot a line, it is a good idea to calculate at least three points, just in case of mistakes in calculations. All points should lie perfectly on the line.

WORKED EXAMPLE 2.4

TO GRAPH A STRAIGHT LINE FROM ITS EQUATION $y = mx + c$

A line has the equation (or formula) $y = 2x - 1$
Plot the line over the interval $x = -2$ to $x = 3$

Solution

Method A

Step 1: $y = 2x - 1$. No rearranging necessary.
Step 2: Since the graph is required for values of x from $x = -2$ to $x = 3$, calculate y for several x-values within this interval. You may choose any x-values you wish, for example $x = -2, -1, 0, 1, 2, 3$. Calculate the corresponding y-coordinates by simply substituting the chosen x-value into the equation (Table 2.2).

Table 2.2 Calculating y-coordinate given the x-coordinate

x	$y = 2x - 1$	Points (x, y)
−2	$y = 2(-2) - 1 = (-4) - 1 = -5$	−2, −5
−1	$y = 2(-1) - 1 = (-2) - 1 = -3$	−1, −3
0	$y = 2(0) - 1 = (0) - 1 = -1$	0, −1
1	$y = 2(1) - 1 = (2) - 1 = 1$	1, 1
2	$y = 2(2) - 1 = (4) - 1 = 3$	2, 3
3	$y = 2(3) - 1 = (6) - 1 = 5$	3, 5

Step 3: Plot the points, draw the line; see Figure 2.12.

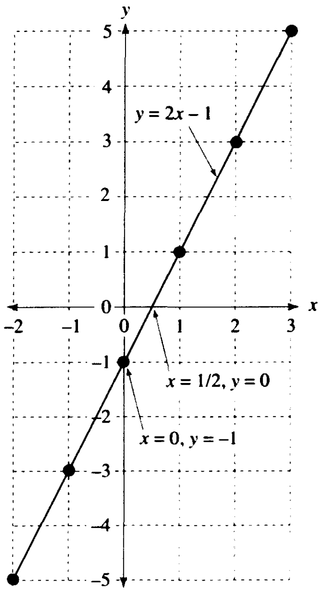


Figure 2.12 Plot the line $y = 2x - 1$

Method B

Step 1: $y = 2x - 1$. No rearranging necessary.
Step 2: **Vertical intercept:** the line cuts the vertical axis at $y = c \rightarrow y = -1$, hence vertical intercept is $(0, -1)$.

Horizontal intercept: the line cuts the horizontal axis at $x = -c/m \rightarrow x = 0.5$, hence horizontal intercept is (0.5, 0).

Note: Rather than memorising formulae, such as $x = -c/m$, $y = 0$ for the horizontal intercept, you may prefer to solve for the x-intercept directly, as follows. Since the line cuts the x-axis when $y = 0$, substitute $y = 0$ into the equation and solve for x:

$$\left.\begin{aligned} y &= 2x - 1 \\ 0 &= 2x - 1 \\ 1 &= 2x \\ \frac{1}{2} &= x \end{aligned}\right\} \rightarrow \text{hence horizontal intercept at } x = 0.5, y = 0$$

Step 3: Plot the vertical and horizontal intercept on the axis, then draw the line through these points. See Figure 2.12.

❐ Graph any line $ax + by + d = 0$

Any equation of the form $ax + by + d = 0$, where a, b and d are constants, is the equation of a straight line. When the equation is rearranged into the form $y = mx + c$, the slope and intercept on the y-axis can be written down immediately. It is helpful to have a rough idea of the graph before you start plotting. Therefore, by writing down m and c you know where the line crosses the vertical axis and whether the slope is positive or negative. Rearranging to obtain slope and intercept:

$$ax + by + d = 0$$

$$ax + by = -d \qquad \text{bringing across the } d$$

$$by = -ax - d \qquad \text{bringing across the } ax$$

$$\frac{1}{b}by = \frac{1}{b}(-ax - d) \qquad \text{dividing each side by } b$$

$$y = \frac{1}{b}\frac{(-ax)}{1} + \frac{1}{b}\frac{(-d)}{1} \qquad \text{multiplying each term inside the bracket by } \frac{1}{b}$$

$$y = -\frac{a}{b}x - \frac{d}{b} \qquad \text{simplifying the fractions}$$

This is a straight line that cuts the vertical axis at $y = -d/b$ with a slope $-a/b$. The slope $-a/b$ will be positive or negative depending on the signs of a and b.

Note: It is not necessary to memorise these formulae for intercept and slope. In numerical examples simply rearrange the equation with y on the LHS (this is called making y the subject of the formula), then read off the value of the intercept and slope, as shown in the following worked example.

WORKED EXAMPLE 2.5

PLOT THE LINE $ax + by + d = 0$

The equation of a line is given as $4x + 2y - 8 = 0$.

(a) Write the equation of the line in the form $y = mx + c$, hence write down the slope and y-intercept. Give a verbal description of slope and intercept.

(b) Which variable should be plotted on the vertical axis and which on the horizontal axis when the equation of the line is written in the form $y = mx + c$?

(c) Plot the line by (i) calculating a table of points (**method A**) and (ii) calculating the intercepts (**method B**).

Solution

(a) Rearrange the equation of the line with y as the only variable on the LHS:

$$4x + 2y - 8 = 0$$

$$2y = 8 - 4x \qquad \text{bringing all non-}y\text{ terms to the RHS}$$

$$\frac{2y}{2} = \frac{8}{2} - \frac{4x}{2} \qquad \text{dividing everything on both sides by 2}$$

$$y = 4 - 2x \qquad \text{simplifying}$$

From the rearranged equation we can read off useful information before going ahead to plot the line. Intercept: the line cuts the y-axis at $y = 4$. So when $x = 0$, $y = 4$. Slope: the line has slope $m = -2$, i.e. the line is falling from left to right, dropping by 2 units for each horizontal increase of 1 unit.

(b) When the equation is written in the form $y = mx + c$ the variable by itself on one side of the equation, y, is usually plotted on the vertical axis, the other variable, x, is plotted on the horizontal axis.

(c)

(i) Calculate a table of points	**(ii) Calculate the intercepts**
Step 1: Done in part (a)	**Step 1:** Done in part (a)
Step 2: From the equation of the line $y = -2x + 4$, calculate y for several x-values, such as: $x = 0, y = -2(0) + 4 = 4$, giving the point (0, 4) $x = 1, y = -2(1) + 4 = 2$, giving the point (1, 2) $x = 2, y = -2(2) + 4 = 0$, giving the point (2, 0) $x = 3, y = -2(3) + 4 = -2$, giving point (3, −2)	**Step 2:** The equation is $y = -2x + 4$ **The line cuts the y-axis** at $y = c$, $y = 4$ Hence the vertical intercept is (0, 4) **The line cuts the x-axis** at $x = -c/m$ $x = \frac{-4}{-2} = 2$ Hence the horizontal intercept is (2, 0)*
Step 3: Plot these points and draw the line through them (Figure 2.13)	**Step 3:** Plot the intercepts, join them and you have the graph of the line (Figure 2.13)

* Alternatively, solve for the x-intercept by using the fact that $y = 0$ at the point where the line cuts the x-axis.

$$\left.\begin{array}{l} y = -2x + 4 \\ 0 = -2x + 4, \text{ since } y = 0 \text{ on the } x\text{-axis} \\ 2x = 4 \text{ i.e. } x = 2 \end{array}\right\} \text{hence the horizontal intercept is } x = 2,\ y = 0$$

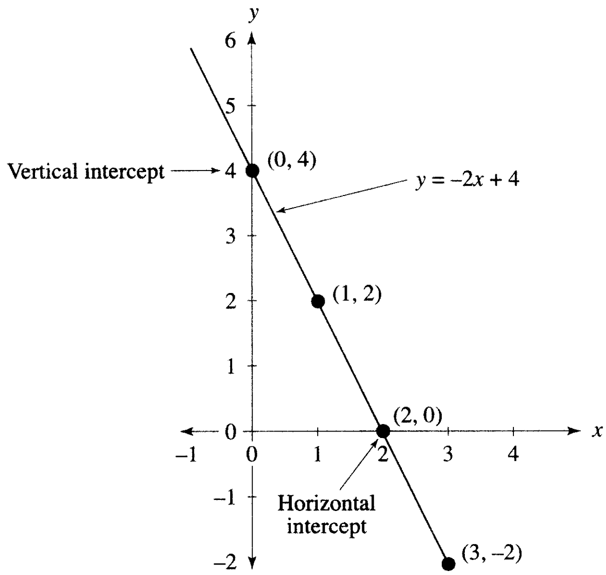


Figure 2.13 Plotting the line $y = -2x + 4$

❒ Some mathematical notations

Take the line $y = mx + c$. In general, when y is written in terms of x we say that '**y is a function of x**'. A function of x may be described as a rule for 'operating' on x. In the case of $y = mx + c$, the rule is: multiply x by the constant, m, then add the constant, c, to obtain the value of y.

x is called the independent variable (in plotting a line from a given equation, any suitable x-value is chosen, as in Worked Example 2.4).

The independent variable, x, is plotted on the horizontal axis.

y is called the dependent variable: since the y-values are calculated from the independently chosen x-values. Usually the dependent variable, y, is the variable which is written by itself on the LHS of the equation.

The dependent variable y, is plotted on the vertical axis.

❒ General notations

'y is a function of x' may be denoted in several ways: $y = \mathrm{f}(x)$; $y = \mathrm{g}(x)$; $y = \mathrm{h}(x)$, etc. where the letters f, g and h are used to distinguish different formulae in x. (The brackets here do not mean multiplication, they mean '**y depends on x**'.) For example:

(a) $y = -20x + 41$ or $y = \mathrm{f}(x) = -20x + 41$

(b) $y = 11x - 12$ or $y = \mathrm{g}(x) = 11x - 12$

Inverse function: When equations are used in applications later, there are times when we write y in terms of x, denoted as $y = \mathrm{f}(x)$, but in other applications we require x in terms of y, denoted as $x = \mathrm{f}^{-1}(y)$. This latter function, $x = \mathrm{f}^{-1}(y)$, is called the inverse function. For example:

$y = 11x - 12$ written symbolically as $y = \mathrm{f}(x) = 11x - 12$

$x = \dfrac{y + 12}{11}$ rearranging, writing x as the variable on the LHS

$x = \dfrac{y}{11} + 1.09$

This is the inverse function written symbolically as,

$$x = \mathrm{f}^{-1}(y) = \frac{y}{11} + 1.09$$

Progress Exercises 2.2 Straight Lines: Equations, Slopes and Intercepts

1. Deduce the equations of the lines in Progress Exercises 2.1, questions 3, 4, 5, 6.

   (3) Slope = 2, y-intercept = 0. (4) Slope = 1, y-intercept = −2.

   (5) Slope = −2, y-intercept = 0. (6) Slope = −1, y-intercept = 2.

2. Given the equations of the lines:

   (a) $y = x + 2$ (b) $y = -4x + 3$ (c) $y = 0.5x - 2$ (d) $2y = 6x + 4$

   (i) Write down the slope and intercepts (horizontal and vertical) for each line.
   (ii) Calculate the values of y when $x = -2, 0, 2, 4, 6$.
   (iii) Plot each line over the interval $x = -2$ to $x = 6$.

3. Write the equation for each of the following lines in the form $y = \mathrm{f}(x)$, hence write down the slope, intercepts (if any):

   (a) $y = 2$ (b) $x = -2$ (c) $5x + y + 4 = 0$ (d) $y = x$ (e) $x - y + 5 = 0$

   Plot the intercepts, hence plot the line by joining the intercepts.

4. Given the equations of the following lines:

   (a) $2y - 5x + 10 = 0$ (b) $x = 10 - 2y$ (c) $y + 5x = 15$

   (i) Write each of the equations in the form $y = \mathrm{f}(x)$.
   (ii) Calculate the intercepts. Plot each line by joining the intercepts.
   (iii) From the graph, show that the magnitude of slope is the ratio: vertical intercept/horizontal intercept.

   From the graph, how would you determine whether the slope is positive or negative?

5. (See question 4) For each line (a), (b), (c)
   (i) Write the equations in the form $x = \mathrm{g}(y)$.
   (ii) Plot each line as $x = \mathrm{g}(y)$, showing both intercepts.

   Compare the graphs, particularly intercepts and slopes, in questions 4 and 5.

6. Determine which of the following points lie on the given line.
   (a) Points (x, y): A = $(1, 3)$, B = $(-1, -1)$, C = $(0, 1)$ and line, $y = 2x + 1$.
   (b) Points (P, Q): A = $(90, 5)$, B = $(8, 10)$, C = $(70, 15)$ and line, $Q = 50 - 0.5P$.
   (c) Points (Q, TC): A = $(2, 14)$, $B = (14, 18)$, $C = (6, 22)$ and line, $TC = 10 + 2Q$.

7. (a) Plot the points (4, 4), (2, 4), (12, 4) and $(-2, 4)$.
   (b) Plot the points (2, 10), (2, 5), (2, 1) and (2, -2).
   The points plotted in parts (a) and (b) each lie on a straight line. Deduce the equation of the line for the points in (a) and (b).

8. On the same diagram plot the following lines in the form $y = \mathrm{f}(x)$:

   (a) $y = -2x + 5$ (b) $y + 2x + 5 = 0$ (c) $0.2y + 0.4x = 2$

   (i) State any properties which the lines (a), (b) and (c) have in common.
   (ii) Do these common properties change when each equation (a), (b) and (c) are written in the form $x = \mathrm{g}(y)$?

2.2 Mathematical Modelling

At the end of this section you should:

- Understand what the term 'mathematical modelling' means
- Understand the importance of economic models
- Understand the interactions between the various parts of an economy: circular flow model

2.2.1 Mathematical modelling

The idea of a 'model' is used in many areas of life to represent aspects of reality. In the broadest sense, models may be classified as physical or abstract.

- **Physical models (static):** such as fashion models, miniature replicates of buildings, cars
- **Physical models (dynamic):** such as the aerodynamics of a car in a wind tunnel
- **Abstract models (static):** such as a chart showing the interrelationships within a large organisation; the design of a car engine; the age/sex structure of a population; the relationship between the price of a good and the quantity demanded, etc. These are called static models, since the model is concerned with reproducing the state of the system.
- **Abstract models (dynamic):** such as the flow of information within an organisation; how the car engine works when switched on; how the population changes in time.

Models may be further classified as deterministic or stochastic:

- A **deterministic model:** the outcome can be determined exactly when all the inputs are given. The models which follow are all deterministic. For example, if P is the price of a lunch and Q is the number of lunches served, then the equation $Q = 20 - 2P$, describes how the number of lunches (quantity) demanded depends on the price. The variables are P and Q. According to this equation or model, when any price, P, is specified, then Q, the number of lunches demanded, may be calculated exactly.

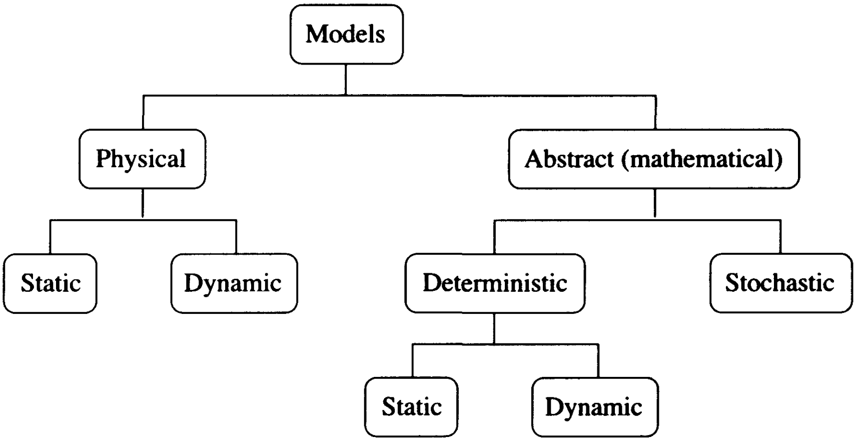


Figure 2.14 The place of mathematical modelling in the scheme of modelling

- **A stochastic model:** the outcome is calculated with a certain probability. For example, when a fair die is tossed, the possible values which turn up are the numbers 1, 2, 3, 4, 5, 6. The probability of getting any one number is 1/6, so these numbers are called random variables (not variables). Equations may be written, based on probability, not to calculate any outcome exactly but rather to calculate the probability of any outcome occurring, such as two sixes on two tosses, etc. In this text, we are not concerned with stochastic models.

Figure 2.14 outlines the place of mathematical modelling in the overall scheme of modelling.

❒ Suggested steps in the construction of a mathematical model

The suggested steps in mathematical modelling are now outlined very briefly, and illustrated graphically in Figure 2.15. Abstract models are usually a mathematical equation or system of equations which attempt to represent or 'model' a real-life situation or system. It is extremely difficult to represent real systems completely and the mathematics required are usually very complex; therefore mathematical models usually start off by representing some aspects of the real system. This requires a good deal of intuition.

Step 1: Understand the real situation in order to ask the pertinent question(s). For example, 'Does price determine the number of units (quantity) of a good demanded?' such as, the number of lunches demanded daily in the Hot Pot restaurant. There are several other possible factors that affect demand: taste; alternatives; income of consumers; substitutes, etc. However, for the present, the effect of these factors will be considered less significant than price.

Step 2: Formulate the mathematical model. Ascertain that the question(s) posed by the model can be treated mathematically.

(a) Decide exactly which variables are important and which have little or no bearing on the situation. In this example, the variables are P and Q, both of which are easily measured.

(b) Describe the problem in terms of mathematical equations. For a normal good, the quantity demanded decreases as the price increases. The simplest mathematical equation is the linear function or straight line, therefore, the proposed equation is of the form: $P = a - bQ$. In the lunches example, suppose it is observed that 50 lunches are demanded

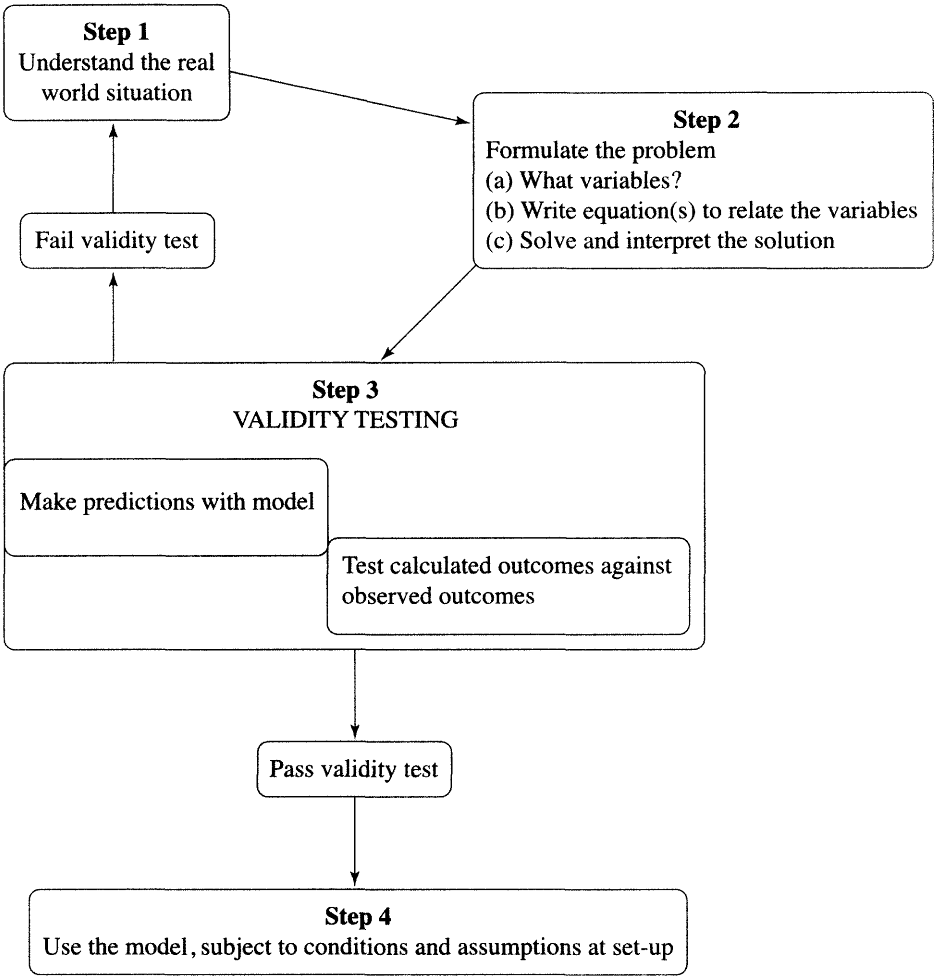

Step 1
Understand the real world situation
Step 2
Formulate the problem
(a) What variables?
(b) Write equation(s) to relate the variables
(c) Solve and interpret the solution
Fail validity test
Step 3
VALIDITY TESTING
Make predictions with model
Test calculated outcomes against observed outcomes
Pass validity test
Step 4
Use the model, subject to conditions and assumptions at set-up

Figure 2.15 Steps in mathematical modelling

when the price is £1, but only 20 lunches when the price is £4. Therefore, the equation of the demand function is: $Q = 60 - 10P$. (Use your knowledge of the straight line to confirm that the observed data is described by this equation.)

(c) Solve the equations and interpret the solution(s). In this example, the solution and interpretation are straightforward. However, in more complex situations, the solution may require complex mathematical methods.

Step 3: Check the validity of the model. Models are generally used to describe the workings of a system and/or to make predictions. Therefore, test the model with real data and check that the outcome calculated by the model compares reasonably with the actual outcome. In the lunches example, check that this equation does predict the quantity demanded when different prices are charged. Remember that you are assuming that the other factors mentioned earlier do not change. For instance, if a competing restaurant closes down, then the 'alternative' assumption changes, so the model for the demand at Hot Pot is no longer valid.

Step 4: If the model does not pass the validity test, then start again, as indicated in Figure 2.15. If the model is satisfactory, then it may be used, subject to the conditions, under which it was set up, remaining constant.

When you have studied the mathematical methods which follow in this text or others, you may be interested in a more in-depth treatment of mathematical modelling. Some references are given in the Bibliography at the end of the book.

2.2.2 Economic models

Economics is a social science which studies how individuals within an economy make economic decisions on the allocation, distribution and utilisation of resources in order to satisfy their needs and wants. As a modern economy is very complex with individuals making millions of possible economic decisions, how can one model such an economic system? (See the circular flow of economic activity.)

Simplified abstractions of reality are used in the form of mathematical models. In this text, we shall refer to mathematical models as economic models since they are used in the context of economics. So, the economic system is studied through the use of economic models.

Economic models fall into two categories:

- **Microeconomics:** this studies the economic decisions of **individual** households and firms. It also analyses how governments can affect these decisions through the use of taxes and how individual markets operate.
  Examples: Individual household demand and individual firm supply of good X.
- **Macroeconomics:** this studies the economy as a whole. It studies the **aggregate** of all economic decisions.
  Examples: Total planned consumption, total planned savings and total planned investment in the economy.

☐ Circular flow of economic activity

A modern economic system is complex with consumers, firms and the government making millions of output and input decisions. The circular flow model, as described in Figure 2.16, is a simplified representation of this economic system (it is an abstraction of reality). It illustrates that economic activity in the economy is circular.

The circular flow model assumes that the economy is composed of:

- Three economic agents: households (consumers), firms (producers) and the government.
- Three markets: the goods market, the labour market and the money market.
- The model shows that both households and firms operate simultaneously in the goods and labour market. Firms are suppliers of goods to the goods market (flow 1) and demanders of labour from the labour market (flow 4). On the other hand, households are demanders of goods from the goods market (flow 2) and suppliers of labour to the labour market (flow 3). Therefore, both households and firms have a dual role in the economy.
- Flows 1–4 illustrate the physical flows of goods and labour in the economy. These physical flows have monetary counterparts. Firms pay wages and salaries to households for

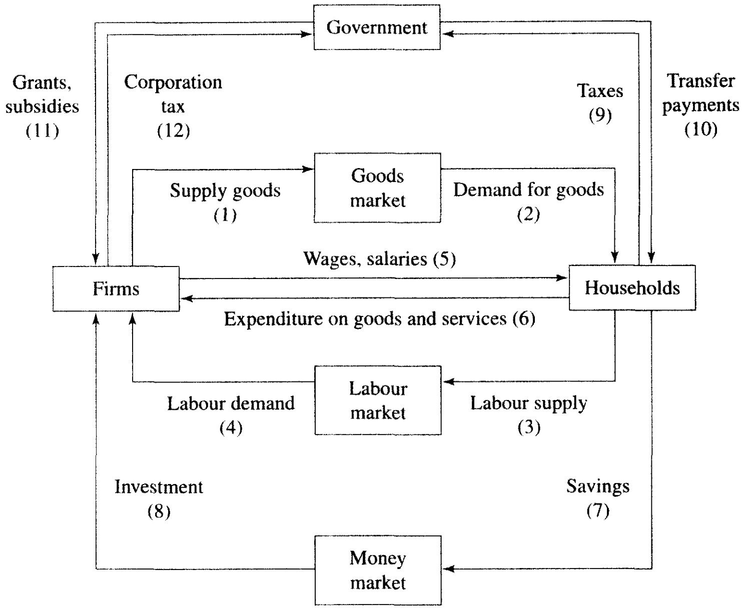

Government
Grants, subsidies (11)
Corporation tax (12)
Taxes (9)
Transfer payments (10)
Goods market
Supply goods (1)
Demand for goods (2)
Wages, salaries (5)
Firms
Households
Expenditure on goods and services (6)
Labour market
Labour demand (4)
Labour supply (3)
Investment (8)
Savings (7)
Money market

Figure 2.16 The circular flow model

supplying their labour services (flow 5). In turn, households use this income for expenditure on the goods and services produced by firms (flow 6).

- Households do not spend all their income, part of it may be saved in financial institutions (flow 7). In turn, these savings may be used to provide funds to firms for investment purposes (flow 8).
- The government has also an important role to play in the economy. Again, the government has a dual role in the model. It takes taxes from both households and firms (flows 9 and 12). These are known as **withdrawals** from the circular flow. In addition, the government puts money into the circular flow through transfer payments (social welfare) and grants/subsidies (flows 10 and 11). These are known as **injections** into the circular flow.

The economic models that follow in the remainder of this text are mathematical in nature. The simplest mathematical function or equation is the straight line which is used to model demand; supply; revenue and cost in Section 2.3.

2.3 Applications: Demand, Supply, Cost, Revenue

At the end of this section you should be familiar with how to model (i) graphically and (ii) algebraically (writing down their equations) the following basic functions:

- Demand and supply
- Cost
- Revenue

2.3.1 Demand and supply

Demand and supply decisions by consumers, firms and the government determine the level of economic activity within an economy. As these decisions play a vital role in business and consumer activity, it is important to possess mathematical tools that enable us to analyse them.

☐ The demand function

There are several variables that influence the demand for a good X. These may be expressed by the general demand function

$$Q = \mathrm{f}(P, Y, P_S, P_C, Ta, A, \ldots)$$

where

Q is the quantity demanded of good X
P is the price of good X
Y is the income of the consumer
P_S is the price of substitute goods
P_C is the price of complementary goods
Ta is the taste or fashion of the consumer
A is the level of advertising

Definitions

A substitute good is one that can be used instead of another good, e.g., trains and buses.
A complementary good is one that is consumed in conjunction with another, e.g. petrol/cars.

The simplest model for the demand function is written as

$$Q = \mathrm{f}(P)$$

Therefore, quantity demanded depends on price only, so long as the other variables upon which demand depends remain constant; that is, $Y, P_S, P_C, Ta, A, \ldots$ are constant.

For example, the demand for good X may be given by the linear equation, $Q = 200 - 2P$. This equation describes the **law of demand**, a basic economic hypothesis which states that there is a negative relationship between quantity demanded and price, that is, when the price of a good increases, the quantity demanded will decrease, all other variables remaining constant. The demand function $Q = 200 - 2P$ is graphed in Figure 2.17(a) with Q, the variable by itself on the LHS of the equation, plotted on the vertical axis and P plotted on the horizontal one. In some applications in economics, however, Q is plotted on the horizontal axis and P on the vertical one. In order to do so, the inverse demand function, $P = \mathrm{f}^{-1}(Q) = \mathrm{g}(Q)$ is required. For example, the inverse demand function of $Q = 200 - 2P$ is given as,

$$\begin{aligned} Q &= 200 - 2P \\ 2P &= 200 - Q \\ P &= 100 - 0.5Q \end{aligned}$$

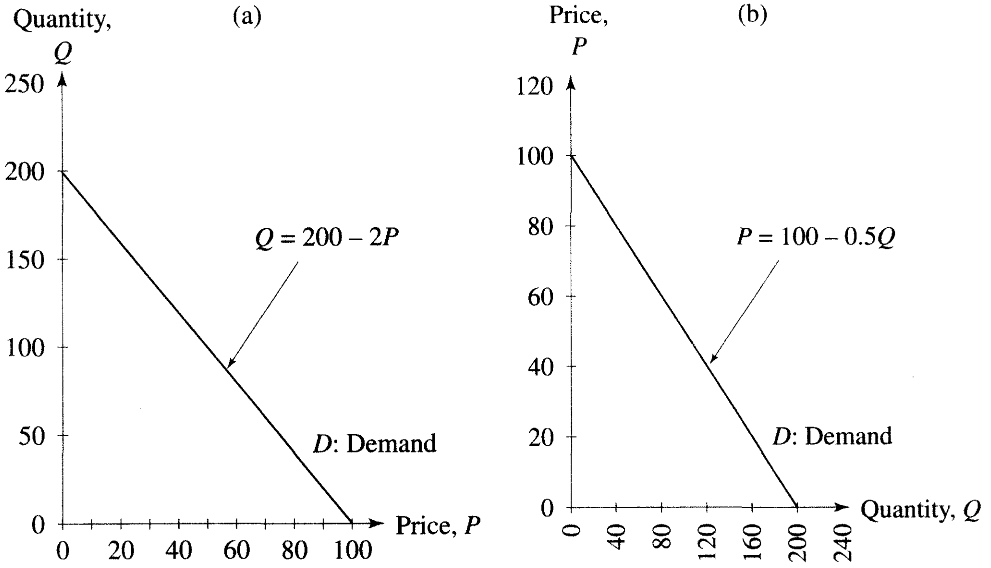


Figure 2.17 Demand functions (a) $Q = f(P)$ and (b) $P = g(Q)$

The inverse demand function, $P = 100 - 0.5Q$, is plotted in Figure 2.17(b). Note that P is written by itself on the LHS of this equation. Most introductory economics texts plot P and Q this way.

Note: For convenience, we do not distinguish between the terms 'demand function' and 'inverse demand function' throughout the remainder of this text. However, it is understood that a function written in the form $Q = f(P)$ refers to the demand function while a function in the form $P = g(Q)$ refers to the inverse demand function.

❒ The equation of the demand function

The demand function, $P = g(Q)$, can be modelled by the simple linear equation

$$P = a - bQ \tag{2.4}$$

where a and b are constants. This is the equation of a straight line. The general format for the equation of a straight line is $y = mx + c$ (or $y = c + mx$); therefore the following can be deduced:

$$\begin{array}{ccccc} P & = & a + & (-b) & Q \\ \vdots & & \vdots & \vdots & \vdots \\ y & = & c + & (m) & x \end{array}$$

$a > 0$: the **vertical intercept** of the demand function is positive ($a > 0$ denotes 'a greater than zero'). The value of a depends on the size of the terms on the RHS of the general demand function, except the price of the good itself.
$(-b) < 0$: the **slope** of the demand function is negative ($(-b) < 0$ denotes '$-b$ less than zero').

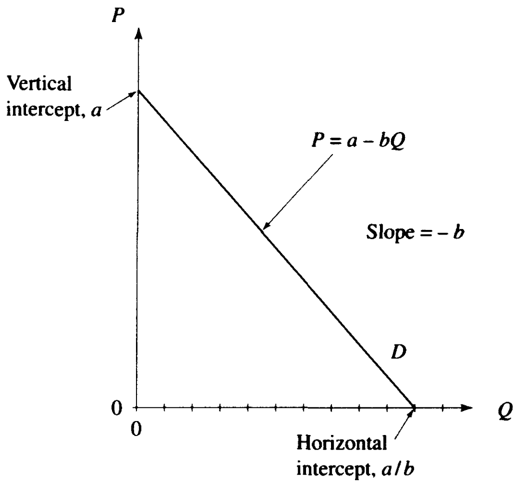


Figure 2.18 Demand function $P = a - bQ$

The negative slope indicates that price drops by b units for each unit increase in quantity. The demand function, $P = a - bQ$ is graphically represented in Figure 2.18.

Note: $P = a - bQ$ intersects the horizontal axis at the point where $P = 0$. Substituting $P = 0$ into the demand function gives

$$0 = a - bQ \quad \rightarrow \quad bQ = a \quad \rightarrow \quad Q = \frac{a}{b}$$

Therefore, the horizontal intercept occurs at $(Q = a/b, P = 0)$.

WORKED EXAMPLE 2.6

LINEAR DEMAND FUNCTION

The demand function is given by the equation $P = 100 - 0.5Q$.

(a) State and give a verbal description of the slope and intercepts.
(b) What is the quantity demanded when $P = 5$?
(c) Plot the demand function $P = 100 - 0.5Q$ for $0 \leq Q \leq 200$.
(d) Find an expression for the demand function in the form $Q = \mathrm{f}(P)$ and graph it.

Solution

(a) The slope and vertical intercept may be deduced as follows,

$$\begin{array}{cccccc} P = & 100 & + & (-0.5) & Q \\ \vdots & \vdots & & \vdots & \vdots \\ P = & a & + & (-b) & Q \\ \vdots & \vdots & & \vdots & \vdots \\ y = & c & + & m & x \end{array}$$

The vertical intercept is 100. This means that $P = 100$ when $Q = 0$. The slope $\Delta P/\Delta Q = -0.5$. This indicates that the price drops by 0.5 units for each successive unit increase in quantity demanded.

The horizontal intercept is calculated by substituting $P = 0$ into the equation of the demand function, that is,

$$P = 100 - 0.5Q \rightarrow 0 = 100 - 0.5Q \rightarrow 0.5Q = 100 \rightarrow Q = \frac{100}{0.5} = 200$$

Therefore, the horizontal intercept is at $(Q = 200, P = 0)$.

(b) The quantity demanded when $P = 5$ is calculated by substituting $P = 5$ into the demand function

$$\begin{aligned} P &= 100 - 0.5Q \\ 5 &= 100 - 0.5Q \\ 0.5Q &= 100 - 5 \\ 0.5Q &= 95 \\ Q &= \frac{95}{0.5} = 190 \end{aligned}$$

(c) **Method A:** To plot the demand function over the range, $0 \leq Q \leq 200$, choose various quantity values within this range. In Table 2.3 equal intervals of 40 units

Table 2.3 Demand schedule

Quantity	Price
0	100
40	80
80	60
120	40
160	20
200	0

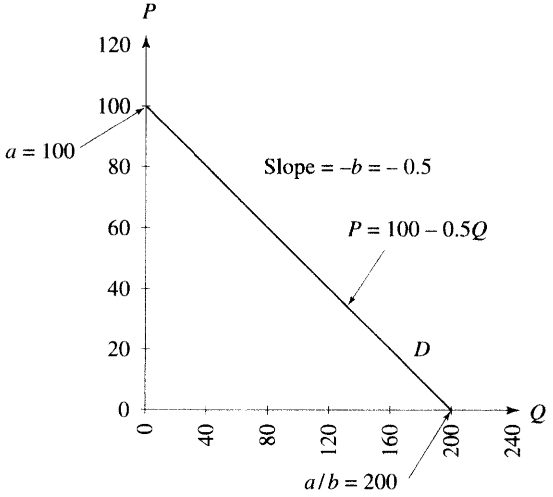


Figure 2.19 Demand function $P = 100 - 0.5Q$

of quantity are used. Substitute these quantity values into the equation of the demand function to derive corresponding values for P. Plot these points and graph the demand function as in Figure 2.19. Table 2.3 and Figure 2.19 are easily set up using Excel.

Method B: Plot the horizontal and vertical intercepts calculated in part (a). Draw a line through these points.

(d) The demand function, $Q = f(P)$ is graphed in Figure 2.20. Its equation is derived as

$$\begin{array}{lcl} P = 100 - 0.5Q & \text{or in general form as} & P = a - bQ \\ 0.5Q = 100 - P & & bQ = a - P \\ Q = 200 - 2P & & Q = \dfrac{a}{b} - \dfrac{1}{b}P \end{array}$$

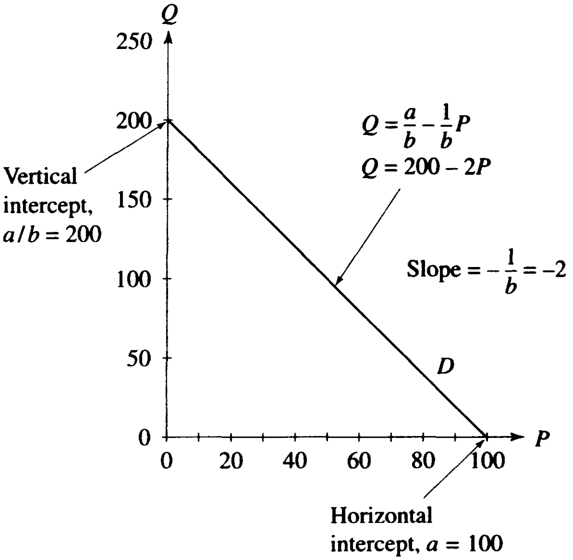


Figure 2.20 Demand function $Q = 200 - 2P$

❒ The supply function

There are several variables that influence the supply of good X. These may be expressed by the general supply function

$$Q = f(P, C, P_0, Te, N, O, \ldots)$$

where

Q is the quantity supplied of good X
P is the price of the good itself
C is the cost of production
P_0 is the price of other goods
Te is the available technology
N is the number of producers in the market
O is other factors, e.g. tax/subsidies

The simplest model for the supply function is written as $Q = f(P)$, that is, quantity supplied depends on price only, so long as the other variables upon which supply depends remain constant. For example, the supply of good X may be given by the linear equation, $Q = 20 + 2P$. This describes **the law of supply**, a basic economic hypothesis which states that there is a positive relationship between quantity supplied and price, that is, when the price of a good increases, the quantity supplied will also increase, all other variables remaining constant.

❒ The equation of the supply function

The supply function $P = h(Q)$ can be modelled by the simple linear equation

$$P = c + dQ \tag{2.5}$$

where c and d are constants.

$c > 0$: the vertical intercept is positive. Its value depends on the size of the terms on the RHS of the general supply function except the price of the good itself.
$d > 0$: the slope of the supply function is positive. Price increases by d units for every unit increase in quantity supplied.

The supply function is graphically represented in Figure 2.21.

Note: $P = c + dQ$ intersects the horizontal axis at the point where $P = 0$. Substituting $P = 0$ into the supply function gives

$$\begin{aligned} P &= c + dQ \\ 0 &= c + dQ \\ -c &= dQ \\ -\frac{c}{d} &= Q \end{aligned}$$

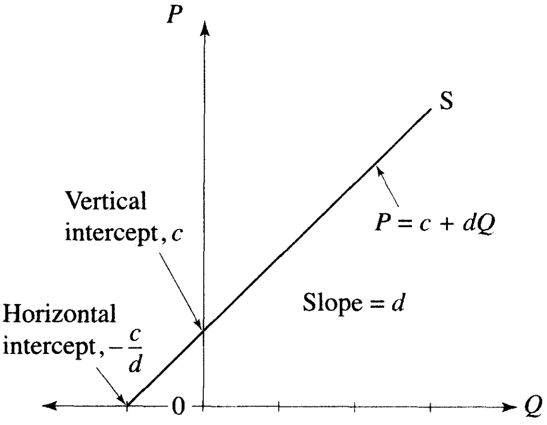


Figure 2.21 Supply function $P = c + dQ$

Therefore, the horizontal intercept occurs at $(Q = -c/d, P = 0)$. A minus quantity is not economically meaningful; therefore, many economic textbooks only illustrate the supply function in the positive quadrant only.

WORKED EXAMPLE 2.7

LINEAR SUPPLY FUNCTION

The supply function is given by $P = 10 + 0.5Q$.

(a) State and verbally describe the slope and intercepts.

(b) Plot the supply function, $P = 10 + 0.5Q$ for $0 \leq Q \leq 100$.

Solution

(a) The vertical intercept is $c = 10$. This means that the firm will supply no units at $P \leq 10$. The slope $\Delta P/\Delta Q = 0.5$. This means that price increases by 0.5 units for every successive unit increase in quantity supplied.

The horizontal intercept is calculated by substituting $P = 0$ into the equation of the supply function, that is,

$$\begin{aligned} P &= 10 + 0.5Q \\ 0 &= 10 + 0.5Q \\ -10 &= 0.5Q \\ -\frac{10}{0.5} &= Q \rightarrow Q = -20 \end{aligned}$$

Therefore, the horizontal intercept is at $(Q = -20, P = 0)$; however, $Q = -20$ is not economically meaningful.

(b) **Method A:** To plot the supply function over the range, $0 \leq Q \leq 100$, choose various quantity values within this range. In Table 2.4 equal intervals of 20 units of quantity are used. Substitute these quantity values into the equation of the supply function to derive corresponding values for P. Plot these points and graph the supply function as in Figure 2.22.

Table 2.4 Supply schedule

Quantity	Price
0	10
20	20
40	30
60	40
80	50
100	60

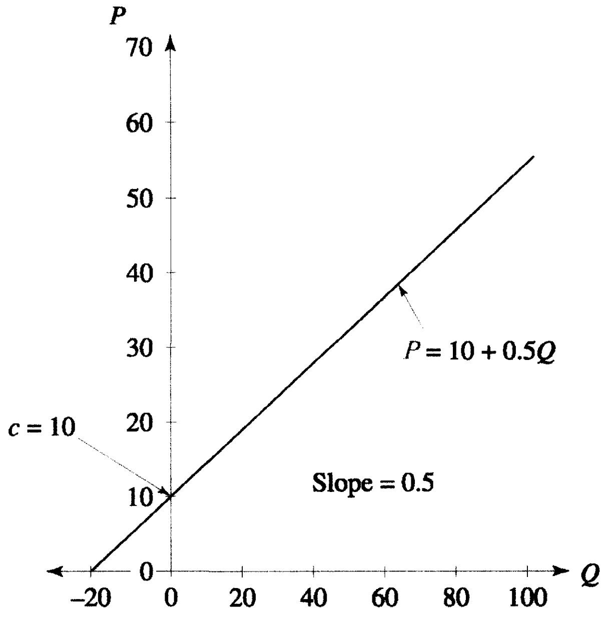


Figure 2.22 Supply function $P = 10 + 0.5Q$

Method B: Plot the horizontal and vertical intercepts calculated in part (a). Draw the supply function by joining these points.

WORKED EXAMPLE 2.8

FURTHER ANALYSIS OF THE LINEAR SUPPLY FUNCTION

A supplier will only start to supply T-shirts when a price greater than £5 per unit is available. He or she will then increase output by 2 units (2 T-shirts) for every unit increase in price.

(a) Plot the supply function in the form $Q = f(P)$.
(b) Write down the equation of the supply function.
(c) Find the value of Q when $P = 15$ from the graph. Confirm your answer from the equation.
(d) Write the equation of the supply function in the form, $P = h(Q)$, i.e. write P in terms of Q. Plot the graph of P in terms of Q.

Solution

(a) Since a graph of the form $Q = f(P)$ is required, plot Q on the vertical axis and P on the horizontal one. In this case the graph crosses the horizontal axis at $P = 5$, that is, the supplier will supply no goods (T-shirts) at a price less than or equal to £5 per unit. Plot the point $P = 5$, $Q = 0$. From this point, draw a line

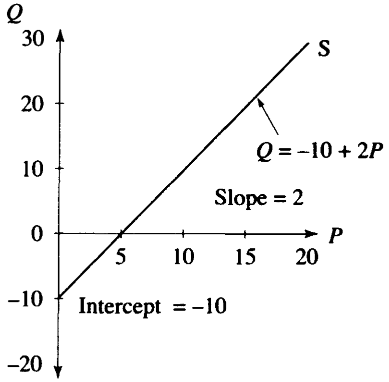


Figure 2.23 Supply function $Q = -10 + 2P$

with a slope of 2 (change in Q per unit change in P). In Figure 2.23 you can see that this line cuts the vertical axis at $Q = -10$.

(b) The general format for the equation of a line is $y = mx + c$. In this case, $y \equiv Q$ and $x \equiv P$. From Figure 2.23, slope $m = 2$ and vertical intercept, $c = -10$. The required equation, therefore, for the supply function is, $Q = -10 + 2P$.

(c) From the graph, when $P = 15$, $Q = 20$. Using the equation, substitute $P = 15$ as follows:

$$Q = -10 + 2P \quad \rightarrow \quad Q = -10 + 2(15) \quad \rightarrow \quad Q = 20$$

(d) The equation of the supply function in the form $P = \text{h}(Q)$ is derived as follows:

$$\begin{aligned} Q &= -10 + 2P \\ -2P &= -10 - Q \\ P &= 5 + 0.5Q \qquad \text{dividing across by } -2 \end{aligned}$$

The graph is given in Figure 2.24. Show how the graph was plotted.

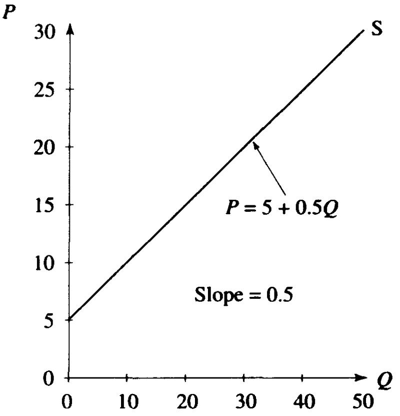


Figure 2.24 Supply function $P = 5 + 0.5Q$

Progress Exercises 2.3 Linear Functions: Supply and Demand

1. For each of the following lines:

   (i) $y = 10 - 2.5x$ (ii) $y = -2.5 + 10x$ (iii) $4y + 2x = 60$

   (a) Calculate the slopes and intercepts.
   (b) Write down the number of units by which y changes when x increases by one unit.
   (c) Plot the lines.

2. The equation of a demand function is given by $Q = 64 - 4P$ where Q is the number of helicopter flights demanded daily; P is the price per flight, in £00s.
   (a) Plot the demand function with Q on the vertical axis.
   (b) What is the change in demand (Q) when price (P) increases by 1 unit?
   (c) What is the demand when $P = 0$?
   (d) What is the price when $Q = 0$?
   Describe the answers to (b), (c) and (d) verbally.

3. The equation of the demand function is given by $P = 65$, where P is the price of bread (per loaf) in pence.
   (a) What is the slope of this demand function? Describe what this means.
   (b) Plot the graph of this demand function.

4. Given the supply function, $P = 500 + 2Q$, where P is the price of a bottle of cognac in French francs, Q is the number of litres supplied.
   (a) Graph the supply function. What is the meaning of the value at the vertical intercept?
   (b) What is the value of Q when $P =$ FF600?
   (c) What is the value of P when $Q = 20$?
   Describe the answers to (a), (b) and (c) verbally.

5. (See question 4)
   (a) Express the supply function given in question 4 in the form $Q = f(P)$.
   (b) Hence, determine the increase in the number of litres supplied for each unit increase in price.

6. The equation of the supply function is given as $P = 50$.
   (a) Write down the slope of the supply function. What is the change in price when the quantity supplied increases by 10 units?
   (b) Plot the graph of this supply function.

7. The equation of the supply function is given as $Q = 1200$, where Q is the number of hospital dinners supplied daily.
   (a) What is the slope of this supply function?
   (b) Plot the graph of this supply function. Describe the situation modelled by the equation verbally.

2.3.2 Cost

Firms incur costs when they employ inputs, such as capital and labour, in order to produce goods for sale in the market. The total cost of producing a good will normally consist of:

(i) Fixed costs, FC: costs that are fixed irrespective of the level of output, e.g. rent on premises.
(ii) Variable costs, VC: costs which vary with the level of output, e.g. each extra unit of a good produced will require additional units of raw materials, labour, etc.

Total cost, TC, is therefore the sum of fixed costs and variable costs:

$$TC = FC + VC \tag{2.6}$$

These costs can be modelled by a linear cost function, such as

$$TC = 20 + 4Q$$

where $FC = 20$ (the vertical intercept) and $VC = 4Q$, where $4 =$ slope of the line.

WORKED EXAMPLE 2.9

LINEAR TOTAL COST FUNCTION

A firm has fixed production costs of £10 and variable production costs of £2 per unit produced.

(a) Write down the equation of the total cost function.
(b) Graph the total cost function.

Solution

(a) $FC = £10$; while $VC = £2Q$, since to produce

1 unit	$VC = 2(1)$
2 units	$VC = 2(2)$
3 units	$VC = 2(3)$
Q units	$VC = 2(Q)$

that is, the total variable costs incurred in producing Q units of this good is £$2Q$. Since $TC = FC + VC$, the total cost incurred in producing Q units is

$$TC = 10 + 2Q$$

Note: This is the same as the line $y = 10 + 2x$, where $y \equiv TC$ and $x \equiv Q$.

(b) Graphically, the TC function is a straight line, with costs measured on the vertical axis and units of the good produced on the horizontal axis. Since the vertical intercept $= 10$ (level of FC) and the slope $= 2$ are known, the total cost function may be plotted. Alternatively, total costs may be plotted by calculating at least two points on the line and joining these points. A number of points are outlined in Table 2.5 and then graphed in Figure 2.25.

Table 2.5 Fixed, variable and total costs

Q	**Variable costs (at £2 per unit)**	$TC = FC + VC$ ($y = 10 + 2Q$)	**Point** (Q, TC) (x, y)
0	0	$TC = 10 + 0 = 10$	(0, 10)
1	$1(2) = 2$	$TC = 10 + 2 = 12$	(1, 12)
2	$2(2) = 4$	$TC = 10 + 4 = 14$	(2, 14)
3	$3(2) = 6$	$TC = 10 + 6 = 16$	(3, 16)
4	$4(2) = 8$	$TC = 10 + 8 = 18$	(4, 18)
5	$5(2) = 10$	$TC = 10 + 10 = 20$	(5, 20)
6	$6(2) = 12$	$TC = 10 + 12 = 22$	(6, 22)

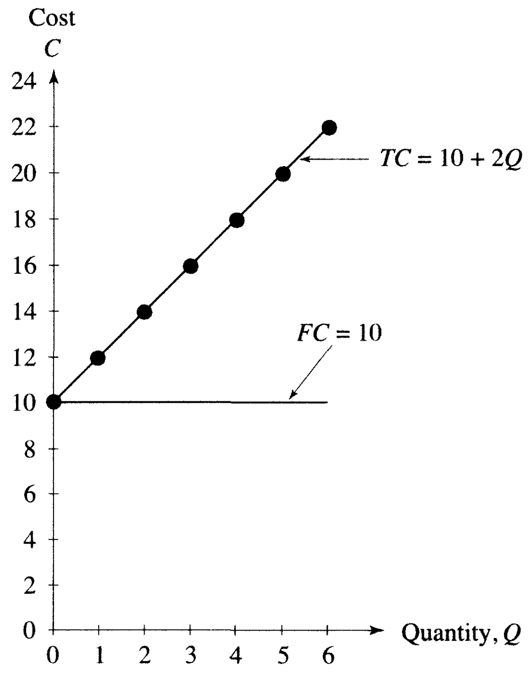


Figure 2.25 Linear total cost function

2.3.3 Revenue

A firm receives revenue when it sells output. The total revenue, TR, received is the price of the good, P, multiplied by the number of units sold, Q, that is,

$$TR = P.Q \tag{2.7}$$

Total revenue may be modelled by a straight line if the price of each unit sold is the same, that is, each firm is a price-taker, denoted by a horizontal demand function. A price-taking firm is known as a perfectly competitive firm. In this case, the total revenue function would take the form

$$TR = P_0 Q$$

where P_0 is the constant price per unit of the good and is represented by the vertical intercept of a horizontal demand function. For example, if $P_0 = 10$, then the total revenue function is

$$\begin{array}{cccccc} TR & = 0 & + & 10 & Q & = 10Q \\ \vdots & \vdots & & \vdots & \vdots & \\ y & = c & + & m & x & = mx \end{array}$$

This is a line through the origin (0, 0) since the vertical intercept, $c = 0$.

WORKED EXAMPLE 2.10

LINEAR TOTAL REVENUE FUNCTION

Suppose that each chicken snack box is sold for £3.50 irrespective of the number of units sold.

(a) Write down the equation of the total revenue function.

(b) Graph the total revenue function.

Solution

(a) Total revenue is price multiplied by the number of units sold, that is,

$$TR = 3.5Q$$

Note that price is constant at £3.50 irrespective of the value of Q.

(b) Total revenue is represented graphically by a straight line, with slope = 3.5 and intercept = 0. It is graphed by calculating values of TR for any values of Q, for example $Q = 0, 2, 4, 6$. These points are outlined in Table 2.6 and then plotted in Figure 2.26.

Table 2.6 Total revenue

Q	$TR = PQ = 3.5Q$	Point (Q, TR)
0	$TR = 3.5(0) = 0$	(0, 0)
2	$TR = 3.5(2) = 7$	(2, 7)
4	$TR = 3.5(4) = 14$	(4, 14)
6	$TR = 3.5(6) = 21$	(6, 21)

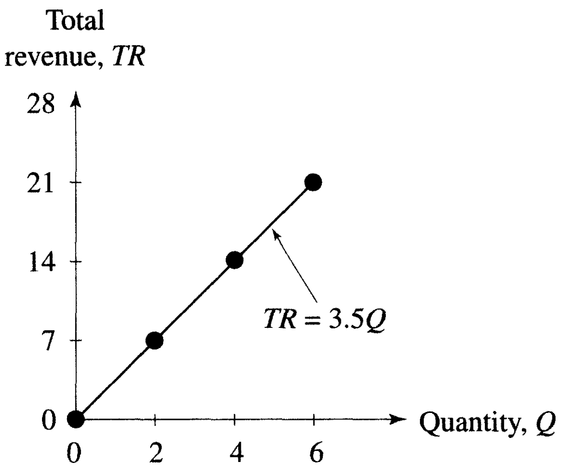


Figure 2.26 Linear total revenue function

Progress Exercises 2.4 Linear Functions: Cost, Revenue

1. A firm faces the following cost function, $TC = 5Q$.
   (a) What is the value of fixed costs?
   (b) Graph the total cost function.
   (c) What is the total cost when $Q = 10$?
   (d) What is the cost of producing each additional unit of this good?
2. A firm sells its product in a perfectly competitive output market (i.e. the price is the same for each unit sold). The total revenue function is given as $TR = 10Q$.
   (a) What is the price per unit charged by the perfectly competitive firm?
   (b) Plot the total revenue function.
3. The canoeing club provides swimming lessons to boost club funds. The club has fixed costs of £250 daily (insurance) when offering these lessons. The variable cost is £25 for each lesson given.
   (a) Write down the equation for total cost and plot its graph for $Q = 0$ to 60.
   (b) Calculate the cost of providing 28 lessons.
   (c) Calculate the number of lessons provided when total costs are £1400.
   (d) Confirm the answers in (b) and (c) graphically.
4. (See question 3)
   If each student is charged £32 for a canoeing lesson,
   (a) Write down the equation for total revenue.
   (b) Calculate the number of students when the total revenue is £1024.
   (c) If the club has 44 students on a given day, determine whether the revenue exceeded costs.

2.4 More Mathematics on the Straight Line

At the end of this section you should be able to:

- Calculate the slope of a line given two points on the line
- Determine the equation of a line given slope and any point on the line
- Determine the equation of a line given any two points on the line

2.4.1 Calculating the slope of a line given two points on the line

Section 2.1 introduced how the slope, m, of a line is measured. This section repeats that exercise in more detail. If two points on a line are known, then the slope is defined as the change in the y-coordinate divided by the change in the x-coordinate given as

$$\text{Slope} = \frac{\text{Change in } y\text{-coordinate}}{\text{Change in } x\text{-coordinate}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.8}$$

WORKED EXAMPLE 2.11

CALCULATING SLOPE GIVEN TWO POINTS ON THE LINE

Calculate the slope of a line given the following two points on the line,

$$(x_1, y_1) \quad \text{and} \quad (x_2, y_2)$$

$$(2, 1) \quad \text{and} \quad (-4, -2)$$

Solution

The slope, m, is calculated using equation (2.8):

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - (1)}{-4 - (2)} = \frac{-3}{-6} = \frac{1}{2} = 0.5$$

This is exactly the same result that you would obtain if you plotted these points on a graph and measured the slope directly from the graph as height, 3, divided by distance, 6 as illustrated in Figure 2.27. Furthermore,

> The slope of a straight line is the **same** when measured between any two points, that is, the slope of a straight line is constant.

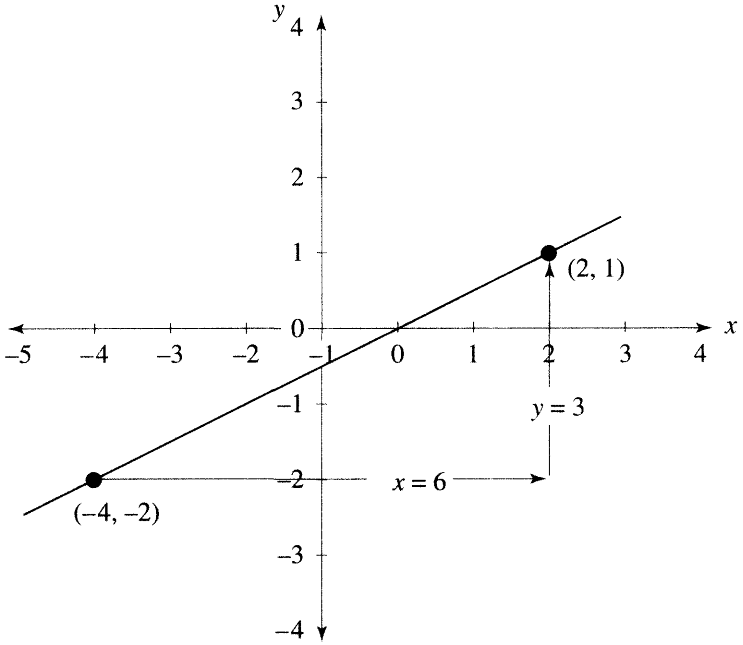


Figure 2.27 Measuring slope given two points

2.4.2 The equation of a line given slope and any point on the line

Since the slope of a line is the same when measured between any two points, such as the points (x_1, y_1) and (x, y), we may write

$$\frac{\Delta y}{\Delta x} = m = \frac{y - y_1}{x - x_1}$$

therefore, $m(x - x_1) = y - y_1$. This equation is usually written as: $y - y_1 = m(x - x_1)$.

$$y - y_1 = m(x - x_1) \qquad \textbf{(2.9)}$$

The equation of a straight line with slope, m, which passes through the specific point (x_1, y_1).

WORKED EXAMPLE 2.12

EQUATION OF A LINE GIVEN SLOPE AND A POINT ON THE LINE

(a) Deduce the equation of a line having a slope of 1.7 and which passes through the point (2, 5).
(b) Plot the graph of the line.

Solution

(a) Using equation (2.9), the equation of the line is deduced as

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= 1.7(x - 2) \\ y - 5 &= 1.7x - 3.4 \\ y &= 1.6 + 1.7x \end{aligned}$$

This line has a slope of 1.7 and it intersects the y-axis at $y = 1.6$.
(b) To plot the line it is necessary to calculate at least two points on the line; however, it is a good idea to calculate three points in case of error. All three points should lie exactly on the line. Using **method** A, take suitable x-values and calculate the corresponding y-coordinates from the equation of the line as shown in Table 2.7. These points are plotted in Figure 2.28.

Table 2.7 Calculating points on the line, $y = 1.6 + 1.7x$

x	$y = 1.6 + 1.7x$	Point (x, y)
−2	$y = 1.6 + 1.7(-2) = -1.8$	(−2, −1.8)
0	$y = 1.6 + 1.7(0) = 1.6$	(0, 1.6)
2	$y = 1.6 + 1.7(2) = 5$	(2, 5)

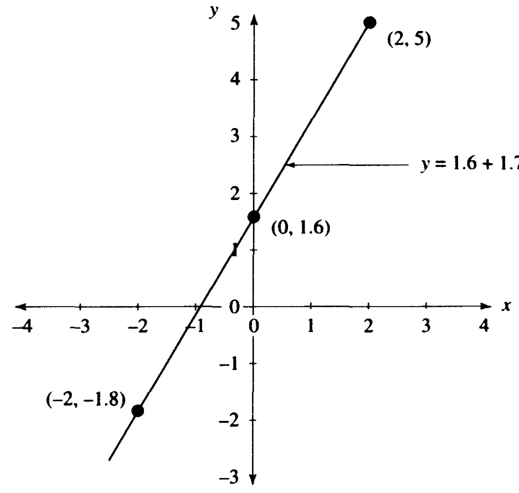


Figure 2.28 Graph of line $y = 1.6 + 1.7x$

2.4.3 The equation of a line given two points

There are two steps involved when deducing the equation of a line when given two points, say points (x_1, y_1) and (x_2, y_2).

Step 1: Calculate the slope of the line using equation (2.8).
Step 2: Determine the equation of the line by substituting the slope (as calculated in step 1), and either point into equation (2.9).

WORKED EXAMPLE 2.13

EQUATION OF A LINE GIVEN TWO POINTS ON THE LINE

(a) A line passes through the points (2, 4) and (6, 1). Deduce the equation of the line.
(b) Plot the graph of the line.

Solution

(a) **Step 1:** The slope of the line is calculated by substituting the points into equation (2.8):

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-4}{6-2} = \frac{-3}{4} = -0.75$$

Step 2: The equation of the line is deduced by substituting this slope, and either point into equation (2.9):

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= -0.75(x - 2) \qquad \text{using point } (2, 4) \\ y - 4 &= -0.75x + 1.5 \\ y &= 5.5 - 0.75x \end{aligned}$$

The equation of the line has a negative slope of −0.75 and y-intercept of 5.5.
(b) Following **method A**, use the equation of the line to calculate any three points as in Table 2.8. These points are then plotted in Figure 2.29.

Table 2.8 Calculating points on the line $y = 5.5 - 0.75x$

x	$y = 5.5 - 0.75x$	Point (x, y)
0	$y = 5.5 - 0.75(0) = 5.5$	(0, 5.5)
2	$y = 5.5 - 0.75(2) = 4$	(2, 4)
4	$y = 5.5 - 0.75(4) = 2.5$	(4, 2.5)

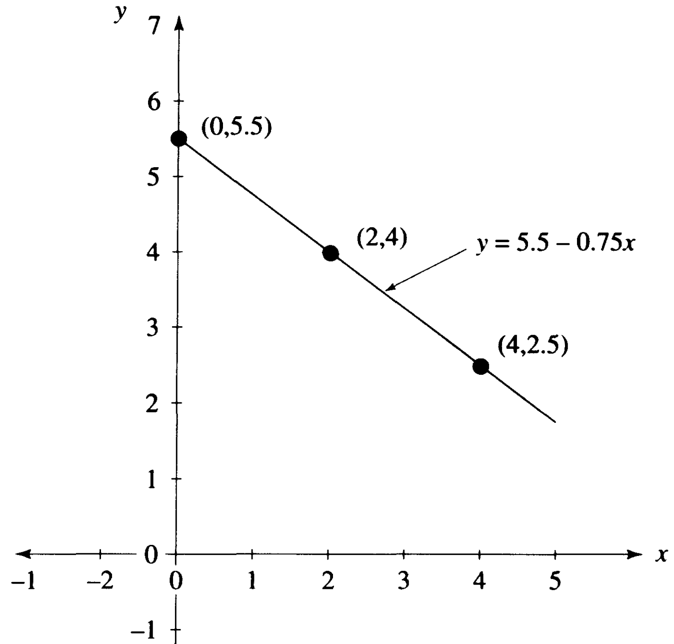


Figure 2.29 Graph of line $y = 5.5 - 0.75x$

☐ Summary

Given one point and slope	Given two points on the line
$y - y_1 = m(x - x_1)$ may be used to determine the equation of a straight line when the slope of the line and a point on the line are given.	Given any two points (x_1, y_1), (x_2, y_2) (1) Calculate the slope: $m = (y_2 - y_1)/(x_2 - x_1)$ (2) Use this slope with one of the points to determine the equation of the line using $y - y_1 = m(x - x_1)$

Progress Exercises 2.5 Determine Equation of a Line from Two Points and Applications

1. (a) Plot the points
(i) (0, 0) and (2, 1), (ii) (2, 3) and (−1, −3).
(b) Measure the slope of the lines formed by joining the points in (i) and (ii).

2. Find the equations of the following lines and, hence plot their graphs.
   (a) A straight line passes through the point (2, 4) and has a slope of 1.
   (b) A straight line passes through the points (2, 4) and (8, 16).
   (c) A straight line intersects the y-axis at $y = 4$, the x-axis at $x = 8$.
3. The variable cost of a product increases by £1.50 for each unit produced while fixed costs are £55.
   (a) Write down the equation for and graph the total cost function.
   (b) Determine the total cost when 8 units are produced, both algebraically and graphically.
4. It is known that the number of lunches demanded is 80 units when the price is £5 and 45 when the price is £12.
   (a) Determine the equation of the demand function in the form $Q = \mathrm{f}(P)$. Plot the graph of the demand function.
   (b) Use the equation of the demand function to calculate the change in demand when the price (i) increases by £3, (ii) decreases by £2.
   (c) Estimate the decrease in price for each lunch when the number of lunches demanded (quantity demanded) increases by 15.
5. A supplier supplies 50 football scarves when the price is £6 each and 90 units when the price is £11 each.
   (a) Determine the equation of the supply function in the form $P = \mathrm{h}(Q)$.
   (b) How many additional scarves are supplied for each successive £1 increase in price?
   (c) Calculate the quantity supplied when the price is £8.5 per scarf.
   (d) Calculate the price when 120 scarves are supplied.
   (e) What is the price below which no scarves are supplied?

2.5 Translations of Linear Functions

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 2.

2.5.1 Meaning of translation

2.5.2 Equation of translated lines

❐ Vertical translations

Worked Example 2.14 Vertical translations of linear functions

Figure 2.30 Vertical translations of the line $y = 1 + 2x$

❐ Calculate the size of the horizontal shift that occurs in conjunction with a vertical translation

❐ Horizontal translations

Worked Example 2.15 Horizontal translations of linear functions

Figure 2.31 Horizontal translations of the line $y = 1 + 2x$

❐ Horizontal and vertical translations together

Worked Example 2.16 Horizontal and vertical translations of linear functions

Figure 2.32 Vertical and horizontal translations of the line $y = 1 + 2x$

2.5.3 Some applications of translations

☐ Excise tax

Worked Example 2.17 The effect of an excise tax on a supply function

Figure 2.33 Effect of an excise tax (10 euros per unit) on the supply function

☐ Value added tax (VAT)

☐ Translating cost functions

Worked Example 2.18 Translating linear cost functions

Figure 2.34 The effect of (i) a subsidy and (ii) changes in fixed costs on a total cost function

Progress Exercises 2.6 Translations of Linear Functions and Applications

Solutions to Progress Exercises 2.6

2.6 Elasticity of Demand, Supply and Income

At the end of this section you should be able to:

- Calculate the coefficient of price elasticity of demand at a point on the demand function
- Illustrate how the coefficient of point elasticity varies along a linear demand function (web site)
- Illustrate that the coefficient of price elasticity of demand depends on price and vertical intercept and not on the slope of the linear demand function
- Calculate the coefficient of price elasticity of demand averaged along an arc on the demand function
- Calculate the coefficient of price elasticity of supply
- Calculate the coefficient of income elasticity of demand (web site)

'The prices of cars increase by 5%'
'The prices of computers decrease by 8%'

Significant as they may seem, the above statements alone are of little value to firms. From the demand function, firms can predict that as price increases, quantity demanded decreases and vice versa. However, this information is not sufficient. What firms require is a numerical value that outlines the responsiveness of quantity demanded to such price changes at various price levels. For example, when the price of a car is £9000, will a 5% increase in price result in a 5% decrease in demand, or a larger percentage decrease in demand? When the price of a car is £19 200, will a 5% increase in price result in a smaller or larger percentage decrease in demand?

This information is invaluable to firms as their total revenue depends on both prices and quantity sold. Will the additional revenue generated by the price increase be offset by the loss in revenue due to the decrease in units sold?

The concept of **elasticity**, as used in economics, is the ratio of the percentage change in the quantity demanded (or supplied) to a percentage change in an economic variable, such as price, income, etc. Elasticity may be used to predict the responsiveness of demand (and supply) to changes in such economic variables.

There are various symbols used to denote elasticity. In this text it is denoted by the symbol ε, pronounced 'epsilon'. Three types of elasticity are introduced: price elasticity of demand, ε_{d}, price elasticity of supply, ε_{s}, and income elasticity of demand, ε_{y}.

2.6.1 Price elasticity of demand

Price elasticity of demand measures the responsiveness (sensitivity) of quantity demanded to changes in the good's own price. It is also referred to as own-price elasticity. It is represented by the general elasticity formula

$$\varepsilon_{\mathrm{d}} = \frac{\text{Percentage change in quantity demanded}}{\text{Percentage change in price}} = \frac{\%\Delta Q_{\mathrm{d}}}{\%\Delta P} = \frac{\dfrac{\Delta Q}{Q}\cdot 100}{\dfrac{\Delta P}{P}\cdot 100} \qquad \textbf{(2.10)}$$

which simplifies to:

$$\varepsilon_{\mathrm{d}} = \frac{\Delta Q}{\Delta P}\cdot\frac{P}{Q} \qquad \textbf{(2.11)}$$

☺ **Remember:** dividing by a fraction is the same as multiplying by the fraction inverted.

The numerical value or coefficient for price elasticity of demand is normally negative since $\Delta Q/\Delta P$ is negative. That is, the linear demand function, $P = a - bQ$ has slope

$$= \frac{\Delta P}{\Delta Q} = -\frac{b}{1}$$

hence, inverting both sides gives

$$\frac{\Delta Q}{\Delta P} = -\frac{1}{b}$$

which is negative.

Note

- The negative sign associated with ε_{d} indicates the direction and magnitude of the responsiveness of one variable w.r.t. another. A negative sign indicates that an increase in one variable is accompanied by a decrease in the other or vice versa. A positive sign indicates that an increase (decrease) in one is accompanied by an increase (decrease) in the other.
- There is a convention in some economic textbooks to give the numerical value of elasticity without the sign. This is known as the absolute value or magnitude of elasticity, $|\varepsilon|$, which only indicates the magnitude of the responsiveness of one variable to a change in another and not the direction of the responsiveness.

In this text, both conventions shall be used as appropriate.

There are two approaches to measuring price elasticity: point elasticity and arc elasticity. Point elasticity measures elasticity at a point on the demand function. Arc elasticity measures elasticity over an interval on the demand function.

❐ Point elasticity of demand

Given the linear demand function, $P = a - bQ$, the formula for point elasticity of demand at any point (P_0, Q_0) is

$$\varepsilon_d = \frac{\Delta Q}{\Delta P}\frac{P}{Q}$$

or

$$\varepsilon_d = -\frac{1}{b}\frac{P_0}{Q_0} \quad \text{since} \quad \text{slope} = \frac{\Delta Q}{\Delta P} = -\frac{1}{b} \tag{2.12}$$

WORKED EXAMPLE 2.19

DETERMINING THE COEFFICIENT OF POINT ELASTICITY OF DEMAND

Given the demand function for computers as $P = 2400 - 0.5$.

(a) Determine the coefficient of point elasticity of demand when (i) $P = 1800$, (ii) $P = 1200$ and (iii) $P = 600$. Give a verbal description of each result.

(b) If the price of computers increases by 12%, calculate the percentage change in the quantity demanded at $P = 1800$, $P = 1200$ and $P = 600$.

(i) First use the definition of elasticity:

$$\varepsilon_d = \frac{\text{percentage change in quantity demanded}}{\text{percentage change in price}} = \frac{\Delta Q_d\ (\%)}{\Delta P\ (\%)}$$

(ii) Then calculate the exact percentage changes and compare them with the answers in (i).

(c) Graph the demand function, indicating where demand is elastic, unit elastic and inelastic.

Solution

(a)

(i) At $P = 1800$ the quantity of computers demanded, Q, is calculated by substituting $P = 1800$ into the demand function:

$$P = 2400 - 0.5Q \rightarrow 1800 = 2400 - 0.5Q \rightarrow 0.5Q = 600 \rightarrow Q = 1200$$

The value of point elasticity of demand at ($P = 1800$, $Q = 1200$) is calculated by substituting these values along with $b = 0.5$ into formula (2.12):

$$\varepsilon_d = -\frac{1}{b}\frac{P_0}{Q_0}$$

$$= -\frac{1}{0.5}\frac{1800}{1200} = -\frac{1800}{600} = -\frac{3}{1} = -3$$

The coefficient of point elasticity of demand is $\varepsilon_d = -3$, which indicates that at the price $P = 1800$ a 1% increase (decrease) in price will cause a 3% decrease (increase) in the quantity of computers demanded. Demand is elastic, $|\varepsilon_d| > 1$.

(ii) and (iii) The calculations for price elasticity of demand for $P = 1200$ and $P = 600$ are carried out in the same way as (i) and are summarised below.

P_0	1800	1200	600
Calculate Q_0	$P_0 = 2400 - 0.5Q_0$ $1800 = 2400 - 0.5Q_0$ $0.5Q_0 = 600$ $Q_0 = 1200$	$P_0 = 2400 - 0.5Q_0$ $1200 = 2400 - 0.5Q_0$ $0.5Q_0 = 1200$ $Q_0 = 2400$	$P_0 = 2400 - 0.5Q_0$ $600 = 2400 - 0.5Q_0$ $0.5Q_0 = 1800$ $Q_0 = 3600$
ε_d for $P = 2400 - 0.2Q$ i.e. $b = 0.5$	$\varepsilon_d = -\frac{1}{b}\frac{P_0}{Q_0}$ $= -\frac{1}{0.5}\frac{1800}{1200} = -3$	$\varepsilon_d = -\frac{1}{b}\frac{P_0}{Q_0}$ $= -\frac{1}{0.5}\frac{1200}{2400}$ $= -\frac{1200}{1200}$ $= -1$ (see notes)	$\varepsilon_d = -\frac{1}{b}\frac{P_0}{Q_0}$ $= -\frac{1}{0.5}\frac{600}{3600}$ $= -\frac{1}{3}$ ≈ -0.33 (notes)
Demand	Demand is elastic $\lvert\varepsilon_d\rvert > 1$	Demand is unit elastic $\lvert\varepsilon_d\rvert = 1$	Demand is inelastic $\lvert\varepsilon_d\rvert < 1$

Notes

$\varepsilon_d = -1$ indicates that at $P = 1200$, a 1% increase (decrease) in price will cause a 1% decrease (increase) in the quantity of computers demanded.

$\varepsilon_d = -0.33$ (correct to 2 decimal places) indicates that at $P = 600$, a 1% increase (decrease) in price will cause a 0.33% decrease (increase) in the quantity of computers demanded.

(b)

(i) The definition

$$\varepsilon_d = \frac{\text{percentage change in quantity demanded}}{\text{percentage change in price}} = \frac{\Delta Q_d\,(\%)}{\Delta P\,(\%)}$$

may be rearranged as

$$\Delta Q\,(\%) = \Delta P\,(\%) \times \varepsilon_d \tag{2.13}$$

- At $P = 1800$ we have calculated that $\varepsilon_d = -3$. To calculate the percentage change in Q, substitute $\varepsilon_d = -3$ and $\Delta P = 12\%$ into equation (2.13):

  $$\Delta Q = 12\% \times -3 = -36\%$$

  The quantity demanded decreases by 36%, a larger percentage decrease than the 12% price increase. Demand is strongly responsive to price change and is described as elastic demand.
- $P = 1200$: $\varepsilon_d = -1$, $\Delta P = 12\%$. Substitute into equation (2.13):

  $$\Delta Q = 12\% \times -1 = -12\%$$

  The quantity demanded decreases by 12%; the percentage increase in price is 12%. This is described as unit elastic demand.
- $P = 600$: $\varepsilon_d = -0.33$, $\Delta P = 12\%$. Substitute into equation (2.13):

  $$\Delta Q \approx 12\% \times -0.33 = -3.96\%$$

  The quantity demanded decreases by 3.96%, a smaller percentage decrease than the 12% price increase. So demand is weakly responsive to price, hence it is described as inelastic demand.

(ii) The calculation of the exact percentage change in Q requires basic arithmetic. Start by calculating the price which results from an increase of 12% on the initial price.

$$\text{Increase } P = 1800 \text{ by } 12\% \rightarrow P_{\text{new}} = \frac{112}{100}P = 1.12(1800) = 2016$$

Then calculate the corresponding values of Q from the equation of the demand function. When $P = 1800$, $Q = 1200$ and when $P = 2016$, $Q = 768$.

Initial P, Q	**New P, Q**	**Percentage change in Q by direct calculation**	ε_d	**$\Delta Q(\%) = \Delta P(\%) \times \varepsilon_d$ by formula (2.13)**
1800, 1200	2016, 768	$\frac{768 - 1200}{1200} \times 100 = -36\%$	-3	$\Delta Q = -3 \times 12 = -36\%$
1200, 2400	1344, 2122	$\frac{2112 - 2400}{2400} \times 100 = -12\%$	-1	$\Delta Q = -1 \times 12 = -12\%$
600, 3600	672, 3456	$\frac{3456 - 3600}{3600} \times 100 = -4\%$	$-\frac{1}{3} \approx -0.33$	$\Delta Q = -\frac{1}{3} \times 12 \approx -4\%$

$\approx$ means approximately equal to

Comment: The exact value of elasticity, $-1/3$, not the rounded value, -0.33, was used for accurate results when $P = 600$. The rounded value of elasticity, -0.33, would give approximately the same answer, -3.96%, as obtained by direct calculation. At prices $P = 1800$ and $P = 1200$, no rounding of ε_d was required; the percentage change in demand was exactly the same, whether calculated directly or calculated by equation (2.13). In general, the computer firm is not only interested in the percentage change in the price of computers, but also in the percentage change in demand which results from the price changes.

(c) The graph of the demand function is given in Figure 2.35.

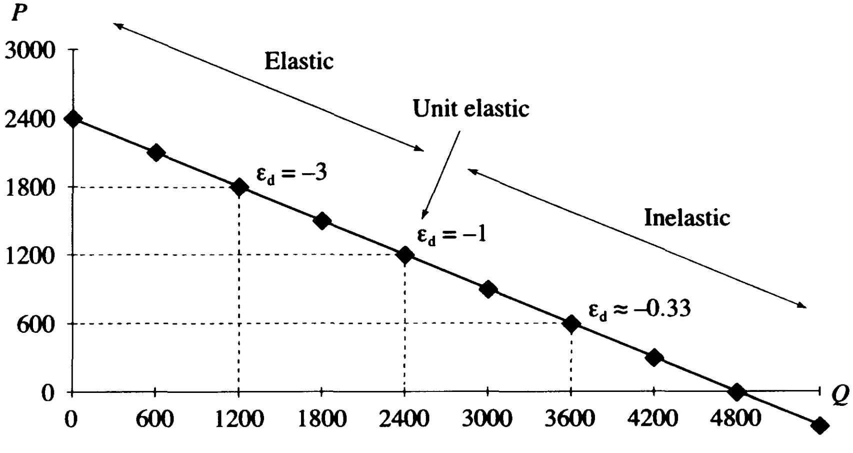


Figure 2.35 Variation of price elasticity of demand with price along the demand function $P = 2400 - 0.5Q$

☐ Coefficient of price elasticity of demand

There are three categories of price elasticity: **elastic**, **inelastic** and **unit elastic**.

Coefficient of price elasticity of demand	**Description**
$-\infty < \varepsilon_d < -1$ Coefficient of price elasticity of demand is greater than minus infinity and less than −1. Note: when $\varepsilon_d < -1$, $\lvert\varepsilon_d\rvert > 1$	Elastic demand: demand is strongly responsive to changes in price; that is, the percentage change in demand is greater than the percentage change in price. When $\varepsilon_d \rightarrow -\infty$, demand for the good is perfectly elastic. The demand function is horizontal with slope = 0; that is, $\frac{\Delta P}{\Delta Q} = \frac{0}{\Delta Q} \rightarrow \frac{\Delta Q}{\Delta P} = \frac{\Delta Q}{0} = \infty \qquad (P \neq 0)$
$-1 < \varepsilon_d < 0$ Coefficient of price elasticity of demand is greater than −1 but less than or equal to zero. Note: $\lvert\varepsilon_d\rvert < 1$	Inelastic demand: demand is weakly responsive to changes in price; that is, the percentage change in demand is less than the percentage change in price. When $\varepsilon_d = 0$, demand for the good is perfectly inelastic. The demand function is vertical with slope = ∞; that is, $\frac{\Delta P}{\Delta Q} = \frac{\infty}{1} \rightarrow \frac{\Delta Q}{\Delta P} = \frac{1}{\infty} = 0 \qquad (Q \neq 0)$
$\varepsilon_d = -1$ Coefficient of price elasticity of demand is equal to −1	Unit elastic demand: the percentage change in demand is equal to the percentage change in price.

The numerical scale for the coefficient of price elasticity of demand is shown in Figure 2.36.

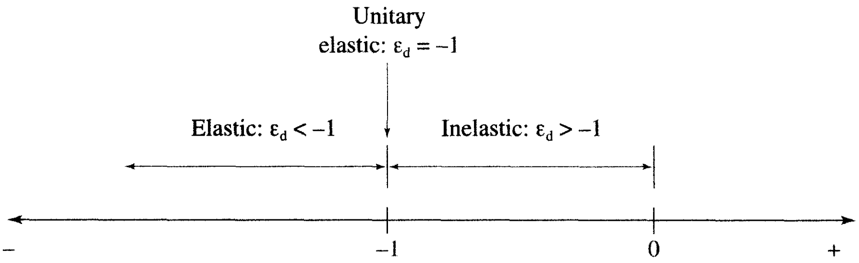

Unitary
elastic: $\varepsilon_d = -1$
Elastic: $\varepsilon_d < -1$
Inelastic: $\varepsilon_d > -1$
−
−1
0
+

Figure 2.36 Numerical scale for the coefficient of price elasticity of demand

Note: ∞ denotes infinity.

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 2.

☐ Point elasticity of demand varies along a demand function

Worked Example 2.20 Calculating the coefficient of point elasticity at different prices

Table 2.9 Coefficient of point elasticity of demand for silicon chips

Figure 2.37 Elasticity varies along a demand function

❑ Point elasticity of demand depends on price and vertical intercept only

Is a relatively flat demand function more elastic than a steeper demand function? Not necessarily. If two demand functions are drawn to the same scale, then

- The point elasticity of demand is the same at any given price for two demand functions having the same vertical intercept, but different slopes.
- The point elasticity of demand is different at any given price for two demand functions having different vertical intercepts (slopes may be the same or different).

To prove the above statements proceed as follows.

Write the price elasticity formula (2.11) for the demand function $P = a - bQ$, in terms of P only:

$$\varepsilon_d = \frac{\Delta Q}{\Delta P}\frac{P}{Q} = \frac{1}{-b}\frac{P}{Q} = \frac{P}{P-a} \tag{2.14}$$

since $P = a - bQ \rightarrow -bQ = P - a$. This equation states that the coefficient of point elasticity of demand is dependent on price, P, vertical intercept, a, and independent of the slope, b, of the linear demand function (no b appears in formula (2.14)).

❑ Arc price elasticity of demand

Arc price elasticity of demand measures the elasticity of demand over an interval on the demand function. Instead of using the price and quantity at a point as in point elasticity, arc elasticity uses the average of the prices and quantities at the beginning and end of the stated interval:

$$\varepsilon_d = \frac{\Delta Q}{\Delta P}\cdot\frac{\frac{1}{2}[P_1 + P_2]}{\frac{1}{2}[Q_1 + Q_2]} = \frac{\Delta Q}{\Delta P}\cdot\frac{P_1 + P_2}{Q_1 + Q_2} \tag{2.15}$$

Formula (2.15) is often referred to as the **midpoints elasticity formula** or **arc-price elasticity formula**.

Note: For linear functions, the arc-price elasticity of demand formula can also be written as

$$\varepsilon_d = \frac{\Delta Q}{\Delta P}\cdot\frac{P_1 + P_2}{Q_1 + Q_2} = -\frac{1}{b}\cdot\frac{P_1 + P_2}{Q_1 + Q_2} \quad \text{since } \frac{\Delta Q}{\Delta P} = -\frac{1}{b}$$

❑ Price elasticity of supply

The same ideas as developed with respect to the price elasticity of demand can be applied to the analysis of the price elasticity of supply at a point on the supply function or averaged along an arc on the supply function. As the supply function is positively sloped, the coefficient of price elasticity of supply will be positive.

The coefficient of price elasticity of supply is given by the formula

$$\varepsilon_s = \frac{\text{Percentage change in quantity supplied}}{\text{Percentage change in price}} = \frac{\%\Delta Q_s}{\%\Delta P} = \frac{\Delta Q}{\Delta P}\cdot\frac{P}{Q} \tag{2.16}$$

Given the supply function, $P = c + dQ$, slope $= d = \Delta P/\Delta Q$. Then, $1/d = \Delta Q/\Delta P$; hence, the point elasticity of supply formula is

$$\varepsilon_s = \frac{1}{d} \cdot \frac{P}{Q}$$

Note: $\varepsilon_s > 1$: supply is elastic; $\varepsilon_s < 1$: supply is inelastic; $\varepsilon_s = 1$: supply is unit elastic.

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 2.

2.6.2 Income elasticity of demand

☐ Coefficient of income elasticity of demand

Figure 2.38 Numerical scale for the coefficient of income elasticity of demand

Worked Example 2.21 Calculating the coefficient of income elasticity of demand

Progress Exercises 2.7 Elasticity for Linear Functions

1. (a) Why is the price elasticity of demand negative?
(b) When the price of a good is £20, the price elasticity of demand is −0.7. Calculate the percentage change in demand Q when
  (i) The price increases by 5%.
  (ii) The price decreases by 8%.

2. Given the demand function, $Q = 250 - 5P$ where Q is the number of children's watches demanded at £P each,
(a) Derive an expression for the point elasticity of demand in terms of P only.
(b) Calculate the point elasticity at each of the following prices, $P = 20; 25; 30$. Describe the effect of price changes (expressed as % changes) on demand at each of these prices.

3. Given the demand function, $P = 60 - 0.2Q$, calculate the arc price elasticity of demand when
(a) Price decreases from £50 to £40 and (b) price decreases from £20 to £10.

4. For the demand functions, (i) $P = a - bQ$, (ii) $Q = c - dP$
(a) Derive expressions for the price elasticity of demand in terms of Q only.
(b) Calculate ε_d for each of the following demand functions at $Q = 100; 500; 900$.

  (i) $Q = 1800 - 0.0P$, (ii) $P = 60 - 0.5Q$.

5. $P = 90 - 0.05Q$ is the demand function for graphics calculators in an engineering college.
(a) Derive expressions for ε_d in terms of (i) P only, (ii) Q only.
(b) Calculate the value of ε_d when the calculators are priced at $P =$ £20; £30; £70.
(c) Determine the number of calculators demanded when $\varepsilon_d = -1$: $\varepsilon_d = 0$.

6. (See question 5)
(a) Explain the difference between the slope of a demand function and the price elasticity of demand.

(b) Use ε_d to calculate the response (% change in Q) to a 10% increase in the price of calculators at each of the following prices: $P =$ £20; £30; £45; £70; £90.
Comment on the results.

7. Given the supply function, $P = 20 + 0.5Q$,
   (a) Calculate the arc price elasticity of supply when price increases from £40 to £60. Interpret your result.
   (b) Calculate the percentage change in quantity supplied in response to a price increase of 10% when $P =$ £40:
      (i) By using the definition of elasticity in equation (2.16)
      (ii) Exactly by using ordinary arithmetic.
   Comment and explain the difference between (i) and (ii).

2.7 Budget and Cost Constraints

At the end of this section you should be familiar with:

- Budget constraints: effect of price and income changes
- Cost constraints: effect of price changes and expenditure limit

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 2.

2.7.1 Budget constraints

❐ Equation of budget constraint

Worked Example 2.22 Graphing and interpreting the budget constraint

Table 2.10 Consumption choices along the budget constraint

Figure 2.39 Budget constraint or budget line

Table 2.11 Effect of changes in prices and income on the budget constraint

Worked Example 2.23 Budget constraint, changes in price and income

Figure 2.40 ΔP_X and its effect on the budget constraint

Figure 2.41 ΔP_Y and its effect on the budget constraint

Figure 2.42 ΔM and its effect on the budget constraint

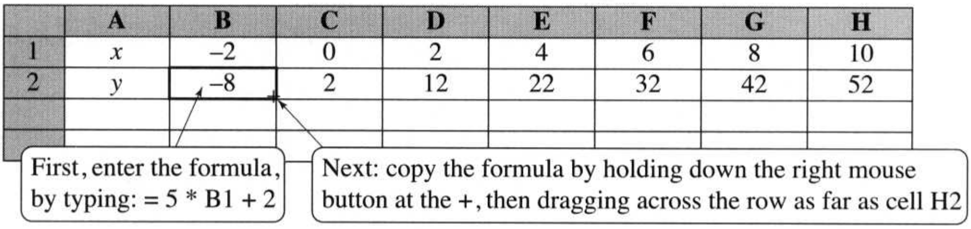
2.7.2 Cost constraints

❐ Equation of the isocost line

Figure 2.43 Isocost line

Progress Exercises 2.8 Linear Budget and Cost Constraints

Solutions to Progress Exercises 2.8



	A	B	C	D	E	F	G	H
1	x	−2	0	2	4	6	8	10
2	y	−8	2	12	22	32	42	52

First, enter the formula, by typing: = 5 * B1 + 2

Next: copy the formula by holding down the right mouse button at the +, then dragging across the row as far as cell H2

Figure 2.44 Excel sheet

2.8 Excel for Linear Functions

You should find Excel extremely useful for carrying out the tedious task of calculating the table of points from which various graphs are plotted. For example, to plot $y = 5x + 2$ from $x = -2$ to $x = 10$, proceed as follows:

- **Set up the table**

1. Use the required values of the independent variable (x), spaced at equal intervals, for the first row of the table, so we shall use $x = -2, 0, 2, 4, 6, 8, 10$ in row 1 as shown in Figure 2.44.
2. In the second row of the table calculate the corresponding y-values using the formula facility in Excel. Point to the cell B2 and type: '= 5*B1 + 2'. The '=' indicates that a formula follows; the formula states: 'multiply the value in B1 by 5, then add 2'. You could enter the actual value in B1 ($x = -2$), but if you then decide to change the x values in row 1, then the formula must be changed as well. With the reference to B1, the formula automatically recalculates y. The second reason for using the reference to the cell B1 is that the first formula may be copied along the row to automatically calculate the other y-values as indicated in Figure 2.44.
3. When several graphs are required on the same diagram, set up additional y-rows (with the appropriate formulae) to calculate the y-coordinates for each graph.

- **Plot the graph(s)**

1. Select the table of coordinates required.
2. Then select the chart wizard button. From the successive menus presented, select the chart type required, enter titles, labels for the axes, etc. In this chapter we require line graphs.
3. The finished graph may not be the size required. However, the finished graph is easily altered. Use the HELP facility in Excel or consult the reference manual.

WORKED EXAMPLE 2.24

USE EXCEL TO SHOW THE EFFECT OF PRICE AND INCOME CHANGES ON A BUDGET CONSTRAINT

Pocket money, £15, may be spent on either pool (60p per hour) or skating (£1.20 per hour).

(a) Write down the equation of the budget constraint. Graph the constraint.

(b) Show by calculation and graphically how the budget constraint changes when:

(i) The price of pool increases to 75p, the other variables do not change.
(ii) The price of skating drops to 90p, the other variables do not change.
(iii) Pocket money decreases to £10, the other variables do not change.

Solution

(a) Let x = number of hours of pool, $P_X = 60$; y = number of hours skating, $P_Y = 120$; $M = 1500$p, the income limit. The budget constraint is given by the equation

$$xP_X + yP_Y = M \quad \rightarrow \quad 60x + 120y = 1500$$

Write the budget constraint in the form $y = c + mx \rightarrow y = 12.5 - 0.5x$.

To plot this line in Excel, determine the range of values of x for which x and y are both positive. What we actually require is the x-intercept to plot the graph for values of x between $x = 0$ and x = x-intercept.

The budget constraint or budget line cuts the x-axis at $y = 0$,

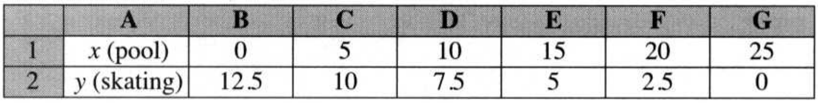
$$0 = 12.5 - 0.5x \quad \rightarrow \quad x = 25$$

Therefore, set up a table in Excel to calculate y (use the formula command) for values of x (at equal intervals), over the range $x = 0$ to 25. See Figure 2.45.

(b) (i) Pool increases to 75p per hour, so fewer hours of pool are affordable. This will cause the x-intercept to decrease towards the origin or, in other words, the budget line pivots downwards from the unchanged vertical intercept. The new budget line is given by the equation

$$xP_X + yP_Y = M \quad \rightarrow \quad 75x + 120y_{hi} = 1500$$

$$\text{or} \quad y_{hi} = 12.5 - 0.625x$$



	A	B	C	D	E	F	G
1	x (pool)	0	5	10	15	20	25
2	y (skating)	12.5	10	7.5	5	2.5	0

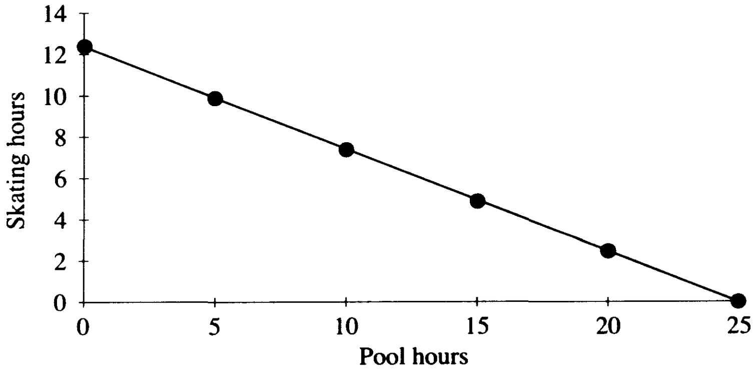
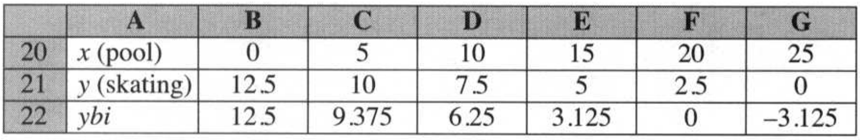


Figure 2.45 Budget constraint: skating versus pool



	A	B	C	D	E	F	G
20	*x* (pool)	0	5	10	15	20	25
21	*y* (skating)	12.5	10	7.5	5	2.5	0
22	*ybi*	12.5	9.375	6.25	3.125	0	−3.125

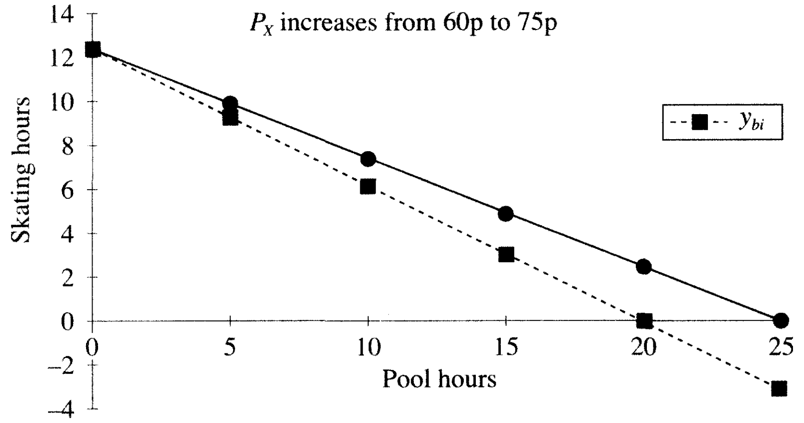


Figure 2.46 Budget constraints and the increase in the price of pool

Again, use Excel to calculate a table of points and plot this budget line along with the original budget line. See Figure 2.46.

Note: The subscript '*bi*' (part (b), section (i) of question) attached to the variable y is used only for referencing the new budget line in the Figure. The same applies to '*bii*' and '*biii*' below.

(ii) Skating decreases from 120p to 90p per hour, so more hours of skating are affordable. This will cause the y-intercept to increase away from the origin, that is, the budget line pivots upwards from the unchanged horizontal intercept. The new budget line is given by the equation

$$xP_X + yP_Y = M \quad \rightarrow \quad 60x + 90y_{bii} = 1500$$
$$\text{or} \quad y_{bii} = 16.67 - 0.67x$$

Again, use Excel to calculate a table of points and plot this budget line along with the original budget line. See Figure 2.47.

(iii) This time income decreases from £15 to £10, so less of everything is affordable. The new budget constraint drops, but runs parallel to the original. The new budget constraint is given by the equation

$$xP_X + yP_Y = M \quad \rightarrow 60x + 120y_{biii} = 1000$$
$$\text{or} \quad y_{biii} = 8.33 - 0.5x$$

Again, use Excel to calculate a table of points and plot this budget line along with the original budget line. See Figure 2.48.

	A	B	C	D	E	F	G
40	*x* (pool)	0	5	10	15	20	25
41	*y* (skating)	12.5	10	7.5	5	2.5	0
42	*ybii*	16.67	13.32	9.97	6.62	3.27	0

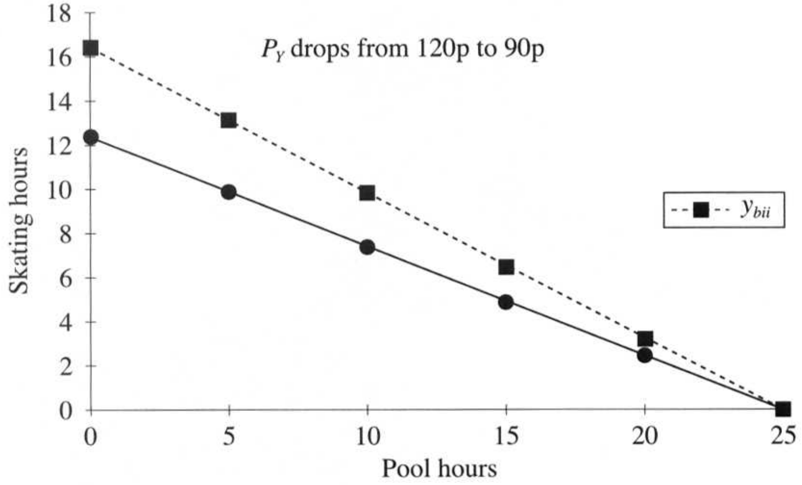


Figure 2.47 Budget constraints and the decrease in the price of skating

	A	B	C	D	E	F	G
60	*x* (pool)	0	5	10	15	20	25
61	*y* (skating)	12.5	10	7.5	5	2.5	0
62	*ybiii*	8.33	5.83	3.33	0.83	−1.67	−4.17

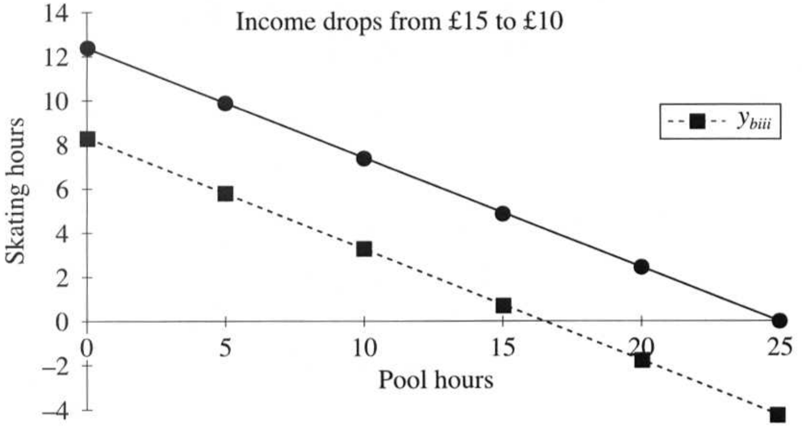


Figure 2.48 Budget constraints and the decrease in pocket money

Progress Exercises 2.9 Excel or Otherwise

1. A line cuts the vertical axis 2 units above the origin and has a slope of 1.
(a) Set up a table which gives five points on this line.
(b) Plot the line.
(c) Write down the equation of the line in the form (i) $y = \mathrm{f}(x)$ and (ii) $P = \mathrm{h}(Q)$.

2. It is known that when the price of a good is £2, 50 units are demanded, but when the price increases to £4 the quantity demanded drops to 45 units (that is, a line contains the points (2, 50) and (4, 45)).
(a) Calculate the change in demand that results from a one-unit increase in price.
(b) Calculate the quantity demanded at prices of £5, £8 and £10 per unit.
(c) Graph the demand function.
(d) Write down the equation of the demand function in the form,

(i) $P = \mathrm{g}(Q)$ and (ii) $Q = \mathrm{f}(P)$.

3. Given the line, $y = 65 - 2x$
(a) Set up a table of points, hence plot the line over the interval $0 \leq x < 34$.
(b) Write down values of and describe verbally the slope and intercepts.

4. A demand function is given by the equation $P = 80 - 4Q$.
(a) Set up a demand schedule for values of Q in the first quadrant (when P and Q are positive).
(b) Plot the demand function.
(c) Write down the values of the slope and intercepts.
(d) Write the demand function in the form $Q = \mathrm{f}(P)$ and plot the graph for values of P in the first quadrant.

5. Repeat question 4 for the supply function $P = 4 + 2.5Q$

6. Given the following demand and supply functions

$$P = 60 - 3.5Q: \qquad Q = 40 - 0.5P: \qquad P = 5 + 1.2Q$$

State which of these functions are demand functions and which are supply functions.
(a) Describe how to set up a table to calculate the values for price elasticity of demand or supply (point).
(b) Set up a table to calculate the values for the arc elasticity for each 5 units increase in P from $P = 0$ to $P = 20$.

7. Pocket money, £5, may be spent on either ice-cream or soft drinks. Ice-cream costs 12p per unit while drinks cost 20p per unit.
(a) Write down the equation of the budget constraint. Graph the constraint.
(b) Show by calculation and graphically how the budget constraint changes if the price of ice-cream drops to 9p, while pocket money and the price of drinks do not change.
(c) Show by calculation and graphically how the budget constraint changes if the price of soft drinks increase to 25p, while pocket money and the price of ice-cream do not change.
(d) If pocket money increases to £7.50, and the price of ice-cream and drinks remain the same, how does the budget constraint alter? Graph the new budget constraint.

8. The demand function for good X is given by the equation

$$Q_X = 120 - 2P_X + 5P_Y + 0.3Y.$$

Calculate the arc elasticity of demand when P_Y increases from

(i) £10 to £20, (ii) £20 to £30, (iii) £30 to £40, (iv) £40 to £50, (v) £50 to £65

while the price of good X and income (Y) remain fixed at £25 and £80 respectively.

2.9 Summary

❐ Mathematics

1. A straight line is uniquely defined by stating the slope (m) and vertical intercept (c).
2. The slope of a line is the 'slant'.
   (a) The slope may be described as the rate of change in y per unit change in x: $m = \Delta y/\Delta x$.
   (b) The slope may be measured between any two points on the line: (x_1, y_1), (x_2, y_2):
   $$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$
3. The equation of a straight line defines the relationship that exists between the x- and y-coordinates for every point on the line. It may be written in several forms:
   (a) $y = mx + c$: vertical intercept $= c$; slope $= m$; horizontal intercept $= (-c)/m$.
   (b) $ax + by + d = 0$ which may be rearranged so that the slope and vertical intercept may be read off:
   $$y = -\frac{d}{b} + \left(\frac{-a}{b}\right)x: \quad c = -\frac{d}{b}, \quad m = \frac{-a}{b}$$
4. The equation of a line may be calculated when certain information is given:
   (a) Given slope $= m$ and vertical intercept $= c$, the equation is $y = mx + c$.
   (b) Given slope $= m$ and one point on the line: (x_1, y_1), the equation is $y - y_1 = m(x - x_1)$.
   (c) Given two points on the line: (x_1, y_1), (x_2, y_2), then calculate slope,
   $$m = \frac{y_2 - y_1}{x_2 - x_1}$$
   The equation is $y - y_i = m(x - x_i)$, where $i = 1$ or 2.
5. Translations of $y = f(x)$:

Vertical translations. ↑ by c units, replace y by $(y - c)$
↓ by c units, replace y by $(y + c)$

corresponding horizontal shift $\Delta x = \frac{\Delta y}{m}$

Horizontal translations. → by c units, replace every x by $(x - c)$
← by c units, replace every x by $(x + c)$

❐ Applications

Demand function: $P = a - bQ$, negative relationship between price per unit and quantity.
Supply function: $P = c + dQ$, positive relationship between price per unit and quantity.

Total revenue: $TR = P \times Q$, price per unit multiplied by quantity sold.

Total cost: $TC = FC + VC = FC + kQ$, where k is the cost of producing each unit.

Supply function when a tax of t per unit is imposed, $P - t = c + dQ$.

Total cost function (linear) when a tax of t per unit is imposed, $TC = FC + (k + t)Q$

Elasticity of demand:

$$\varepsilon_{\mathrm{d}} = \frac{\%\text{ change in quantity demanded}}{\%\text{ change in price per unit}} = \frac{\Delta Q}{\Delta P}\frac{P}{Q}$$

Point elasticity of demand for the demand function, $P = a - bQ$, is

$$\varepsilon_{\mathrm{d}} = -\frac{1}{b}\frac{P}{Q}$$

Arc elasticity of demand between price and quantity (P_1, Q_1), (P_2, Q_2) is

$$\varepsilon_{\mathrm{d}} = \frac{\Delta Q}{\Delta P}\frac{P_1 + P_2}{Q_1 + Q_2}$$

Income elasticity of demand:

$$\varepsilon_{Y} = \frac{\Delta Q}{\Delta Y}\frac{Y_1 + Y_2}{Q_1 + Q_2}$$

Budget constraint or budget line: $xP_X + yP_Y = M$

Cost constraint: $wL + rK = C$

Excel: Useful for calculating tables and graph plotting, hence, visually a good sense of slope, positive, negative, steep, flat can be gained. Also very useful for plotting and comparing several functions when one or more of the components change, such as the prices in the budget line. All these tasks can be carried out by hand, but are slow, time consuming and open to human error.

Test Exercises 2

1. A straight line has a slope of 2 and intersects the y-axis at $y = -2$. Deduce the equation of the line and plot it.
2. Determine the slope and y-intercept (if any) for each of the following lines:

   (a) $x + 2y - 4 = 0$ (b) $2Q = P + 8$ (c) $P - 2Q = 4 + 3Q$

   (d) $5Q + P = P - 5$ (e) $y = 1$ (f) $x = 4$ (g) $y = x$

3. Deduce the equation of the following lines given slope and/or a point(s):

   (a) $m = 1.4$, (1, 3) (b) $m = -0.5$, (−2, 5) (c) (3, 1), (5, 3)

4. Given the following demand functions

$$Q = 25 - 5P \qquad \text{Demand function (1)}$$

$Q = 80 - 2.5P$ Demand function (2)

(a) Plot the graphs of the demand functions.
(b) Calculate P when $Q = 15$ from the graph. Confirm your answer algebraically.
(c) Calculate the price elasticity of demand at $Q = 15$
(d) Write the functions in the form $P = f(Q)$ and plot the graph.

5. (a) The quantity demanded of a product is 102 units when $P = 0$. If the demand, Q, drops by five units for every unit increase in price, write down the equation of the demand function in the form $Q = f(P)$ and plot its graph.
(b) From the graph in (a), calculate the arc price elasticity of demand when the quantity demanded increases from 55 to 65. Confirm your answer algebraically.

6. A supplier will start producing a product when a price of £24 per unit is available and output, Q, will increase by five units for each unit increase in price.
(a) Plot the graph of the supply function, plotting Q on the horizontal axis.
(b) Deduce the equation of the supply function.
(c) From the graph calculate Q when $P = 45$. Confirm your answer algebraically.

7. A supply function is given by the equation $20P = 80 + 5Q$. Deduce the equation of the supply function if a tax of £1.50 is imposed on the price of each unit produced.

8. A student has £140 per month to spend on football and on cinema. If each attendance at a football match costs £12, while outing to the cinema costs £8,
(a) Write down the equation of the budget constraint. Graph the constraint.
(b) Analyse the effect on the constraint if the price of a football outing increases to £20.

CHAPTER 3

Simultaneous Equations

In this chapter the methods for solving simultaneous linear equations are introduced. These methods are used to determine equilibrium outcomes in various markets such as the goods, labour and money markets. In addition, national income equilibrium is analysed and the *IS-LM* model is introduced. This chapter is divided into the following sections:

3.1 Solving simultaneous linear equations
3.2 Equilibrium and break-even
3.3 Consumer and producer surplus
3.4 National income model and the *IS-LM* model
3.5 Excel for simultaneous linear equations
3.6 Summary

3.1 Solving Simultaneous Linear Equations

At the end of this section you should be familiar with:

- How to solve two simultaneous equations in two unknowns using:
  (a) Algebra
  (b) Graphical methods
- Determining when two equations in two unknowns have:
  (a) A unique solution
  (b) No solution
  (c) Infinitely many solutions
- Solving three simultaneous equations in three unknowns.

In many applications, both practical and theoretical, there will be several equations with several variables or unknowns. These are referred to generally as simultaneous equations. This section introduces some methods to solve simultaneous linear equations.

☺ **Reminder**

A solution of an equation in an unknown, say x, is simply the value for x for which the left-hand side (LHS) of the equation is equal to the right-hand side (RHS).
For example, consider the equation $x + 4 = 6$
The solution of this equation is $x = 2$
$x = 2$ is the only value of x for which the LHS = RHS.
The statement '$x = 2$ satisfies the equation' is another way of saying that $x = 2$ is a solution.

The solution, therefore, of a set of simultaneous equations is a set of values for the variables which satisfy all the equations.

3.1.1 Two equations in two unknowns

A standard method for solving two linear equations in two unknowns is outlined in Worked Examples 3.1 to 3.3.

WORKED EXAMPLE 3.1

SOLVING SIMULTANEOUS EQUATIONS 1

Given the simultaneous equations

$$x + 3y = 4 \qquad \textbf{(1)}$$

$$-x + 2y = 6 \qquad \textbf{(2)}$$

(a) Solve for x and y algebraically.
(b) Solve for x and y graphically.

Solution

(a) **Method:** Eliminate x from the system of equations by adding equations (1) and (2). The two equations reduce to a single equation in which the only unknown is y. Solve for y, then substitute the value of y into either of the original equations and solve for x:

Step 1:

$$\begin{array}{rl} x + 3y = 4 & (1) \\ \underline{-x + 2y = 6} & (2) \\ 0 + 5y = 10 & \text{adding} \\ y = \dfrac{10}{5} = 2 & \text{solving for } y \end{array}$$

Step 2: Solve for x by substituting $y = 2$ into either equation (1) or equation (2):

$$-x + 2(2) = 6 \qquad \text{substituting } y = 2 \text{ into equation (2)}$$
$$-x = 6 - 4$$
$$x = -2$$

Step 3: Check the solution $x = -2$, $y = 2$ by substituting these values into equations (1) and (2) and confirm that both equations balance.

- Substitute $x = -2$ and $y = 2$ into equation (1):

$$x + 3y = 4$$
$$(-2) + 3(2) = 4 \qquad \text{substituting in } x = -2 \text{ and } y = 2$$
$$-2 + 6 = 4$$
$$4 = 4 \qquad \text{so equation (1) balances and } x = -2,\ y = 2 \text{ is a solution}$$

- Substitute $x = -2$ and $y = 2$ into equation (2):

$$-x + 2y = 6$$
$$-(-2) + 2(2) = 6 \qquad \text{substituting in } x = -2 \text{ and } y = 2$$
$$2 + 4 = 6$$
$$6 = 6 \qquad \text{so equation (2) balances and } x = -2,\ y = 2 \text{ is a solution}$$

Since the point $(-2, 2)$ satisfies equations (1) and (2), then this point is at the point of intersection of the lines represented by equations (1) and (2) as shown in (b).
(b) The two lines are plotted in Figure 3.1. The point of intersection is the solution. The coordinates of this point are $x = -2$ and $y = 2$. In this case, it is a **unique solution**, that is, the lines intersect at only one point. This point is on the first line so it satisfies equation (1) and also on the second line, so it satisfies equation (2).

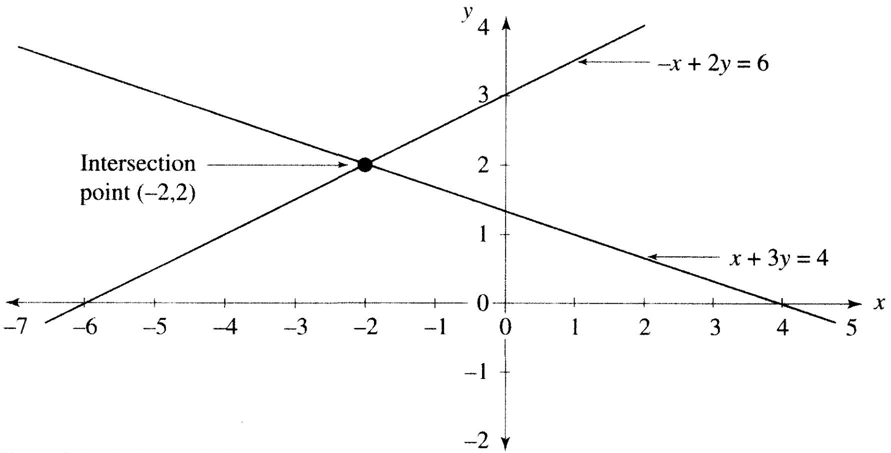


Figure 3.1 Unique solution

WORKED EXAMPLE 3.2

SOLVING SIMULTANEOUS EQUATIONS 2

Given the simultaneous equations

$$2x + 3y = 12.5 \qquad \textbf{(1)}$$

$$-x + 2y = 6 \qquad \textbf{(2)}$$

(a) Solve for x and y algebraically.
(b) Solve for x and y graphically.

Solution

(a) In this example, neither the x nor the y terms are the same. However, all terms on both sides of any equation may be multiplied by a constant without affecting the solution of the equation. So, if equation (2) is multiplied by 2, the x terms in both equations will be the same with opposite signs. Then, proceed as in Worked Example 3.1.

Step 1: Eliminate x from the system of equations by multiplying equation (2) by 2 and then add the equations; $2x$ and $-2x$ cancel to leave a single equation in one unknown, y:

$$\begin{array}{rl}
2x + 3y = 12.5 & \text{(1) as given} \\
\underline{-2x + 4y = 12} & (2) \times 2 \\
0 + 7y = 24.5 & \text{adding} \\
y = \dfrac{24.5}{7} = 3.5 & \text{solving for } y
\end{array}$$

Step 2: Solve for the value of x by substituting $y = 3.5$ into either equation (1) or equation (2):

$$\begin{array}{rl}
-x + 2(3.5) = 6 & \text{substituting } y = 3.5 \text{ into (2) to find the value of } x \\
-x = 6 - 7 & \\
-x = -1 \rightarrow x = 1 &
\end{array}$$

Step 3: Check the solution $x = 1$, $y = 3.5$, by substituting these values into equations (1) and (2) and confirm that both equations balance.

- Substitute $x = 1$ and $y = 3.5$ into equation (1):

$$\begin{array}{rl}
2x + 3y = 12.5 & \\
2(1) + 3(3.5) = 12.5 & \text{substituting in } x = 1 \text{ and } y = 3.5 \\
2 + 10.5 = 12.5 & \\
12.5 = 12.5 & \text{so (1) balances and } x = 1, y = 3.5 \text{ is a solution}
\end{array}$$

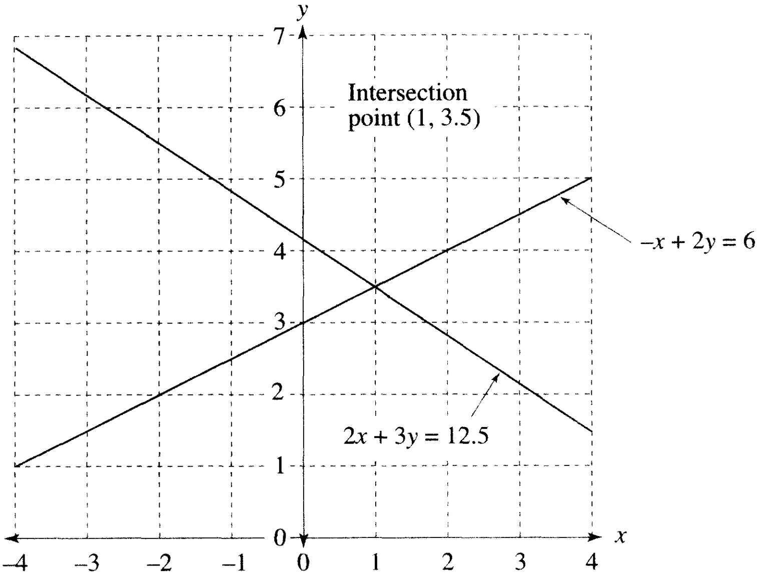


Figure 3.2 Unique solution

- Substitute $x = 1$ and $y = 3.5$ into equation (2):

$$\begin{aligned} -x + 2y &= 6 \\ -(1) + 2(3.5) &= 6 \qquad \text{substituting in } x = 1 \text{ and } y = 3.5 \\ -1 + 7 &= 6 \\ 6 &= 6 \qquad \text{so (2) balances and } x = 1, y = 3.5 \text{ is a solution} \end{aligned}$$

Therefore, the solution of equations (1) and (2) is at the point of intersection of the lines represented by equations (1) and (2), as shown in (b).

(b) The two lines are plotted in Figure 3.2. The point of intersection is the solution. The coordinates of this point are $x = 1$ and $y = 3.5$. In this case it is a **unique solution**, that is, the lines intersect at only one point. This point is on the first line, so it satisfies equation (1), and also on the second line, so it satisfies equation (2).

WORKED EXAMPLE 3.3

SOLVING SIMULTANEOUS EQUATIONS 3

Given the simultaneous equations

$$2x + 3y = 2 \qquad \textbf{(1)}$$

$$5x + 2y = 6 \qquad \textbf{(2)}$$

Solve for x and y algebraically.

Solution

In these two equations, neither the x nor the y terms are the same. If equation (1) is multiplied by 2 and equation (2) is multiplied by −3, the y terms in both equations will be the same with opposite signs. Then proceed as in Worked Example 3.1 above.

Step 1: Eliminate y terms from the system of equations:

$$\begin{array}{rl} 4x + 6y = 4 & (1) \times 2 \\ \underline{-15x - 6y = -18} & (2) \times -3 \\ -11x = -14 & \text{adding} \\ x = \dfrac{-14}{-11} = 1.2727 & \text{solving for } x \end{array}$$

Step 2: Solve for y by substituting $x = 1.2727$ into either equation (1) or equation (2):

$$\begin{array}{rl} 2(1.2727) + 3y = 2 & \text{substituting } x = 1.2727 \text{ into equation (1)} \\ 3y = 2 - 2.5454 & \\ y = \dfrac{-0.5454}{3} = -0.1818 & \end{array}$$

Step 3: Checking the solution ($x = 1.2727, y = -0.1818$) is left as an exercise for the reader.

3.1.2 Unique, infinitely many and no solutions of simultaneous equations

A set of simultaneous equations may have

- A unique solution
- No solution
- Infinitely many solutions

☐ Unique solution

This occurs when a set of equations has one set of values which satisfy all equations. This was illustrated in Worked Examples 3.1 to 3.3.

☐ No solution

This occurs when a set of equations has no set of values which satisfy all equations.

WORKED EXAMPLE 3.4

SIMULTANEOUS EQUATIONS WITH NO SOLUTION

Given the simultaneous equations

$$y = 1 + x$$
$$y = 2 + x$$

(a) Solve for x and y algebraically.
(b) Solve for x and y graphically.

Solution

$$\begin{array}{ll} y = 1 + x & \\ \underline{y = 2 + x} & \\ 0 = -1 & \text{subtracting equations} \end{array}$$

$0 = -1$ is not possible, therefore, there is no solution. Even from a purely practical point of view, you can see that there is no way that both these equations can both be true. How can y be equal to $(1 + x)$ and $(2 + x)$ at the same time?

Note:

A false statement (or a contraction) like $0 = -1$ indicates a set of equations with no solution

(b) The two equations are plotted in Figure 3.3. The lines will never meet since they are parallel and thus will never have a point (solution) in common.

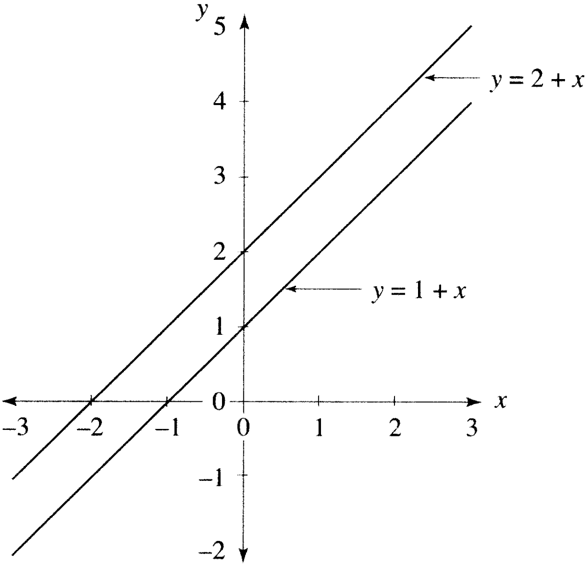


Figure 3.3 No solution

❐ Infinitely many solutions

A set of equations has infinitely many solutions when there is an infinite number of sets of values that satisfy all equations.

WORKED EXAMPLE 3.5

SIMULTANEOUS EQUATIONS WITH INFINITELY MANY SOLUTIONS

Given the simultaneous equations

$$y = 2 - x \tag{1}$$

$$2y = 4 - 2x \tag{2}$$

(a) Solve for x and y algebraically.
(b) Solve for x and y graphically.

Solution

When equation (2) is divided by 2, the result is exactly the same as equation (1), since,

$$\frac{2y}{2} = \frac{4}{2} - \frac{2x}{2} \quad \rightarrow \quad y = 2 - x$$

So, equations (1) and (2) are the same! There is only one equation in two unknowns. If x is given any value, the corresponding y-value can be calculated.

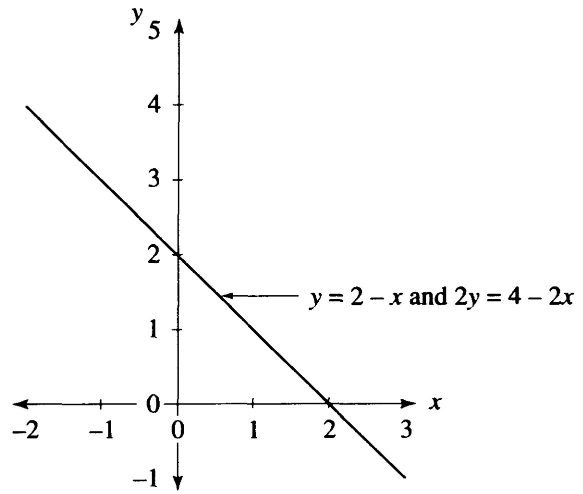


Figure 3.4 Infinitely many solutions

For example, when

$$x = 2 \quad \rightarrow \quad y = 2 - 2 = 0$$
$$x = 3 \quad \rightarrow \quad y = 2 - 3 = -1$$
$$x = 5 \quad \rightarrow \quad y = 2 - 5 = -3 \text{ etc.}$$

There is an infinite number of (x, y) pairs which satisfy equations (1) and (2).
(b) Equations (1) and (2) are plotted in Figure 3.4. Note that these equations represent coincident lines, therefore every point on one line is also a point on the other line. Since a line has infinitely many points, there are infinitely many solutions or points in common.

3.1.3 Three simultaneous equations in three unknowns

The methods used above to solve two equations in two unknowns may be extended to three equations in three unknowns, four equations in four unknowns, etc. The strategy is to eliminate one of the variables first by adding multiples of equations to other equations and hence reducing the problem to two equations in two unknowns. This is best demonstrated by Worked Example 3.6.

WORKED EXAMPLE 3.6

SOLVE THREE EQUATIONS IN THREE UNKNOWNS

Solve the equations

$$2x + y - z = 4 \qquad \textbf{(1)}$$
$$x + y - z = 3 \qquad \textbf{(2)}$$
$$2x + 2y + z = 12 \qquad \textbf{(3)}$$

Solution

The simplest approach is to add equation (3) to equation (1), and hence eliminate z, giving an equation in x and y. Then add equation (3) to equation (2), eliminating z again, giving another equation in x and y.

$$\begin{array}{rl} 2x + y - z = 4 & (1) \\ \underline{2x + 2y + z = 12} & (3) \\ 4x + 3y + 0 = 16 & (4) = (1) + (3) \end{array}$$

$$\begin{array}{rl} x + y - z = 3 & (2) \\ \underline{2x + 2y + z = 12} & (3) \\ 3x + 3y + 0 = 15 & (5) = (2) + (3) \end{array}$$

Equations (4) and (5) are the usual two equations in two unknowns, so solve for x and y. Then solve for z later.

$$\begin{array}{rcll} 4x + 3y & = & 16 & (4) \\ -3x - 3y & = & -15 & (6) = (5) \times -1 \\ \hline x & = & 1 & (4) + (6) \end{array}$$

So, $x = 1$. Substitute $x = 1$ into equations (4), (5) or (6) to solve for y. Substituting $x = 1$ into equation (4) gives $4(1) + 3y = 16 \rightarrow y = 4$. Finally, find z by substituting $x = 1, y = 4$ into any of the equations (1), (2) or (3). For example, substituting into equation (2),

$$1 + 4 - z = 3 \quad \rightarrow \quad z = 2$$

Therefore, the values which satisfy all three equations (1), (2) and (3) are $x = 1, y = 4, z = 2$.

Progress Exercises 3.1 Simultaneous Equations: Solving for Two and Three Unknowns

Solve the following simultaneous equations:

1. $y = x$, $y = 3 - x$

2. $x + y = 10$, $x - y = 4$

3. $x + y = 19$, $x - 8y = 10$

4. $3y + 2x = 5$, $4y - x = 3$

5. $5x - 2y = 11$, $3x + 3y = 15$

6. $y = 2x + 3$, $y = 7 - 2x$

7. $2y + 4x - 1 = 7$, $3y - 2x = 12$

8. $x + 2y = 10$, $y = -0.5x + 5$

9. $4x - y = 12$, $2y - 3x = 11.2$

10. $5x - 2y = 15$, $15x - 45 = 6y$

11. $2x - 5y = 7$, $2 = 3x - 2.5y$

12. $4P - 3Q = 4$, $2Q + 1.5P = 20$

13. $5 + 2P = 6Q$, $5P + 8Q = 25$

14. $x - y + z = 0$, $2y - 2z = 2$, $-x + 2y + 2z = 29$

15. $P_1 - 3P_2 = 0$, $5P_2 - P_3 = 10$, $P_1 + P_2 + P_3 = 8$

3.2 Equilibrium and Break-even

At the end of this section you should be familiar with:

- Equilibrium in the goods market and labour market
- Price controls, government intervention in various markets
- Market equilibrium for substitute and complementary goods
- Taxes, subsidies and their distribution between producer and consumer
- Break-even analysis

The method of simultaneous equations is now applied to the determination of equilibrium conditions in various markets; for example the goods, labour and money markets. In addition, situations are considered that prevent the occurrence of equilibrium. Furthermore, the analysis also considers factors that change the state of equilibrium from one position to another.

Note: A state of equilibrium within a model is a situation that is characterised by a lack of tendency to change.

3.2.1 Equilibrium in the goods and labour markets

❐ Goods market equilibrium

Goods market equilibrium (market equilibrium) occurs when the quantity demanded (Q_d) by consumers and the quantity supplied (Q_s) by producers of a good or service are equal. Equivalently, market equilibrium occurs when the price that a consumer is willing to pay (P_d) is equal to the price that a producer is willing to accept (P_s). The equilibrium condition, therefore, is expressed as

$$Q_d = Q_s \quad \text{and} \quad P_d = P_s \tag{3.1}$$

Note: In equilibrium problems, once the equilibrium condition is stated, Q and P are used to refer to the equilibrium quantity and price respectively.

WORKED EXAMPLE 3.7

GOODS MARKET EQUILIBRIUM

The demand and supply functions for a good are given as

$$\text{Demand function:} \quad P_d = 100 - 0.5Q_d \tag{3.2}$$

$$\text{Supply function:} \quad P_s = 10 + 0.5Q_s \tag{3.3}$$

Calculate the equilibrium price and quantity algebraically and graphically.

Solution

Market equilibrium occurs when $Q_d = Q_s$ and $P_d = P_s$. Since the functions are written in the form $P = f(Q)$ with P as the only variable on the LHS of each equation, it is easier to equate prices, thereby reducing the system to an equation in Q only; hence, solve for Q:

$$P_d = P_s$$

$$100 - 0.5Q = 10 + 0.5Q \quad \text{equating equations (3.2) and (3.3)}$$

$$100 - 10 = 0.5Q + 0.5Q$$

$$90 = Q \quad \text{equilibrium quantity}$$

Now solve for the equilibrium price by substituting $Q = 90$ into either equation (3.2) or (3.3):

$$P = 100 - 0.5(90) \qquad \text{substituting } Q = 90 \text{ into equation (3.2)}$$

$$P = 55 \qquad \text{equilibrium price}$$

Check the solution, $Q = 90$, $P = 55$ by substituting these values into either equation (3.2) or (3.3). This exercise is left to the reader.

Figure 3.5 illustrates market equilibrium at point E_0 with equilibrium quantity, 90, and equilibrium price, £55. The consumer pays £55 for the good which is also the price that the producer receives for the good. There are no taxes (what a wonderful thought!).

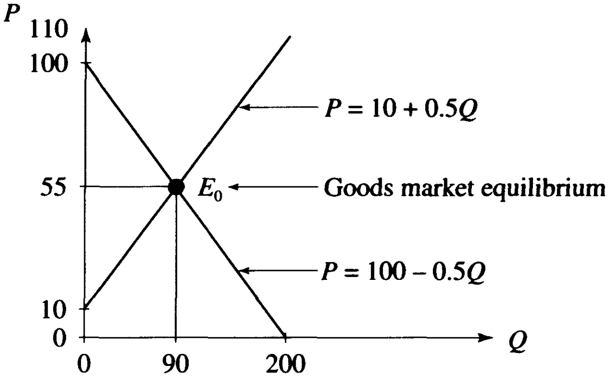


Figure 3.5 Goods market equilibrium

❒ Labour market equilibrium

Labour market equilibrium occurs when the labour demanded (L_d) by firms is equal to the labour supplied (L_s) by workers or, equivalently, when the wage that a firm is willing to offer (w_s) is equal to the wage that workers are willing to accept (w_d). Labour market equilibrium, therefore, is expressed as

$$L_d = L_s \quad \text{and} \quad w_d = w_s \tag{3.4}$$

Again, in solving for labour market equilibrium, once the equilibrium condition is stated, L and w refer to the equilibrium number of labour units and the equilibrium wage respectively.

WORKED EXAMPLE 3.8

LABOUR MARKET EQUILIBRIUM

The labour demand and supply functions are given as

$$\text{Labour demand function:} \quad w_d = 9 - 0.6L_d \tag{3.5}$$

$$\text{Labour supply function:} \quad w_s = 2 + 0.4L_s \tag{3.6}$$

Calculate the equilibrium wage and equilibrium number of workers algebraically and graphically. (In this example 1 worker $\equiv$ 1 unit of labour.)

Solution

Labour market equilibrium occurs when $L_d = L_s$ and $w_d = w_s$. Since the functions are written in the form $w = f(L)$, equate wages, thereby reducing the system to an equation in L only; hence, solve for L:

$$w_d = w_s$$

$$9 - 0.6L = 2 + 0.4L \quad \text{equating equations (3.5) and (3.6)}$$

$$9 - 2 = L$$

$$7 = L \quad \text{equilibrium number of workers}$$

Now solve for w by substituting $L = 7$ into either equation (3.5) or (3.6):

$$w = 9 - 0.6(7) \quad \text{substituting } L = 7 \text{ into equation (3.5)}$$

$$w = 4.8 \quad \text{equilibrium wage}$$

Figure 3.6 illustrates labour market equilibrium at point E_0 with equilibrium number of workers, 7, and equilibrium wage, £4.80. Each worker receives £4.80 per hour for his or her labour services which is also the wage that the firm is willing to pay.

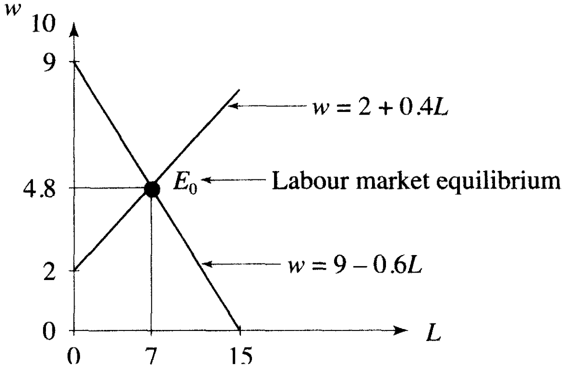


Figure 3.6 Labour market equilibrium

3.2.2 Price controls, government intervention in various markets

In reality, markets may fail to achieve market equilibrium due to a number of factors; for example, the intervention of governments, the existence of firms with monopoly power. Government intervention in the market through the use of price controls is now analysed.

❐ Price ceilings

Price ceilings are used by governments in cases where they believe that the equilibrium price is too high for the consumer to pay. Thus, price ceilings operate **below market equilibrium** and are aimed at protecting consumers. Price ceilings are also known as maximum price controls, where the price is not allowed to go above the maximum or 'ceiling' price (for example, rent controls or maximum price orders).

WORKED EXAMPLE 3.9

GOODS MARKET EQUILIBRIUM AND PRICE CEILINGS

The demand and supply functions for a good are given by

$$\text{Demand function:} \quad P_d = 100 - 0.5Q_d \qquad \textbf{(3.7)}$$

$$\text{Supply function:} \quad P_s = 10 + 0.5Q_s \qquad \textbf{(3.8)}$$

(a) Analyse the effect of the introduction of a price ceiling of £40 in this market.
(b) Calculate the profit made by black marketeers if a black market operated in this market.

Solution

(a) The demand and supply functions are the same as those in Worked Example 3.7 where the equilibrium price and quantity were £55 and 90 units respectively. The price ceiling of £40 is below the equilibrium price of £55. Its effect is analysed by comparing the levels of quantity demanded and supplied at $P = £40$.

The quantity demanded at $P = 40$ is

$$P_d = 100 - 0.5Q_d \quad \text{equation (3.7)}$$

$$40 = 100 - 0.5Q_d \quad \text{substituting } P = 40$$

$$0.5Q_d = 100 - 40$$

$$Q_d = 120$$

The quantity supplied at $P = 40$ is

$$P_s = 10 + 0.5Q_s \quad \text{equation (3.8)}$$

$$40 = 10 + 0.5Q_s \quad \text{substituting } P = 40$$

$$-0.5Q_s = 10 - 40$$

$$-0.5Q_s = -30$$

$$Q_s = 60$$

Since the quantity demanded ($Q_d = 120$) is greater than the quantity supplied ($Q_s = 60$), there is an excess demand (XD) of: $XD = Q_d - Q_s = 120 - 60 = 60$. This is also referred to as a shortage in the market. It is illustrated in Figure 3.7.

(b) The existence of price ceilings often leads to the establishment of black markets where goods are sold illegally at prices above the legal maximum. Black marketeers would buy the 60 units supplied at the controlled price of £40 per unit. However, as there is a shortage of goods, consumers are willing to pay a higherprice for these 60 units. The price that consumers are willing to pay is calculated from the demand function for $Q = 60$. Substitute $Q = 60$ into the demand function:

$$P_d = 100 - 0.5Q_d$$

$$P_d = 100 - 0.5(60) = 100 - 30 = 70$$

So, $P_d = 70$ is the price consumers are willing to pay.

Therefore, black marketeers buy the 60 units at the maximum price of £40 per unit, costing them 60 × £40 = £2400, and then sell these 60 units at £70 per unit, generating revenue of 60 × £70 = £4200. Their profits (π) is the difference between revenue and costs:

$$\begin{aligned}\pi &= TR - TC \\ &= (70 \times 60) - (40 \times 60) \\ &= 4200 - 2400 \\ &= 1800\end{aligned}$$

This is illustrated as the dashed area in Figure 3.7.

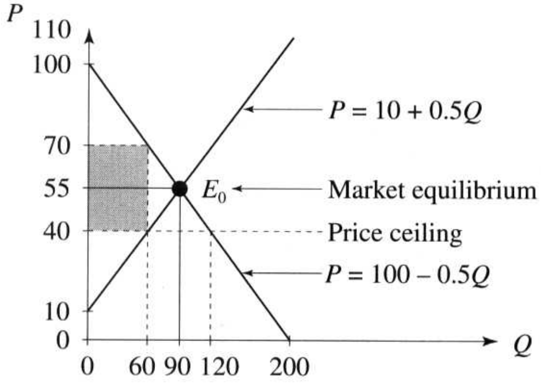


Figure 3.7 Price ceiling and black market

❐ Price floors

Price floors are used by governments in cases where they believe that the equilibrium price is too low for the producer to receive. Thus, price floors operate **above market equilibrium** and are aimed at protecting producers. Price floors are also known as minimum prices, where the price is not allowed to go below the minimum or 'floor' price (for example, the Common Agricultural Policy (CAP) in the European Union and minimum wage laws).

WORKED EXAMPLE 3.10

LABOUR MARKET EQUILIBRIUM AND PRICE FLOORS

Given the labour demand and supply functions as

Labour demand function: $w_d = 9 - 0.6L_d$ **(3.9)**

Labour supply function: $w_s = 2 + 0.4L_s$ **(3.10)** analyse the effect on the labour market if the government introduces a minimum wage law of £6 per hour.

Solution

The labour demand and supply functions are the same as those in Worked Example 3.8 where the equilibrium wage and units of labour were £4.80 per hour and 7 labour units respectively. The minimum wage law (price floor) of £6 is above market equilibrium. Its effect is analysed by comparing the levels of labour demanded and supplied at $w = 6$.

Labour demanded at $w = 6$ is	Labour supplied at $w = 6$ is
$w_d = 9 - 0.6L_d$ equation (3.9)	$w_s = 2 + 0.4L_s$ equation (3.10)
$6 = 9 - 0.6L_d$ substituting $w = 6$	$6 = 2 + 0.4L_s$ substituting $w = 6$
$0.6L_d = 3$ $L_d = 5$	$4 = 0.4L_s$ $10 = L_s$

Since labour supplied ($L_s = 10$) is greater than labour demanded ($L_d = 5$), there is an excess supply of labour of $XS = L_s - L_d = 10 - 5 = 5$. This is also referred to as a surplus, that is, there is unemployment in the labour market. The graphical illustration of this result is left to the reader.

Progress Exercises 3.2 Determination of Equilibrium for Linear Functions

1. (a) Determine the equation of the line which has a slope $m = 1.5$, and which passes through the point $x = 4$, $y = 12$.
(b) A supplier is known to supply a quantity of goods $Q = 30$ when the market price P is 25 per unit. If the quantity supplied increases by 3 for each unit increase in price, determine the equation of the supply function in the form $Q = f(P)$.
2. The demand and supply functions for a good (jeans) are given by:

$$\text{Demand function:} \quad P_d = 50 - 3Q_d$$

$$\text{Supply function:} \quad P_s = 14 + 1.5Q_s$$

where P is the price of a pair of jeans; Q is the number of pairs of jeans.
(a) Calculate the equilibrium price and quantity.
(b) Calculate the level of excess supply ($Q_s - Q_d$) when $P = 38$.
3. (See question 2)
(a) Calculate the level of excess demand ($Q_d - Q_s$) when $P = 20$.
(b) Calculate the profit made on the black market if a maximum price of £20 per pair is imposed.

4. The demand and supply functions for labour are given by:

$$\text{Labour demand function: } w_{\mathrm{d}} = 70 - 4L$$

$$\text{Labour supply function: } w_{\mathrm{s}} = 10 + 2L$$

(a) Calculate the equilibrium number of workers employed and the equilibrium wage per hour.
(b) Calculate the excess demand for labour $(L_{\mathrm{d}} - L_{\mathrm{s}})$ when $w = 20$.
(c) Calculate the excess supply for labour $(L_{\mathrm{s}} - L_{\mathrm{d}})$ when $w = 40$.

3.2.3 Market equilibrium for substitute and complementary goods

Complementary goods are goods that are consumed together (for example, cars and petrol, computer hardware and computer software). One good cannot function without the other. On the other hand, substitute goods are consumed instead of each other (for example, coffee versus tea; bus versus train on given routes).

The general demand function is now written as,

$$Q = \mathrm{f}(P, P_{\mathrm{s}}, P_{\mathrm{c}})$$

that is, the quantity demanded of a good is a function of the price of the good itself and the prices of those goods that are substitutes and complements to it.

Note: In this case, P_{s} refers to the price of substitute goods, not to be confused with P_{s} which is used to refer to the supply price of a good.

Consider two goods, X and Y. The demand function for good X is written differently depending on whether good Y is a substitute to X or a complement to X.

X and Y are substitutes

$$Q_X = a - bP_X + dP_Y$$

Note the positive sign before dP_Y

There is a positive relationship between the quantity demanded of good X and the price of good Y, since

$$Q_X = (a + dP_Y) - bP_X$$

Therefore, as P_Y increases, so does Q_X.

Similarly, the demand function for good Y is

$$Q_Y = \alpha - \beta P_Y + \delta P_X$$

Example: if train fares increase, individuals will reduce their demand for train journeys and increase their demand for bus journeys.

X and Y are complements

$$Q_X = a - bP_X - dP_Y$$

Note the negative sign before dP_Y

There is a negative relationship between the quantity demanded of good X and the price of good Y, since

$$Q_X = (a - dP_Y) - bP_X$$

Therefore, as P_Y increases, Q_X decreases.

Similarly, the demand function for good Y is,

$$Q_Y = \alpha - \beta P_Y - \delta P_X$$

Example: if car prices increase, individuals will reduce their demand for cars and consequently the demand for petrol decreases.

WORKED EXAMPLE 3.11

EQUILIBRIUM FOR TWO SUBSTITUTE GOODS

Find the equilibrium price and quantity for two substitute goods X and Y given their respective demand and supply equations as,

$$Q_{dX} = 82 - 3P_X + P_Y \qquad \textbf{(3.11)}$$

$$Q_{sX} = -5 + 15P_X \qquad \textbf{(3.12)}$$

$$Q_{dY} = 92 + 2P_X - 4P_Y \qquad \textbf{(3.13)}$$

$$Q_{sY} = -6 + 32P_Y \qquad \textbf{(3.14)}$$

Solution

The equilibrium condition for this two-goods market is

$$Q_{dX} = Q_{sX} \quad \text{and} \quad Q_{dY} = Q_{sY}$$

Therefore, the equilibrium prices and quantities are calculated as follows:

$$82 - 3P_X + P_Y = -5 + 15P_X \qquad \text{equating equations (3.11) and (3.12)}$$

$$-18P_X + P_Y = -87 \qquad \text{simplifying} \qquad \textbf{(3.15)}$$

and

$$92 + 2P_X - 4P_Y = -6 + 32P_Y \qquad \text{equating equations (3.13) and (3.14)}$$

$$2P_X - 36P_Y = -98 \qquad \text{simplifying} \qquad \textbf{(3.16)}$$

Equations (3.15) and (3.16) are two equations in two unknowns, P_X and P_Y. Therefore, solve these simultaneous equations for the equilibrium prices, P_X and P_Y.

$$\begin{array}{rll} -18P_X + P_Y &= -87 & \text{equation (3.15)} \\ \underline{18P_X - 324P_Y} &\underline{= -882} & \text{equation (3.16) multiplied by 9} \\ -323P_Y &= -969 & \\ P_Y &= 3 & \end{array}$$

Solve for P_X by substituting $P_Y = 3$ into either equation (3.15) or equation (3.16):

$$\begin{array}{rll} -18P_X + 3 &= -87 & \text{substituting } P_Y = 3 \text{ into equation (3.15)} \\ -18P_X &= -90 & \\ P_X &= 5 & \end{array}$$

Now, solve for Q_X and Q_Y

Solve for Q_X by substituting $P_X = 5$ and $P_Y = 3$ into either equation (3.11) or equation (3.12) as appropriate:

$$\begin{array}{rll} Q_X &= -5 + 15P_X & \text{using equation (3.12)} \\ Q_X &= -5 + 15(5) & \text{substituting } P_X = 5 \\ Q_X &= 70 & \end{array}$$

Solve for Q_Y by substituting $P_Y = 3$ and $P_X = 5$ into either equation (3.13) or equation (3.14) as appropriate:

$$Q_Y = -6 + 32P_Y \quad \text{using equation (3.14)}$$

$$Q_Y = -6 + 32(3) \quad \text{substituting } P_Y = 3$$

$$Q_Y = 90$$

The equilibrium prices and quantities in this two goods market are

$$P_X = 5, \quad Q_X = 70, \quad P_Y = 3, \quad Q_Y = 90$$

3.2.4 Taxes, subsidies and their distribution

Taxes and subsidies are another example of government intervention in the market. A tax on a good is known as an indirect tax. Indirect taxes may be either:

- A fixed amount per unit of output (excise tax); for example, the tax imposed on petrol and alcohol. This will translate the supply function vertically upwards by the amount of the tax.
- A percentage of the price of the good; for example, value added tax. This will change the slope of the supply function. The slope will become steeper since a given percentage tax will be a larger absolute amount the higher the price.

❐ Fixed tax per unit of output

When a tax is imposed on a good, two issues of concern arise:

- How does the imposition of the tax affect the equilibrium price and quantity of the good?
- What is the distribution (incidence) of the tax, that is, what percentage of the tax is paid by consumers and producers respectively?

In these calculations

- The consumer always pays the equilibrium price.
- The supplier receives the equilibrium price minus the tax.

WORKED EXAMPLE 3.12

TAXES AND THEIR DISTRIBUTION

The demand and supply functions for a good are given as

$$\text{Demand function: } P_d = 100 - 0.5Q_d \tag{3.17}$$

$$\text{Supply function: } P_s = 10 + 0.5Q_s \tag{3.18}$$

(a) Calculate the equilibrium price and quantity.

(b) Assume that the government imposes a fixed tax of £6 per unit sold.

(i) Write down the equation of the supply function, adjusted for tax.

(ii) Find the new equilibrium price and quantity algebraically and graphically.

(iii) Outline the distribution of the tax, that is, calculate the tax paid by the consumer and the producer.

Solution

(a) The equilibrium quantity and price are 90 units and £55 respectively.

☺ **Remember:** the equilibrium price of £55 (with no taxes) means that the price the consumer pays is equal to the price that the producer receives.

(b)

(i) ☺ **Remember:** Translations, Chapter 2.

The tax of £6 per unit sold means that the effective price received by the producer is $(P_s - 6)$. The equation of the supply function adjusted for tax is

$$\begin{aligned} P_s - 6 &= 10 + 0.5Q \\ P_s &= 16 + 0.5Q \qquad \textbf{(3.19)} \end{aligned}$$

The supply function is translated vertically upwards by 6 units (with a corresponding horizontal leftward shift). This is illustrated in Figure 3.8 as a line parallel to the original supply function.

(ii) The new equilibrium price and quantity are calculated by equating the original demand function, equation (3.17), and the supply function adjusted for tax, equation (3.19):

$$\begin{aligned} P_d &= P_s \\ 100 - 0.5Q &= 16 + 0.5Q \qquad \text{equating equations (3.17) and (3.19)} \\ Q &= 84 \end{aligned}$$

Substitute the new equilibrium quantity, $Q = 84$ into either equation (3.17) or equation (3.19) and solve for the new equilibrium price:

$$\begin{aligned} P &= 100 - 0.5(84) \qquad \text{substituting } Q = 84 \text{ into equation (3.17)} \\ P &= 58 \end{aligned}$$

The point (84, 58) is shown as point E_1 in Figure 3.8.

(iii) The consumer always pays the equilibrium price, therefore the consumer pays £58, an increase of £3 on the original equilibrium price with no tax, which was £55. This means that the consumer pays 50% of the tax. The producer receives the new equilibrium price, minus the tax, so the producer receives £58 − £6 = £52, a reduction of £3 on the original equilibrium price of £55. This also means that the producer pays 50% of the tax.

In this example, the tax is evenly distributed between the consumer and producer. The reason for the 50 : 50 distribution is due to the fact that the slope of the demand function is equal to the slope of the supply function (ignoring signs). This suggests that changes in the slope of either the demand or supply functions will alter this distribution.

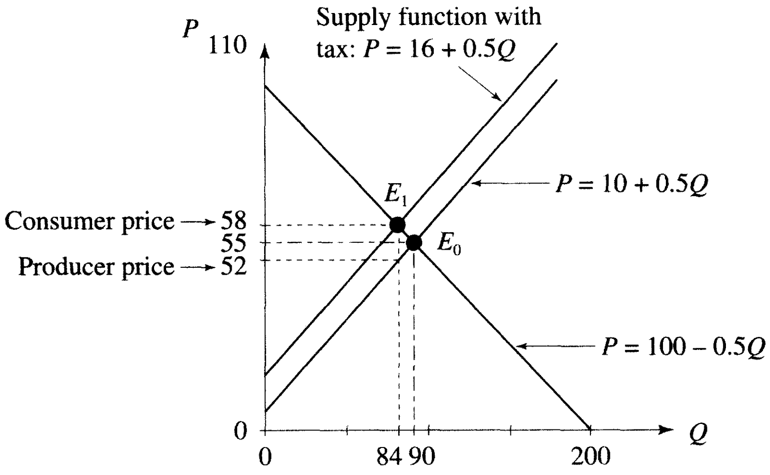


Figure 3.8 Goods market equilibrium and taxes

❐ Subsidies and their distribution

Similar ideas may be analysed with respect to subsidies and their distribution. In the case of subsidies, one would be interested in analysing how the **benefit** of the subsidy is distributed between the producer and consumer.

In the analysis of subsidies, a number of important points need to be highlighted:

- A subsidy per unit sold will translate the supply function vertically downwards, that is, the price received by the producer is (P + subsidy).
- The equilibrium price will decrease (the consumer pays the new lower equilibrium price). The price that the producer receives is the new equilibrium price **plus** the subsidy.
- The equilibrium quantity increases.

WORKED EXAMPLE 3.13

SUBSIDIES AND THEIR DISTRIBUTION

The demand and supply functions for a good (£P per ton of potatoes) are given as

$$\text{Demand function: } P_d = 450 - 2Q_d \qquad \textbf{(3.20)}$$

$$\text{Supply function: } P_s = 100 + 5Q_s \qquad \textbf{(3.21)}$$

(a) Calculate the equilibrium price and quantity.

(b) The government provides a subsidy of £70 per unit (ton) sold:

(i) Write down the equation of the supply function, adjusted for the subsidy.

(ii) Find the new equilibrium price and quantity algebraically and graphically.

(iii) Outline the distribution of the subsidy, that is, calculate how much of the subsidy is received by the consumer and the supplier.

Solution

(a) The solution to this part is given over to the reader. Show that the equilibrium quantity and price are 50 units and £350 respectively.

(b) (i) With a subsidy of £70 per unit sold, the producer receives $(P_s + 70)$. The equation of the supply function adjusted for subsidy is

$$P_s + 70 = 100 + 5Q$$

$$P_s = 30 + 5Q \qquad \textbf{(3.22)}$$

The supply function is translated vertically downwards by 70 units. This is illustrated in Figure 3.9 as a line parallel to the original supply function.

(ii) The new equilibrium price and quantity are calculated by equating the original demand function, equation (3.20), and the supply function adjusted for the subsidy, equation (3.22):

$$P_d = (P_s + \text{subsidy})$$

$$450 - 2Q = 30 + 5Q \qquad \text{equating equations (3.20) and (3.22)}$$

$$Q = 60$$

Substitute the new equilibrium quantity $Q = 60$ into either equation (3.20) or equation (3.22) and solve for the new equilibrium price:

$$P = 450 - 2Q$$

$$P = 450 - 2(60) \qquad \text{substituting } Q = 60 \text{ into equation (3.20)}$$

$$P = 330$$

The point $(P = 330,\ Q = 60)$ is shown as point E_1 in Figure 3.9.

(iii) The consumer always pays the equilibrium price, therefore, the consumer pays £330, a decrease of £20 on the equilibrium price with no subsidy (£350). This means that the consumer receives 20/70 of the subsidy. The producer

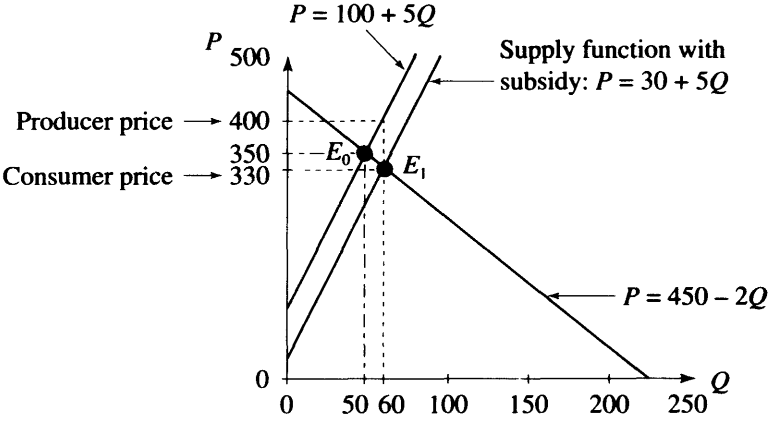


Figure 3.9 Goods market equilibrium and subsidies

receives the equilibrium price, plus the subsidy, so the producer receives £330 + £70 = £400, an increase of £50 on the original price of £350. The producer receives 50/70 of the subsidy.

In this case, the subsidy is not evenly distributed between the consumer and producer; the producer receives a greater fraction of the subsidy than the consumer. The reason? The slope of the supply function is greater than the slope of the demand function (ignoring signs).

❐ Distribution of taxes/subsidies

The fraction of the tax/subsidy that the consumer pays/receives is given by the equation

$$\frac{|m_d|}{|m_d| + |m_s|}$$

The fraction of the tax/subsidy that the producer pays/receives is given by the equation

$$\frac{|m_s|}{|m_s| + |m_d|}$$

where m_d and m_s are the slopes of the demand and supply functions respectively. See appendix to this chapter for proof of these formulae.

3.2.5 Break-even analysis

The break-even point for a good occurs when total revenue is equal to total cost.

Break-even point: total revenue = total cost

WORKED EXAMPLE 3.14

CALCULATING THE BREAK-EVEN POINT

The total revenue and total cost functions are given as follows:

$$TR = 3Q \tag{3.23}$$

$$TC = 10 + 2Q \tag{3.24}$$

(a) Calculate the equilibrium quantity algebraically and graphically at the break-even point.

(b) Calculate the value of total revenue and total cost at the break-even point.

Solution

(a) The break-even point is algebraically solved by equating total revenue, equation (3.23), and total cost, equation (3.24):

$$3Q = 10 + 2Q$$
$$Q = 10$$

The equilibrium quantity at the break-even point is $Q = 10$. This is illustrated in Figure 3.10.

(b) The value of total revenue and total cost at the break-even point is calculated by substituting $Q = 10$ into the respective revenue and cost functions:

$$TR = 3Q = 3(10) = 30$$
$$TC = 10 + 2Q = 10 + 2(10) = 30$$

At $Q = 10$, $TR = TC = 30$.

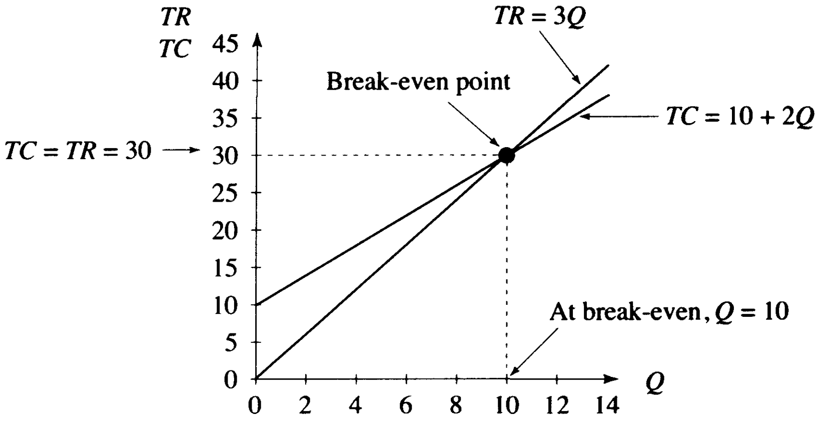


Figure 3.10 Break-even point

Progress Exercises 3.3 Equilibrium, Break-even

1. The following demand and supply functions for a safari holiday package are:

Demand function: $Q = 81 - 0.05P$

Supply function: $Q = -24 + 0.025P$

(a) Calculate the equilibrium price and quantity, algebraically and graphically.

(b) Graph the supply and demand function, showing the equilibrium.

2. (See question 1)

The government imposes a tax of £120.

(a) Write down the equation of the supply function adjusted for tax, hence graph it on the diagram in 1(b).

(b) Calculate the equilibrium price and quantity when the tax is imposed.
(c) Outline the distribution of the tax, i.e. calculate the tax paid by the consumer and by the travel agent.

3. The demand and supply functions for two complementary products X and Y: pitching wedges and putters (for pitch and putt), respectively are given as:

$$Q_{dX} = 190 - 2P_X - 2P_Y \qquad Q_{dY} = 240 - 2P_X - 4P_Y$$

$$Q_{sX} = -10 + 2P_X \qquad Q_{sY} = -40 + P_Y$$

Find the equilibrium price and quantity for each good.

4. The demand and supply functions for golf lessons at Greens Club are:

$$\text{Demand function:} \quad P = 200 - 5Q$$

$$\text{Supply function:} \quad P = 92 + 4Q$$

(a) Calculate the equilibrium price and quantity algebraically and graphically.
(b) The government imposes a tax of £9 per lesson:
   (i) Write down the equation of the supply function adjusted for tax, hence graph it on the same diagram as in part (a).
   (ii) Calculate the equilibrium price and quantity when the tax is imposed.
   (iii) Outline the distribution of the tax, i.e. how much of the tax is paid by the customer and the club (supplier).

5. The demand function for a perfectly competitive firm (same price charged for each good) is given as $P = 30$. The firm has fixed costs of £200 and variable costs of £5 per unit sold.
(a) Calculate the equilibrium quantity at the break-even point.
(b) Calculate the value of total revenue and total cost at the break-even point.

6. The demand and supply functions for free-range Christmas turkeys are given by the equations:

$$P_d = 80 - 0.4Q_d \quad \text{and} \quad P_s = 20 + 0.4Q_s$$

(a) Calculate the equilibrium price and quantity.
(b) If the government provides a subsidy of £4 per bird:
   (i) Rewrite the equation of the supply function to include the subsidy.
   (ii) Calculate the new equilibrium price and quantity.
   (iii) Outline the distribution of the subsidy, i.e. how much of the subsidy is received by the customer and by the supplier.

7. A firm, which makes travel alarm clocks has a total cost function $TC = 800 + 0.2Q$.
(a) If the price of the clock is £6.6, write down the equation of the total revenue function. Calculate the number of clocks which must be made and sold to break-even.
(b) When the firm charges a price P for each alarm clock the break-even point is $Q = 160$. Write down the equation for break-even, hence calculate the price charged per clock.
(c) Graph the total revenue functions (a) and (b) with the total cost on the same diagram, showing each break-even point.

8. The supply and demand functions for complementary goods (jeans and shirts) are given by the equations:

$$P_{dX} = 100 - 5Q_X - Q_Y \qquad P_{dY} = 240 - 10Q_X - 8Q_Y$$

$$P_{sX} = 50 + Q_X \qquad P_{sY} = 40 + 2Q_Y$$

Find the equilibrium price and quantity for each good.

3.3 Consumer and Producer Surplus

At the end of this section you should be familiar with:

- The meaning of consumer and producer surplus
- How to measure consumer and producer surplus.

3.3.1 Consumer and producer surplus

❐ Consumer surplus (*CS*)

This is the difference between the expenditure a consumer is willing to make on successive units of a good from $Q = 0$ to $Q = Q_0$ and the actual amount spent on Q_0 units of the good at the market price of P_0 per unit. To explain how consumer surplus is calculated geometrically consider the demand function, $P = 100 - 0.5Q$ which is graphed in Figure 3.11.

To calculate consumer surplus when the market price is 55, proceed as follows. When the market price per unit is 55 the consumer will purchase 90 units, since, according to the demand function, $55 = 100 - 0.5Q \rightarrow 0.5Q = 100 - 55 = 45 \rightarrow Q = 90$. The consumer, therefore, spends a total of $P \times Q = (55)(90) = £4950$. This is equivalent to the area of the rectangle $0P_0E_0Q_0$ (since area = length × breath = $P \times Q$).

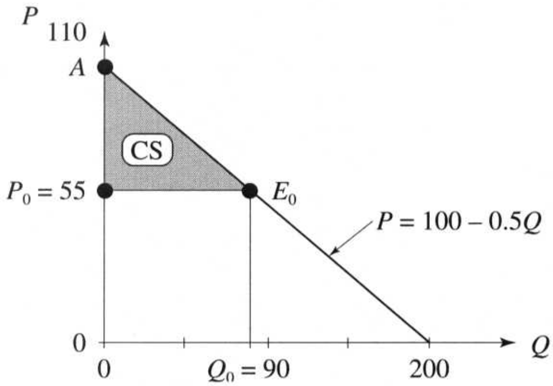


Figure 3.11 Consumer surplus

The consumer pays the same price, 55, for each of the 90 units purchased; however, he or she is willing to pay more than 55 for each of the units preceding the 90th when the good was scarcer, $Q < 90$. (These higher prices, which the consumer is willing to pay, may be calculated from the demand function.) The total amount which the consumer is willing to pay for the first 90 units is given by the area under the demand function between $Q = 0$ and $Q = 90$, that is, area $0AE_0Q_0$.

Since consumer surplus is the difference between the amount that the consumer is willing to pay and the amount that the consumer actually pays, then

$$\begin{aligned} CS &= 0AE_0Q_0 - 0P_0E_0Q_0 = AP_0E_0 \\ &= 0.5(90)(55) = 2475 \quad \text{(See Area of triangles below)} \end{aligned}$$

Area AP_0E_0 is a benefit to the consumer as the amount which the consumer is willing to pay exceeds the amount which is actually paid.

❒ Producer surplus (*PS*)

This is the difference between the revenue the producer receives for Q_0 units of a good when the market price is P_0 per unit and the revenue that the producer was willing to accept for successive units of the good from $Q = 0$ to $Q = Q_0$. To explain how producer surplus is calculated geometrically consider the supply function, $P = 10 + 0.5Q$ which is graphed in Figure 3.12.

To calculate producer surplus when the market price is 55 per unit, proceed as follows. When the market price per unit is 55 the producer will supply 90 units since

$$P = 10 + 0.5Q \quad \rightarrow \quad 55 = 10 + 0.5Q \quad \rightarrow \quad 45 = 0.5Q \quad \rightarrow \quad 90 = Q$$

So, the producer receives 55 for each of the 90 units, giving a total revenue of $P \times Q = (55)(90) = £4950$. This is equivalent to the area of rectangle $0P_0E_0Q_0$.

The producer, however, is willing to supply each unit, up to the 90th unit, at prices less than 55. (These lower prices may be calculated from the equation of the supply function.) The revenue the producer is willing to accept for units below the ninetieth unit is given by the area under the supply function between $Q = 0$ and $Q = 90$, that is, area $0BE_0Q_0$.

Since producer surplus is the difference between the revenue the producer receives at $Q = 90$ and the revenue that the producer was willing to accept for units supplied up to

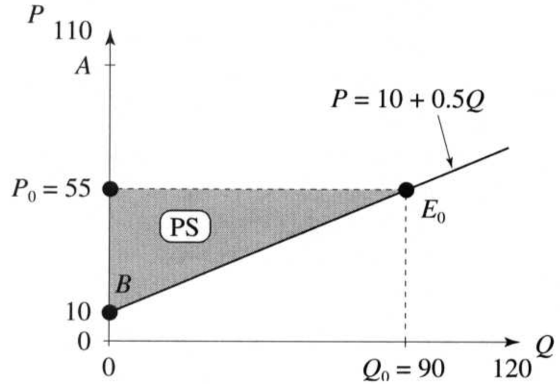


Figure 3.12 Producer surplus

the ninetieth, then

$$PS = 0P_0E_0Q_0 - 0BE_0Q_0 = BP_0E_0$$
$$= 0.5(90)(45) = 2025 \quad \text{(See Area of triangles below)}$$

Therefore, area BP_0E_0 is a benefit to the producer.

3. The demand and supply functions for a product (helicopter rides) are given by:

$$\text{Demand function:} \quad Q = 50 - 0.1P$$
$$\text{Supply function:} \quad Q = -10 + 0.1P$$

   (a) Calculate the equilibrium price and quantity. Plot the demand and supply functions in the form $P = g(Q)$. Illustrate graphically the consumer and producer surplus at equilibrium.
   (b) Calculate the consumer surplus at equilibrium.
   (c) Calculate the producer surplus at equilibrium.
   (d) Calculate the total surplus at equilibrium.

4. (See question 3) The price per helicopter ride decreases to £250.
   (a) Calculate the number of helicopter rides demanded at the reduced price £250.
   (b) Calculate the consumer surplus at £250, graphically illustrating your answer. What is the change in *CS* as a result of the price reduction?

5. (See question 3) The price per helicopter ride decreases to £250.
   (a) Calculate the number of helicopter rides supplied at the reduced price £250.
   (b) Calculate the producer surplus at £250, graphically illustrating your answer. What is the change in *PS* as a result of the price reduction?

3.4 The National Income Model and the *IS-LM* Model

At the end of this section you should be familiar with:

- The national income model: national income equilibrium and expenditure multipliers.
- The *IS-LM* model: determination of equilibrium national income and interest rates.

3.4.1 National income model

National income, Y, is the total income generated within an economy from all productive activity over a given period of time, usually one year. Equilibrium national income occurs when aggregate national income, Y, is equal to aggregate planned expenditure, E, that is,

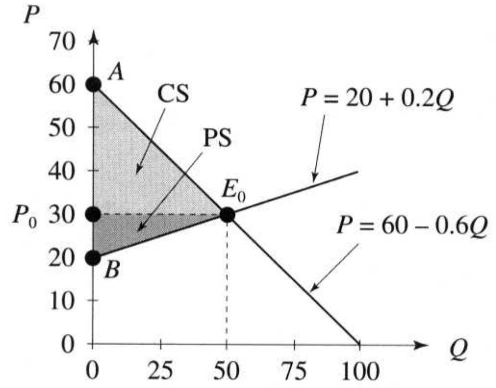


Figure 3.14 Consumer and producer surplus

(b) Consumer and producer surplus at market equilibrium are calculated as follows:

At $P = 30$, $Q = 50$ $\quad$ CS = triangle $AP_0E_0 = 0.5 \times 50 \times 30 = 750$

At $P = 30$, $Q = 50$ $\quad$ PS = triangle $BP_0E_0 = 0.5 \times 50 \times 10 = 250$

(c) Total surplus is the sum of consumer and producer surplus; therefore

$$TS = CS + PS = 750 + 250 = 1000$$

Progress Exercises 3.4 CS/PS for Linear Functions

1. (a) Define (i) consumer surplus, (ii) producer surplus, (iii) total surplus at the equilibrium price P_0. Illustrate all three surpluses graphically.
   (b) Define (i) consumer surplus, (ii) producer surplus, (iii) total surplus at price P_A, which is below the equilibrium price P_0. Illustrate all three surpluses graphically.
2. The demand and supply functions for seats on a certain weekend bus route are given by:

$$\text{Demand function:} \quad P = 58 - 0.2Q$$

$$\text{Supply function:} \quad P = 4 + 0.1Q$$

(a) Calculate the equilibrium price and quantity. Plot the demand and supply functions and illustrate consumer and producer surplus at equilibrium.

(b) Calculate:

(i) The amount consumers pay for bus journeys at equilibrium.
(ii) The amount consumers are willing to pay for bus journeys up to equilibrium.
(iii) The consumer surplus (CS); hence show that the CS = (ii) − (i).

(c) Calculate:

(i) The amount the producer (bus company) receives for bus journeys at equilibrium.
(ii) The amount the producer is willing to accept for bus journeys up to equilibrium.
(iii) The producer surplus (PS); hence show that the PS = (i) − (ii).

3. The demand and supply functions for a product (helicopter rides) are given by:

$$\text{Demand function:} \quad Q = 50 - 0.1P$$
$$\text{Supply function:} \quad Q = -10 + 0.1P$$

   (a) Calculate the equilibrium price and quantity. Plot the demand and supply functions in the form $P = g(Q)$. Illustrate graphically the consumer and producer surplus at equilibrium.
   (b) Calculate the consumer surplus at equilibrium.
   (c) Calculate the producer surplus at equilibrium.
   (d) Calculate the total surplus at equilibrium.
4. (See question 3) The price per helicopter ride decreases to £250.
   (a) Calculate the number of helicopter rides demanded at the reduced price £250.
   (b) Calculate the consumer surplus at £250, graphically illustrating your answer. What is the change in CS as a result of the price reduction?
5. (See question 3) The price per helicopter ride decreases to £250.
   (a) Calculate the number of helicopter rides supplied at the reduced price £250.
   (b) Calculate the producer surplus at £250, graphically illustrating your answer. What is the change in PS as a result of the price reduction?

3.4 The National Income Model and the *IS-LM* Model

At the end of this section you should be familiar with:

- The national income model: national income equilibrium and expenditure multipliers.
- The *IS-LM* model: determination of equilibrium national income and interest rates.

3.4.1 National income model

National income, Y, is the total income generated within an economy from all productive activity over a given period of time, usually one year. Equilibrium national income occurs when aggregate national income, Y, is equal to aggregate planned expenditure, E, that is,

$$Y = E \tag{3.25}$$

Note: In the discussion which follows it is assumed that all expenditure is planned expenditure.

Aggregate expenditure, E, is the sum of households' consumption expenditure, C; firms' investment expenditure, I; government expenditure, G; foreign expenditure on domestic exports, X **minus** domestic expenditure on imports, M, that is,

$$E = C + I + G + X - M \tag{3.26}$$

Note: Expenditure on imports is income lost to the economy, hence the minus sign.

Therefore, substituting equation (3.26) into equation (3.25) gives the equation for equilibrium national income:

$$Y = C + I + G + X - M \tag{3.27}$$

That is, equilibrium national income exists when total income is equal to total expenditure.

Note: Aggregate expenditure on goods and services (E) is one method of measuring national income. Alternatively, national income may be measured by aggregating total income received by firms and individuals (total income) or aggregating total production (total output). For the purposes of this text, these differences in measuring national income may be ignored.

❐ Steps for deriving the equilibrium level of national income

Step 1: Express expenditure in terms of income, Y: $E = \mathrm{f}(Y)$
Step 2: Substitute expenditure, expressed as a function of Y, into the RHS of the equilibrium condition, $Y = E$. Solve the equilibrium equation for the equilibrium level of national income, Y_e.

Graphical solution: The point of intersection of the equilibrium condition, $Y = E$ (the 45° line through the origin), and the expenditure equation, $E = C + I + G + X - M$ gives the equilibrium level of national income.

Note: When graphing the national income equations, Y is plotted on the horizontal axis. As a reminder; 'E on the vertical, Y on the horizontal', all equations shall be written in the form $E = \mathrm{f}(Y)$.

❐ Equilibrium level of national income when $E = C + I$

Initially, the model assumes the existence of only two economic agents, households and firms operating in a closed economy (no foreign sector) with no government sector and no inflation. Households' consumption expenditure, C, is modelled by the equation, $C = C_0 + bY$, where C_0 is autonomous consumption, that is, consumption which does not depend on income. $b(0 < b < 1)$ is called the marginal propensity to consume. $b = MPC = \Delta C/\Delta Y$ measures the change in consumption per unit increase in income. Firm's investment expenditure is autonomous, $I = I_0$.

WORKED EXAMPLE 3.16

EQUILIBRIUM NATIONAL INCOME WHEN $E = C + I$

In a two-sector economy, autonomous consumption expenditure, $C_0 = £50$m, autonomous investment expenditure, $I_0 = £100$m and $b = 0.5$.
(a) Determine (i) the equilibrium level of national income, Y_e, and (ii) the equilibrium level of consumption, C_e, algebraically.
(b) Plot the consumption function, $C = C_0 + bY$, the expenditure function, $E = C + I_0$ and the equilibrium condition, $Y = E$ on the same diagram.

Hence, determine the equilibrium level of national income, Y_e, and the equilibrium level of consumption, C_e.

(c) Given that $Y = C + S$, determine the equilibrium level of savings. Plot the savings function. Plot the investment function on the same diagram. Comment.

Solution

(a) (i) **Step 1:** Households' consumption expenditure and firms' investment expenditure are the only components of aggregate expenditure, therefore

$$E = C + I = C_0 + bY + I_0 = 50 + 0.5Y + 100 = 150 + 0.5Y$$

Step 2: At equilibrium, $Y = E$ (equation 3.25), therefore,

Example:	In general:	
$Y = 150 + 0.5Y$	$Y = (C_0 + bY) + I_0$	
$Y - 0.5Y = 150$	$Y - bY = C_0 + I_0$	
$0.5Y = 150$	$Y(1 - b) = C_0 + I_0$	
$Y = \dfrac{150}{0.5}$	$Y = \dfrac{C_0 + I_0}{1 - b}$	
$Y_e = \dfrac{1}{0.5}150 = 300$	$Y_e = \dfrac{1}{1 - b}(C_0 + I_0)$	**(3.28)**

The equilibrium level of national income $Y_e = 300$.

(ii) When the equilibrium level of income has been found, the equilibrium level of consumption is calculated directly from the consumption function,

$$\begin{aligned} C_e &= C_0 + bY_e \\ &= 50 + 0.5(300) = 50 + 150 = 200 \end{aligned} \qquad \textbf{(3.29)}$$

(b) The consumption function $C = 50 + 0.5Y$, the expenditure function, $E = 150 + 0.5Y$ and the equilibrium condition $Y = E$ are plotted in Figure 3.15 with E plotted vertically and Y plotted horizontally. The equilibrium condition $Y = E$, is represented by a 45° line from the origin, provided the number scale is the same on both axes. Graphically, equilibrium national income Y_e is illustrated in Figure 3.15(a) with the equilibrium point occurring at the point of intersection of the expenditure equation and the line $Y = E$. The point of intersection is at $Y = 300 = E$. Graphically, the equilibrium level of consumption, C_e, is at the point of intersection of the consumption function and the vertical line $Y_e = 300$.

(c) Since $C_e = 200$, the equilibrium level of savings $S_e = Y_e - C_e = 300 - 200 = 100$. The savings function $S = -C_0 + (1 - b)Y = -50 + 0.5Y$ is plotted in Figure 3.15(b). Graphically, investment expenditure is illustrated by a horizontal line in Figure 3.15(b). Notice that at equilibrium national income,

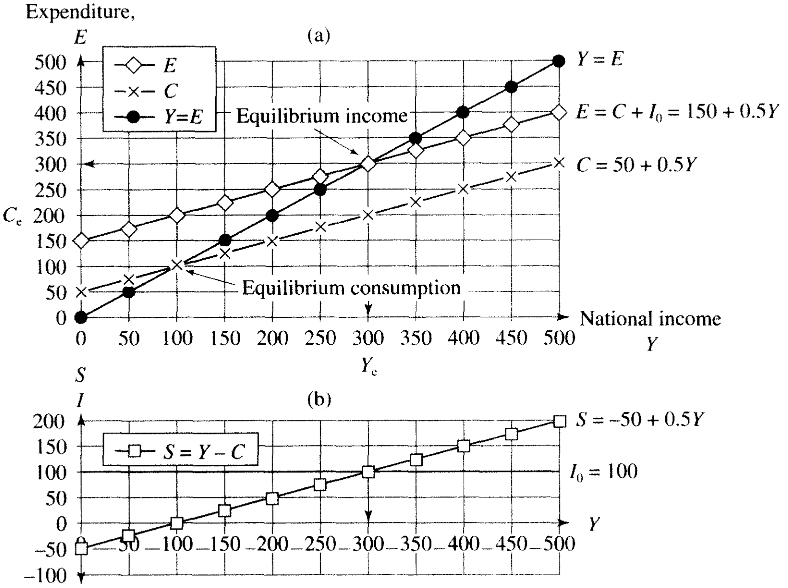


Figure 3.15 Equilibrium national income with consumption and investment

savings is equal to investment,

$$S_e = I$$

Therefore, in this example, one can also say that equilibrium national income occurs when savings (leakages) is equal to investment (injections).

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 3.

❑ Expenditure multiplier

Table 3.1 Relationship between the expenditure multiplier and MPC, b

Worked Example 3.17 Effect of changes in MPC and I_0 on Y_e

❑ Government expenditure and taxation: $E = C + I + G$

❑ Expenditure multiplier with taxes

Worked Example 3.18 Equilibrium national income and effect of taxation

Figure 3.16 Equilibrium national income and effect of taxation

❑ Foreign trade

Worked Example 3.19 Expenditure multiplier with imports and trade balance

☐ Table 3.2 Summary of national income model

Progress Exercises 3.5, questions 5, 6, 7

Solutions to questions 5, 6, 7

Progress Exercises 3.5 The National Income Model

1. Given the condition for equilibrium national income $Y = E$, and the expenditure equation, $E = C$ where $C = C_0 + bY$:
   (a) Describe the constants C_0 and b. Are there any restrictions on the range of values which b can assume?
   (b) Find an expression for the equilibrium level of income (the reduced form).
   (c) Deduce how the equilibrium level of income changes as:
      (i) b increases
      (ii) b decreases.
2. Given the national income model $Y = E$; $E = C + I$ where $C = 280 + 0.6Y$, $I_0 = 80$:
   (a) Write down the value of the intercept and slope of the expenditure equation.
   (b) Graph the equilibrium equation and the expenditure equation on the same diagram; hence determine the equilibrium level of income Y_e (i) graphically, (ii) algebraically.
   (c) How will the equilibrium level of income change if the marginal propensity to consume increases to 0.9?
3. Given the national income model $Y = E$; $E = C + I$; where $C = 280 + 0.6Y_d$, $I_0 = 80$, $T = 0.2Y$ (that is, $t = 0.2$):
   (a) Write down the value of the intercept and slope of the expenditure equation.
   (b) Graph the equilibrium equation and the expenditure equation on the same diagram; hence, determine the equilibrium level of income Y_e (i) graphically, (ii) algebraically.
   (c) If the marginal propensity to consume increases to 0.9, how will
      (i) The expenditure equation change?
      (ii) The equilibrium level of income change?
4. Assuming that the initial level of national income is £800m, calculate the new level of national income when $b = 0.6$ and $\Delta I = £500$m.

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 3.

3.4.2 *IS-LM* model: determination of equilibrium national income and interest rates

☐ *IS* schedule

☐ *LM* schedule

☐ Equilibrium national income and equilibrium interest rates

Worked Example 3.20 *IS-LM* analysis

Figure 3.17 Equilibrium income and interest rates

Progress Exercises 3.6 *IS-LM* Analysis

Solutions to Progress Exercises 3.6

3.5 Excel for Simultaneous Linear Equations

If you found Excel useful for plotting graphs in Chapter 2 you should find it equally useful in this chapter as the same skills are required in graph plotting. However, since this chapter is mainly concerned with finding solutions to simultaneous equations, you will need to show the solution, that is, the point of intersection of the lines on the graph. To ensure that the solution is shown on the graph, solve the equations algebraically first. Once the solution is known, the axis may then be scaled by choosing a range of x-values which includes the x-value of the solution.

Alternatively, if the equations are not easy to solve algebraically then a table of y-values calculated for each of the simultaneous equations $y_1 = f(x)$, $y_2 = g(x)$ may be used to detect whether the solution $(y_1 = y_2)$ is included in the table. If $y_1 > y_2$ at the start of the table, the first function is greater than the second; if $y_1 < y_2$ at the end of the table, the first function is less than the second, or vice versa, then the solution, $y_1 = y_2$, is included.

WORKED EXAMPLE 3.21

COST, REVENUE, BREAK-EVEN, PER UNIT TAX WITH EXCEL

A firm receives £2.5 per unit for a particular good. The fixed costs incurred are £44 while each unit produced costs £1.4.

(a) Write down the equations for (i) total revenue and (ii) total cost.

(b) Calculate the break-even point algebraically.

(c) If the government imposes a tax of £0.70 per unit, recalculate the break-even point. Show the graphical solutions to parts (b) and (c) on the same diagram (using Excel).

Solution

(a) (i) $TR = P \times Q = 2.5Q$

(ii) $TC = FC + VC = 44 + 1.4Q$

(b) Break-even occurs when $TR = TC$

- Algebraically:

$$2.5Q = 44 + 1.4Q$$
$$1.1Q = 44$$
$$Q = 40$$

When $Q = 40$, then $TR = TC = 100$.

(c) If a tax per unit is imposed, either the total revenue function or the total cost function may be adjusted for the tax as follows:

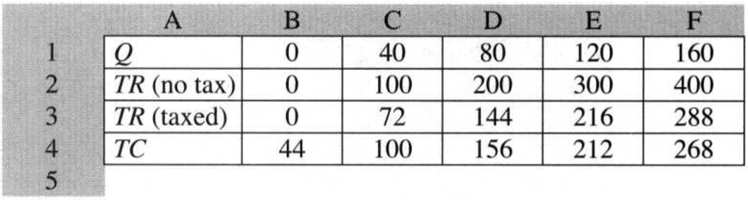
- Algebraically:

The net revenue per unit is (price – tax): $TR = (2.5 - 0.7)Q = 1.8Q$

Break-even is at $TR = TC \rightarrow 1.8Q = 44 + 1.4Q \rightarrow Q = 110$

- Graphically:

To show the break-even points on a graph, choose values of Q such as $Q = 0$ to $Q = 160$. Set up the table of points in Excel and plot the graph as shown in Figure 3.18. (Since the graph is a straight line, a minimum of two points is required.)



	A	B	C	D	E	F
1	*Q*	0	40	80	120	160
2	*TR* (no tax)	0	100	200	300	400
3	*TR* (taxed)	0	72	144	216	288
4	*TC*	44	100	156	212	268
5						

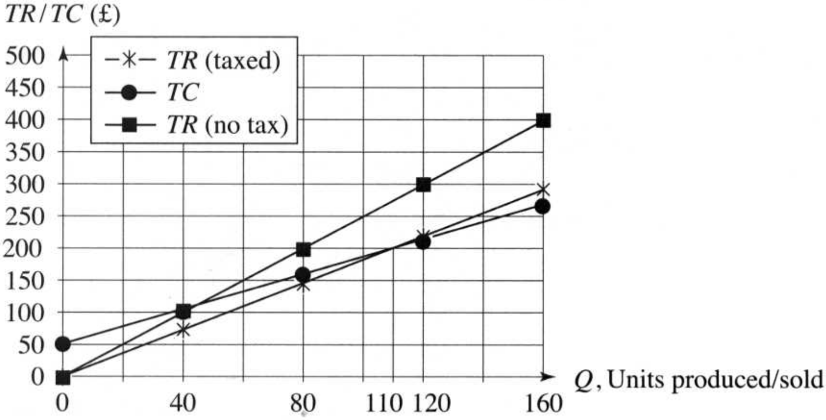


Figure 3.18 Break-even with tax and no tax

WORKED EXAMPLE 3.22

DISTRIBUTION OF TAX WITH EXCEL

(a) Show graphically that the equilibrium price and quantity for the demand and supply functions given by the pairs of equations (i) and (ii) is the same.

$$\text{(i)}\ \begin{array}{l} P_d = 120 - 2Q \\ P_s = 10 + 2Q \end{array} \quad \text{and} \quad \text{(ii)}\ \begin{array}{l} P_d = 120 - 2Q \\ P_s = 37.5 + Q \end{array}$$

(b) If a tax of 20 is imposed on each unit produced, recalculate the equilibrium price for (i) and (ii) above, hence determine the distribution of the tax. Show the distribution of tax graphically.

(c) Can you deduce a general rule describing how the distribution of the tax changes according to the function with the flatter slope?

	A	B	C	D	E	F
1	Q	0	10	20	30	40
2	P_d	120	100	80	60	40
3	P_{s1}	10	30	50	70	90
4	P_{s2}	37.5	47.5	57.5	67.5	77.5

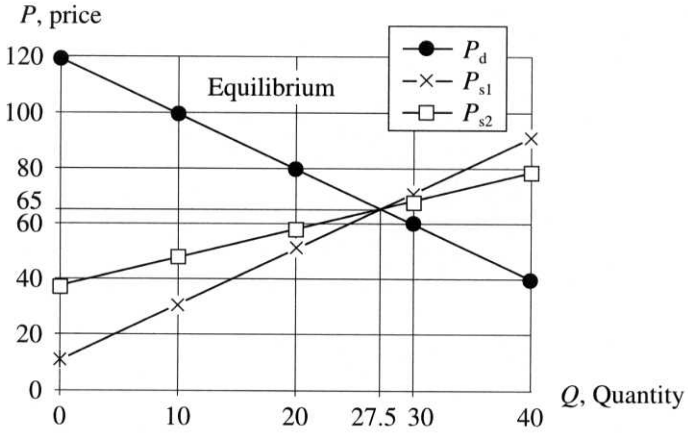


Figure 3.19 Market equilibrium

Solution

(a) The table of points and the graphs of equations (i) and (ii) are shown in Figure 3.19. The equilibrium point for each pair is the same, $Q = 27.5$, $P = 65$.
(b) When a tax of 20 is imposed, the price the supplier receives is the (original price − tax); therefore, replace P in the supply functions by $(P - 20)$.

(i) With the tax, the set of equations is now

$$P_d = 120 - 2Q, \quad \text{stays the same}$$

$$P_s - 20 = 10 + 2Q \quad \rightarrow \quad P_s = 30 + 2Q$$

Solve for equilibrium at $Q = 22.5$, $P = 75$.

The consumer (who always pays the equilibrium price) pays $(P_{e(\text{tax})} - P_e) = 75 - 65 = 10$ more than before the tax. The producer receives $(P_{e(\text{tax})} - \text{tax}) = 75 - 20 = 55$. This is 10 units less than before the tax was imposed. So when the slope of the demand and supply functions are equal the distribution of tax is 50 : 50.

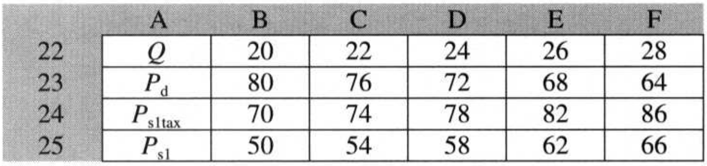
A table of values is set up to plot this pair of graphs with the untaxed supply function. The range of Q-values is selected so that the graph focuses on the original and the new equilibrium points as shown in Figure 3.20.

(ii) With the tax, the set of equations is now

$$P_d = 120 - 2Q$$

$$P_s - 20 = 37.5 + Q \quad \rightarrow \quad P_s = 57.5 + Q$$

Solve for equilibrium at $Q = 20.83$, $P = 78.33$.



	A	B	C	D	E	F
22	Q	20	22	24	26	28
23	P_d	80	76	72	68	64
24	P_{s1tax}	70	74	78	82	86
25	P_{s1}	50	54	58	62	66

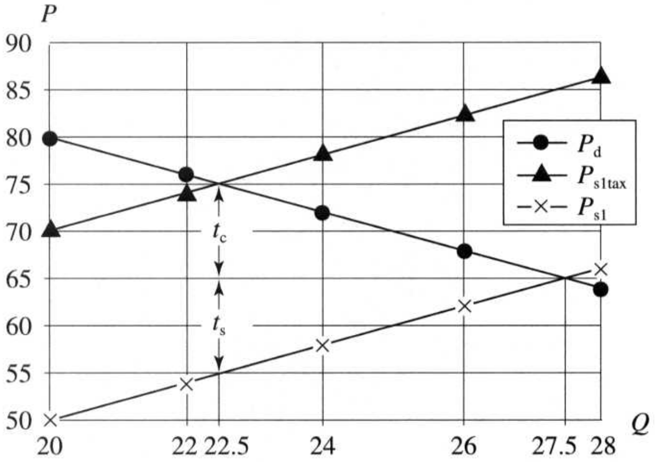
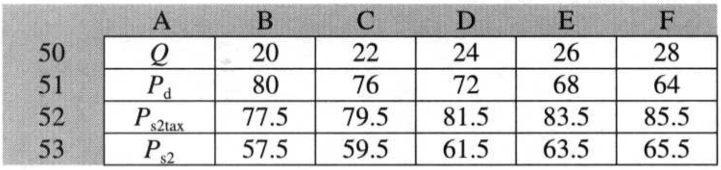


Figure 3.20 Distribution of tax for equations (i)



	A	B	C	D	E	F
50	Q	20	22	24	26	28
51	P_d	80	76	72	68	64
52	P_{s2tax}	77.5	79.5	81.5	83.5	85.5
53	P_{s2}	57.5	59.5	61.5	63.5	65.5

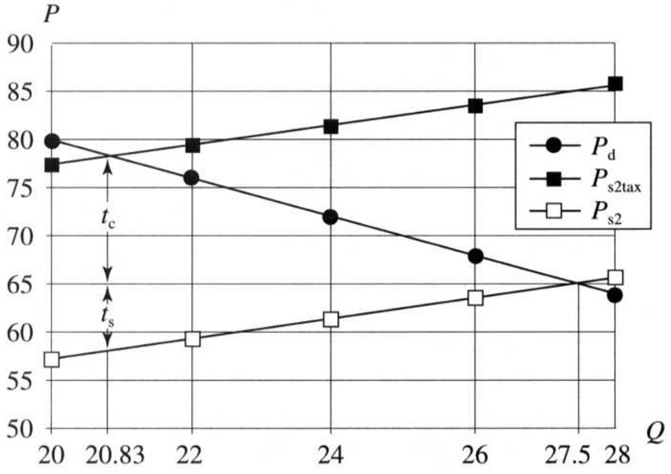


Figure 3.21 Distribution of tax for equations (ii)

The consumer pays $(P_{\mathrm{e(tax)}} - P_{\mathrm{e}}) = 78.33 - 65 = 13.33$ more than before the tax. The producer receives $(P_{\mathrm{e(tax)}} - \text{tax}) = 78.33 - 20 = 58.33$. This is 6.67 units less than before the tax was imposed. As expected, the function whose slope is greater in magnitude pays the greater share of the tax.

A table of values is set up to plot this pair of graphs with the untaxed supply function. The range of Q-values is selected so that the graph focuses on the original and the new equilibrium points as shown in Figure 3.21.
(c) ☺ **Remember:** The distribution of tax for linear functions is given as

$$\text{Consumer pays } \frac{|m_{\mathrm{d}}|}{|m_{\mathrm{d}}| + |m_{\mathrm{s}}|} \times \text{tax} \qquad \text{Producer pays } \frac{|m_{\mathrm{s}}|}{|m_{\mathrm{s}}| + |m_{\mathrm{d}}|} \times \text{tax}$$

See Appendix to chapter. The distribution of tax is graphically illustrated in Figures 3.20 and 3.21.

Progress Exercises 3.7 Excel or Otherwise

1. Solve the following simultaneous equations (i) algebraically, (ii) graphically with Excel.

(a) $\begin{array}{l} x + y - 3 = 0 \\ 2x - y = 15 \end{array}$ (b) $\begin{array}{l} 3.8P - 0.75Q = 12 \\ P = 5Q - 6 \end{array}$

2. Set up a table in Excel to find the approximate solution to the following equations:

$P - 8Q = 120$
$3Q - 1.5P + 270 = 0$

Hence, determine the solution
(a) algebraically (b) graphically

3. Given the demand and supply functions,
$P = 124 - 4.5Q$ and $Q = -16.5 + 0.5P$, respectively.
(a) Find the equilibrium point (i) graphically, (ii) algebraically.
(b) If a tax of 30 per unit is imposed calculate
(i) the equilibrium point, (ii) the distribution of tax.
Show the distribution of tax graphically.

4. On the same diagram plot
(a) (i) the 45° line $Y = E$, (ii) $E = C + I$, where $C = 125 + 0.65Y$ and $I = 20$. (b) (i) the 45° line $Y = E$, (ii) $E = C + I$, where $C = 125 + 0.65Y_{\mathrm{d}}$, $I = 20$, $T = 0.2Y$. Hence, determine the equilibrium level of income (i) graphically, (ii) algebraically.

3.6 Summary

❐ Mathematics

The solution of a set of simultaneous equations is the value of x and y which satisfies all equations.

(a) To solve the equations algebraically, eliminate all but one variable, solve for this one variable, then solve for the other(s).
(b) To solve the equations graphically, plot the graphs. The solution is given by the coordinates of the point of intersection of the graphs.

Simultaneous equations may have:

(i) A unique solution
(ii) No solution
(iii) Infinitely many solutions

❐ Applications

- **Goods market equilibrium:** $Q_d = Q_s$ and $P_d = P_s$
- **Labour demand:** $w_d = a - bL$: a negative relationship between the number of labour units and the wage rate (price per unit).
- **Labour supply:** $w_s = c + dL$: a positive relationship between the number of labour units and the wage rate (price per unit).
- **Labour market equilibrium:** $L_d = L_s$ and $w_d = w_s$

  Revise price ceilings, black market profits, price floors.
- **Equilibrium for complementary and substitute goods**

  when $Q_{dX} = Q_{sX}$ and $Q_{dY} = Q_{sY}$: $P_{dX} = P_{sX}$ and $P_{dY} = P_{sY}$: X and Y are substitutes → $Q_X = a - bP_X + dP_Y$; X and Y are complements → $Q_X = a - bP_X - dP_Y$
- **Tax and its distribution between consumer and producer:**

  When a tax per unit is imposed the price to the supplier is (P – tax)

  $$P_s = c + dQ \quad \rightarrow \quad P_s - \text{tax} = c + dQ.$$

  The fraction of tax paid by the consumer and producer respectively:

  $$\frac{|m_d|}{|m_d| + |m_s|} \text{ (consumer)}, \ \frac{|m_s|}{|m_s| + |m_d|} \text{ (producer)}$$

- **Break-even**: $TR = TC$
- **Consumer surplus (*CS*)** is the difference between the expenditure a consumer is willing to make on successive units of a good from $Q = 0$ to $Q = Q_0$ and the actual amount spent on Q_0 units of the good at the market price P_0 per unit.
- **Producer surplus (*PS*)** is the difference between the revenue the producer receives for Q_0 units of a good when the market price is P_0 per unit and the revenue that the producer was willing to accept for successive units of the good from $Q = 0$ to $Q = Q_0$.

  Revise the effect of price increases and decreases on CS and PS.
- **National income model:**

  Equilibrium exists when income (Y) = expenditure (E)

  Expenditure may consist of (i) consumption: $C = C_0 + bY$, less tax, (ii) investment, I_0, (iii) government expenditure, G_0, (iv) exports, X_0, (v) less imports, $M = M_0 + mY$, hence $E = C + I + G + X - M$

  To find the level of income (Y_e) at which equilibrium exists, solve the equation $Y = E$ for Y. The solution is Y_e. Hence, equilibrium consumption and taxation: $C_e = C_0 + bY_e$: $T_e = tY_e$.

Revise the reduced expressions for Y_e, with multipliers. These are formulae from which T_e may be calculated directly for various standard national income models.

- ***IS/LM* model**

  Equilibrium in the goods market when $Y = E$ (national income), but consider investment as a function of interest rate: $I = I_0 - dr$, thus

$$Y = \frac{1}{1 - b(1 - t)} \cdot (C_0 + I_0 - dr + G_0)$$

hence the equation: $r = \mathrm{f}(Y)$. This is the *IS* schedule.
Equilibrium in the money market when money supply = money demand

Money demand: $M_d = M_d^T + M_d^P + M_d^S = L_1 + L_2 = kY + (a - hr)$
Money supply: $M_s = M_0$
Equilibrium, $M_s = M_d \rightarrow M_0 = kY + a - hr$

This equation may also be written as, $r = \mathrm{g}(Y)$. This is the *LM* schedule. The goods and money market are simultaneously in equilibrium for the values of r and Y which satisfy the simultaneous *IS* and *LM* equations: for example,

IS schedule: $r = 32 - 0.014Y$
LM schedule: $r = -2.0 + 0.0025Y$

- **Excel:** Useful as described in Chapter 2. In this chapter points of intersection and equilibrium points may be viewed. It should prove helpful for problems on the national income model to view the effect of various sectors on the equilibrium level of income.

Appendix

- **Distribution of tax paid by the consumer and producer**

Consumer pays $(P_{e2} - P_{e1})$of the tax: t_c
Supplier (producer) pays $(P_{e1} - P)$ of the tax: t_s

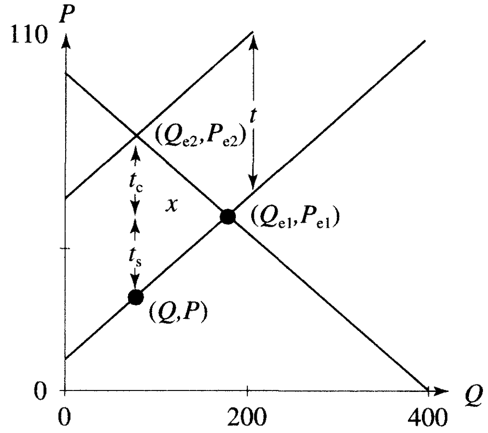


Figure 3.22 Distribution of tax

$$\left.\begin{array}{lll} |m_\mathrm{d}| = \dfrac{t_\mathrm{c}}{x} & \rightarrow & x = \dfrac{t_\mathrm{c}}{|m_\mathrm{d}|} \\[2ex] |m_\mathrm{s}| = \dfrac{t_\mathrm{s}}{x} & \rightarrow & x = \dfrac{t_\mathrm{s}}{|m_\mathrm{s}|} \end{array}\right\} \dfrac{t_\mathrm{c}}{|m_\mathrm{d}|} = \dfrac{t_\mathrm{s}}{|m_\mathrm{s}|}$$

Therefore

$$\begin{aligned} t_\mathrm{c}|m_\mathrm{s}| &= t_\mathrm{s}|m_\mathrm{d}| \\ t_\mathrm{c}|m_\mathrm{s}| &= (t - t_\mathrm{c})|m_\mathrm{d}| \qquad t_\mathrm{c} + t_\mathrm{s} = t \\ t_\mathrm{c}(|m_\mathrm{s}| + |m_\mathrm{d}|) &= t|m_\mathrm{d}| \\ t_\mathrm{c} &= t\frac{|m_\mathrm{d}|}{|m_\mathrm{s}| + |m_\mathrm{d}|} \qquad t_\mathrm{s} = t\frac{|m_\mathrm{s}|}{|m_\mathrm{s}| + |m_\mathrm{d}|} \end{aligned}$$

Test Exercises 3

1. Solve the following simultaneous equations (i) algebraically, (ii) graphically.

   (a) $\begin{aligned} 2x + y &= 12 \\ x - y &= 15 \end{aligned}$ (b) $\begin{aligned} 3.5P - Q &= 12 \\ P &= 5Q - 6 \end{aligned}$

2. Given the demand and supply functions, $P = 50 - 1.5Q$ and $Q = -11 + 0.5P$, respectively
   (a) Find the equilibrium point (i) graphically, (ii) algebraically
   (b) If a tax of 15.05 per unit is imposed calculate,
   (i) the equilibrium point, (ii) the distribution of tax.
   Show the distribution of tax graphically.

3. On the same diagram plot
   (a) the 45° line $Y = E$, $E = C + I$, where $C = 100 + 0.8Y$ and $I = 50$
   (b) the 45° line $Y = E$, $E = C + I$, where $C = 100 + 0.8Y_\mathrm{d}$, $I = 50$, $T = 0.2Y$
   Determine the equilibrium level of income for (a) and (b) (i) graphically, (ii) algebraically. Comment on the effect of taxation.

4. (See question 3) Determine the equilibrium levels of consumption and taxation (i) graphically, (ii) algebraically

Go to the web site www.wiley.co.uk/bradley2ed for Questions 5, 6 and 7.

8. Solve the simultaneous equations

$$x + y + z = 9: \qquad 2x - y + 4z = 19: \qquad 3x + 6y - z = 16$$

CHAPTER

4

Non-linear Functions and Applications

This chapter introduces the student to several types of non-linear functions which are frequently used in economics, management and business studies. At the end of this chapter you should be able to:

- Recognise the general form of the equation representing a non-linear function as well as the main characteristics of the graph representing the function
- Manipulate non-linear functions algebraically, particularly in economic applications such as demand, supply, revenue, cost and profit
- Use exponentials and logs in a range of applications, such as production and consumption
- Plot quadratic, cubic and other functions using Excel.

The chapter is divided into the following sections:

4.1 Quadratic, cubic and other polynomial functions

4.2 Exponential functions

4.3 Logarithmic functions

4.4 Hyperbolic functions of the form $a/(bx + c)$

4.5 Excel for non-linear functions

4.6 Summary

4.1 Quadratic, Cubic and Other Polynomial Functions

At the end of this section you should be able to:

- Solve quadratic equations
- Sketch any quadratic function
- Use quadratics in economics, for example total revenue and profit
- Recognise any cubic or other polynomial function
- Use cubic equations in economics, for example cost and break-even.

Linear functions, covered in the previous three chapters, are very good introductory models for demand, supply, cost, etc.; however, these models provide limited representations of real-life situations, hence the need for more versatile non-linear models. For example, the total revenue function in Worked Example 2.10 was given as $TR = 3.5Q$. This models a linear total revenue function representative of a perfectly competitive firm whose demand function is given as $P = 3.5$. On the other hand, if the firm is a monopolist and assuming that the demand function for the monopolist is given as $P = a - bQ$, for example $P = 50 - 2Q$, then total revenue is

$$TR = P \times Q$$

$$TR = (50 - 2Q)Q$$

$$TR = 50Q - 2Q^2 \tag{4.1}$$

Equation (4.1) is a quadratic function where TR is represented by an inverted U-shaped curve (see Figure 4.1).

Another example which demonstrates the limitations of linear models is the total cost function, $TC = 10 + 2Q$, as given in Worked Example 2.9. It is not reasonable to assume that costs rise by the same amount (2 units) for each extra unit of output produced, whether the extra unit is the 2nd or the 200th. It is more realistic to assume that after the initial outlay, the cost of producing an extra unit will eventually decrease, that is, total costs rise at a decreasing rate, and possibly at an increasing rate further on, when further outlay for expansion is required. This situation is best described by a cubic function, $TC = aQ^3 - bQ^2 + cQ + d$, ($a$, b, c and d are constants) as illustrated in Figure 4.2.

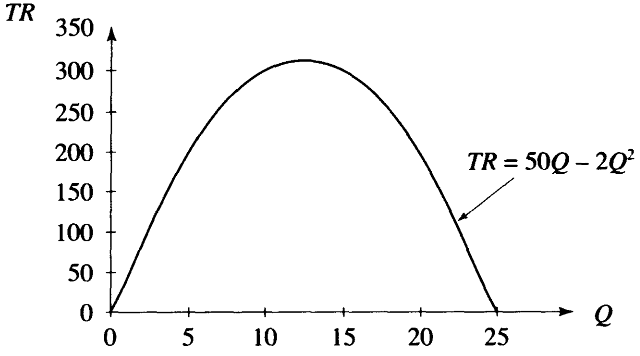


Figure 4.1 Non-linear total revenue function

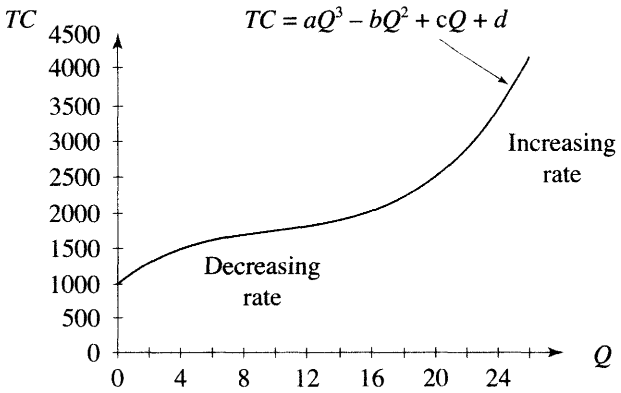


Figure 4.2 A cubic total cost function

Warning!

The following introduction to quadratic and cubic equations may appear to have little to do with economics or business. However, later in the text, particularly in Chapters 6 and 7, you will discover that the methods for finding maximum revenue, minimum costs, maximum profits, etc. often involve the solution of quadratic equations.

4.1.1 Solving quadratic equation

A quadratic equation has the general form

$$ax^2 + bx + c = 0$$

where a, b and c are constants. The solution of this equation may be found by using the formula (the '**minus** b' formula),

$$x = \frac{-(b) \pm \sqrt{(b)^2 - 4(a)(c)}}{2(a)} \tag{4.2}$$

Notice that a, b and c are all placed inside brackets, so that the values of a, b and c, with signs included, are substituted into the brackets. This should help avoid mistakes with negative signs.

☺ **Remember** that a solution of a quadratic equation (or any equation) is simply the value of x for which the right-hand side of the equation is equal to the left-hand side. The solutions of quadratic equations are also known as the **roots** of the quadratic equation.

☐ The roots of quadratic equations: an overview

An equation of the form

$$ax^2 + bx + c = 0 \tag{4.3}$$

is defined as a quadratic equation. This is the most general form of a quadratic equation and is solved by means of the 'minus b' formula; see formula (4.2). However, the only condition for equation (4.3) to be classified as a quadratic is that $a \neq 0$. Equations (4.4), (4.5) and (4.6) are also quadratic equations:

$$ax^2 = 0 \quad \text{quadratic with } b = 0 \text{ and } c = 0 \tag{4.4}$$

$$ax^2 + bx = 0 \quad \text{quadratic with } c = 0 \tag{4.5}$$

$$ax^2 + c = 0 \quad \text{quadratic with } b = 0 \tag{4.6}$$

These latter quadratic equations may be solved without recourse to the 'minus b' formula, as demonstrated in Worked Example 4.1.

WORKED EXAMPLE 4.1

SOLVING LESS GENERAL QUADRATIC EQUATIONS

Solve the following equations:

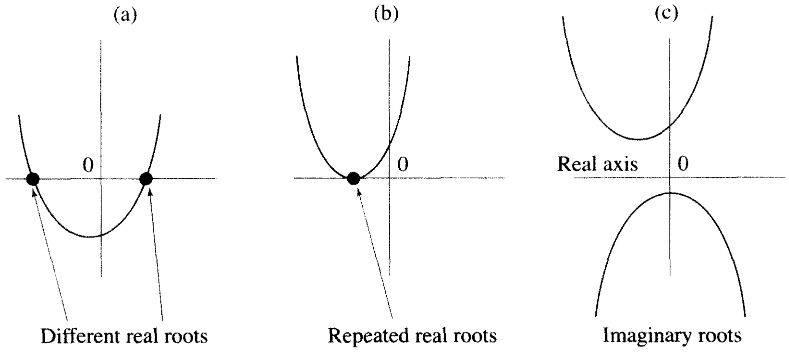
(a) $5x^2 = 0$ (b) $2x^2 - 32 = 0$ (c) $2x^2 + 32 = 0$ (d) $2x^2 - 32x = 0$

Solution

(a)	(b)	(c)	(d)
$5x^2 = 0$ $x^2 = \frac{0}{5}$ $x^2 = 0$ $x = \pm 0$	$2x^2 - 32 = 0$ $2x^2 = 32$ $x^2 = 16$ $x = \pm 4$	$2x^2 + 32 = 0$ $2x^2 = -32$ $x^2 = -16$ $x = \pm\sqrt{-16}$ * $x = \pm 4i$	$2x^2 - 32x = 0$ $x\underbrace{(2x - 32)} = 0$ $\downarrow$ $x = 0$ or $(2x - 32) = 0$ $x = \frac{32}{2} = 16$
A repeated real root (or solution)	Two real roots (or solutions)	Two imaginary roots (or solutions)	Two real roots (or solutions)
$x = 0$, repeated	$x = -4$, $x = 4$	$x = 4i$, $x = -4i$	$x = 0$ and $x = 16$

We define the imaginary number i, such that

$$(i)^2 = -1 \rightarrow \sqrt{(i)^2} = \sqrt{-1} \rightarrow i = \sqrt{-1}.$$



Different real roots   Repeated real roots   Imaginary roots

Figure 4.3 Quadratics: (a) real roots, (b) repeated roots, (c) complex roots

Notice that there are always two solutions of a quadratic equation, where the solutions may be:

1. Two different solutions (real roots)
2. Two identical solutions (real roots), that is, a repeated solution (repeated real root)
3. Two imaginary or complex roots (no real roots—these are of no interest in this text).

❐ Reasons for three different types of solutions (roots)

The value of the term inside the square root sign ($\sqrt{}$) in the 'minus b' formula gives rise to different types of roots:

$$x = \frac{-(b) \pm \sqrt{(b)^2 - 4(a)(c)}}{2(a)}$$

- $b^2 - 4ac > 0$, then the term under the square root is positive, leading to two different solutions (real roots). The graph will cut the x-axis at two different points as shown in Figure 4.3(a).
- $b^2 - 4ac = 0$, then the term under the square root is zero, leading to two identical solutions (two identical real roots). The graph will touch the x-axis at one point only as shown in Figure 4.3(b).
- $b^2 - 4ac < 0$, then the term under the square root is negative, leading to two different complex roots. The graph will never cut or touch the x-axis as shown in Figure 4.3(c).

Quadratic equations with each type of root are solved in Worked Example 4.2.

WORKED EXAMPLE 4.2

SOLVING QUADRATIC EQUATIONS

Solve each of the following equations, classifying the type of solutions (roots) found as real and different, real and repeated or complex:

(a) $x^2 + 6x + 5 = 0$ (b) $x^2 + 6x + 9 = 0$ (c) $x^2 + 6x + 10 = 0$

Solution

Use equation (4.2), the 'minus b' formula, to solve for the values of x.

(a)	(b)	(c)
$x^2+6x+5=0$	$x^2+6x+9=0$	$x^2+6x+10=0$
$x=\frac{-(b)\pm\sqrt{(b)^2-4(a)(c)}}{2(a)}$	$x=\frac{-(b)\pm\sqrt{(b)^2-4(a)(c)}}{2(a)}$	$x=\frac{-(b)\pm\sqrt{(b)^2-4(a)(c)}}{2(a)}$
$a=1,\ b=6,\ c=5$	$a=1,\ b=6,\ c=9$	$a=1,\ b=6,\ c=10$
$x=\frac{-(6)\pm\sqrt{(6)^2-4(1)(5)}}{2(1)}$	$x=\frac{-(6)\pm\sqrt{(6)^2-4(1)(9)}}{2(1)}$	$x=\frac{-(6)\pm\sqrt{(6)^2-4(1)(10)}}{2(1)}$
$x=\frac{-6\pm\sqrt{36-20}}{2}$	$x=\frac{-6\pm\sqrt{36-36}}{2}$	$x=\frac{-6\pm\sqrt{36-40}}{2}$
$x=\frac{-6\pm\sqrt{16}}{2}$	$x=\frac{-6\pm\sqrt{0}}{2}$	$x=\frac{-6\pm\sqrt{-4}}{2}$
$x=\frac{-6\pm 4}{2}$	$x=\frac{-6\pm 0}{2}$	$x=\frac{-6\pm 2i}{2}$
root 1 $=\frac{-6+4}{2}=\frac{-2}{2}=-1$	root 1 $=\frac{-6+0}{2}=\frac{-6}{2}=-3$	root 1 $=\frac{-6+2i}{2}=-3+i$
root 2 $=\frac{-6-4}{2}=\frac{-10}{2}=-5$	root 2 $=\frac{-6-0}{2}=\frac{-6}{2}=-3$	root 2 $=\frac{-6-2i}{2}=-3-i$
Different solutions (real roots)	Repeated solutions (real roots)	Complex roots (of no interest in this text)

Progress Exercises 4.1 Solve Quadratic Equations

Solve the following quadratic equations:

1. $x^2-6x+5=0$
2. $2Q^2-7Q+5=0$
3. $-Q^2+6Q-5=0$
4. $Q^2+6Q+5=0$
5. $P^2-7=0$
6. $Q^2-6Q+9=0$
7. $Q^2-6Q-9=0$
8. $Q^2=6Q$
9. $x^2-6x=7+3x$
10. $P^2+12=3$
11. $P+10=11P^2-P+1$
12. $Q^2-8Q=Q^2-2$
13. $12=P^2-2P+12$
14. $5+P=4P^2-4+P$
15. $-4P^2-2P+1=P$

4.1.2 Properties and graphs of quadratic functions: $f(x)=ax^2+bx+c$

Many of the properties of functions may be deduced from their graphs. The properties of quadratics are illustrated in the sketches of quadratic functions in Worked Examples 4.3, 4.4 and 4.5.

To plot the graph of a quadratic function:

- Calculate a table of x, y values (remember, y is the short way of indicating $f(x)$). Be extremely careful with the negative values. Remember, a minus sign is a (-1).
- Plot the calculated points and join the points by a smooth curve.
- The points of intersection with the x-axis are the roots of the quadratic equation $f(x) = 0$

Since the graph crosses the x-axis when $y = 0$, that is, at $0 = ax^2 + bx + c$, then the solution of this equation, $0 = ax^2 + bx + c$, gives the x-coordinates of the points of intersection with the x-axis, i.e. the roots of $f(x) = 0$.

❐ Graphical representation of the roots of a quadratic

- Distinct real roots cut the x-axis in two distinct points. See Figure 4.3(a).
- Repeated roots touch the x-axis at one point. See Figure 4.3(b).
- Complex roots never touch the x-axis. See Figure 4.3(c).

WORKED EXAMPLE 4.3

SKETCHING A QUADRATIC FUNCTION $f(x) = \pm x^2$

(a) Plot the graphs of $y = x^2$ and $y = -x^2$ on the same diagram.
(b) Comment on the general properties of the graph.
What is the effect on the graph of multiplying the right-hand side of each equation by (-1)?

Solution

(a) Calculate a table of (x, y) points as in Table 4.1. Then graph these points as in Figure 4.4.

(b) From Figure 4.4, note the following. The graphs are symmetrical about the x-axis; that is, the graph of $y = -x^2$ is simply the graph of $y = x^2$ inverted. Therefore, multiplying the right-hand-side of each equation by (-1) simply reflects the graph through the x-axis.

Table 4.1 Calculation of points for $y = x^2$ and $y = -x^2$

x	$f(x) = x^2$	$f(x) = (-1)x^2$
-3	$(-3)^2 = 9$	$(-1)(-3)^2 = -9$
-2	$(-2)^2 = 4$	$(-1)(-2)^2 = -4$
-1	$(-1)^2 = 1$	$(-1)(-1)^2 = -1$
0	$(0)^2 = 0$	$(-1)(0)^2 = 0$
1	$(1)^2 = 1$	$(-1)(1)^2 = -1$
2	$(2)^2 = 4$	$(-1)(2)^2 = -4$
3	$(3)^2 = 9$	$(-1)(3)^2 = -9$

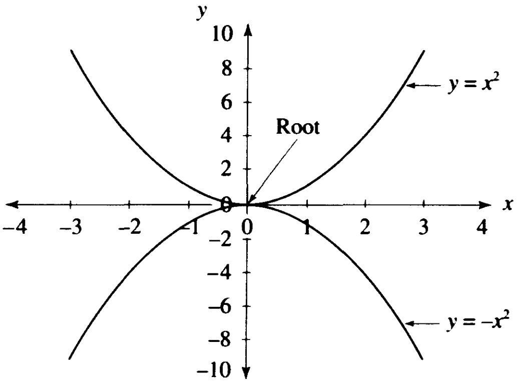


Figure 4.4 $y = x^2$ and $y = -x^2$

Go to the web site www.wiley.co.uk/bradley2ed for Worked Example 4.4.

❑ To sketch any quadratic $y = ax^2 + bx + c$

In the following worked examples, quadratic curves are sketched by calculating a table of points for the range of x values given.

WORKED EXAMPLE 4.5

SKETCHING ANY QUADRATIC EQUATION

(a) Sketch the graph of $y = 2x^2 - 7x - 9$ for values of x from -2 to 6.
(b) Calculate the points of intersection with the axes; hence, find the roots and mark the roots on the diagram.
(c) Estimate the coordinates of the turning point from the graph.
(d) Measure the difference between the x-coordinate of the turning point and each root.

Solution

(a) Calculate the y-values given x, ranging from $x = -2$ to 6. The results are given in Table 4.3 and sketched in Figure 4.6.
(b) The points of intersection with the axes are calculated as follows:

- The graph cuts the y-axis when $x = 0$, that is, at $y = -9$.
- The graph cuts the x-axis at $y = 0$, that is, at $2x^2 - 7x - 9 = 0$.

Solve this quadratic equation, to find roots $x = -1$ when $y = 0$; $x = 4.5$ when $y = 0$.
These roots (solutions) are indicated in Figure 4.6.

Table 4.3 Calculation of points for $y = 2x^2 - 7x - 9$

x	y
−2	13
−1	0
0	−9
1	−14
2	−15
3	−12
4	−5
5	6
6	21

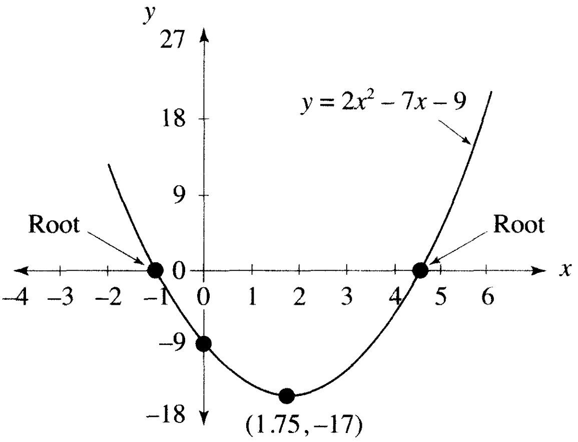


Figure 4.6 Graph for Worked Example 4.5

(c) From the graph the minimum point is at $x = 1.75$, $y = -17$ approximately. These points can be found exactly using differentiation in Chapter 6.

(d) The two roots are equal distance, 2.75, on either side of the x-coordinate of the minimum point.

❐ Summary to date

Quadratics of the form: $y = ax^2 + bx + c$:

- Are continuous curves (no breaks)
- Have only one turning point. If a is positive, the turning point is a minimum, if a is negative, the turning point is a maximum (turning points are estimated graphically in this chapter, but maximum and minimum points will be calculated in Chapter 6).
- Are symmetrical about the vertical line drawn through the turning point.
- y-intercept $= c$

- Are wider than the $y = x^2$ graph if $a < 1$, and narrower than the $y = x^2$ graph if $a > 1$
- The roots of the quadratic are the solutions of $ax^2 + bx + c = 0$, graphically the roots are the points where the graph crosses the x-axis. The roots are symmetrical about the vertical line drawn through the turning point.

Progress Exercises 4.2 Graphs and Solutions of Quadratic Equations

1. Graph each of the following functions on the same diagram over the range -2 to 2.

   (a) $y = x^2$ (b) $y = -3x^2$ (c) $y = 0.5x^2$

2. Graph each of the following functions on the same diagram.

   (a) $P = (Q + 2)^2$ (b) $P = (Q - 1)^2$ (c) $P = Q^2 + 1$

   What is the relationship between each graph (a), (b) and (c) to $P = Q^2$?
3. The following functions were given in Worked Example 4.2.

   (a) $y = x^2 + 6x + 5$ (b) $y = x^2 + 6x + 9$ (c) $y = x^2 + 6x + 10$

   (i) Plot each graph for $-8 < x < 2$, indicating

   - The points of intersection with the axes
   - The roots of the quadratic
   - Turning points

   (ii) Calculate the points of intersection with the axes algebraically.
   (iii) What is the relationship between graphs (a), (b) and (c)?
4. Plot the graph of $P = Q^2$.
   (a) Sketch the graph of $P = Q^2$, when translated vertically downwards by 9 units.
   (b) Write down the equation of the translated graph. Calculate the points of intersection with the axes.
5. (a) Plot the graph of the quadratic $y = x(10 - x)$. Mark in the points of intersection with the axes and estimate the turning points.
   (b) Calculate the points of intersection with the axes.
6. Given the following quadratic functions,

   (a) $P = -Q^2$ (b) $P = -Q^2 + 4$ (c) $P = -(Q - 3)^2 + 4$

   (i) Plot each graph for $-3 < x < 6$, indicating

   - The points of intersection with the axes
   - The roots of the quadratic
   - Turning points

   (ii) Calculate the points of intersection with the axes algebraically.
   (iii) What is the relationship between graphs (a), (b) and (c)?

7. (a) Sketch three possible graphs of a quadratic function which has roots at $x = 2$ and $x = 6$.
   (b) Write down an expression for any quadratic equation which has roots at $x = 2$ and $x = 6$. What characteristic of the quadratic equation determines whether the graph is a minimum type or a maximum type?

4.1.3 Quadratic functions in economics

❐ Non-linear supply and demand functions

In Chapter 3, market equilibrium was calculated for linear demand and supply functions. Now, market equilibrium is calculated for non-linear demand and supply functions. This is demonstrated in Worked Example 4.6.

WORKED EXAMPLE 4.6

NON-LINEAR DEMAND AND SUPPLY FUNCTIONS

The supply and demand functions for a particular market are given by the equations

$$P_s = Q^2 + 6Q + 9 \qquad \text{and} \qquad P_d = Q^2 - 10Q + 25$$

(a) Sketch the graph of each function over the interval $Q = 0$ to $Q = 5$.
(b) Find the equilibrium price and quantity graphically and algebraically.

Solution

(a) Calculate the P values for $Q = 0$ to 5. The results are given in Table 4.4 and sketched in Figure 4.7. Note that the vertical and horizontal intercepts of the demand function are $P = 25$ (when $Q = 0$) and $Q = 5$ (when $P = 0$) respectively. The vertical intercept of the supply function is $P = 9$ (when $Q = 0$).

Table 4.4 Points for $P_s = Q^2 + 6Q + 9$ and $P_d = Q^2 - 10Q + 25$

Q	$P = Q^2 + 6Q + 9$	$P = Q^2 - 10Q + 25$
0	9	25
1	16	16
2	25	9
3	36	4
4	49	1
5	65	0

(b) From the graph, market equilibrium is at the point of intersection of the two functions, estimated at $Q = 1$, $P = 16$. This solution may be found exactly using algebra. Algebraically, market equilibrium exists when $P_d = P_s$ and $Q_d = Q_s$. In

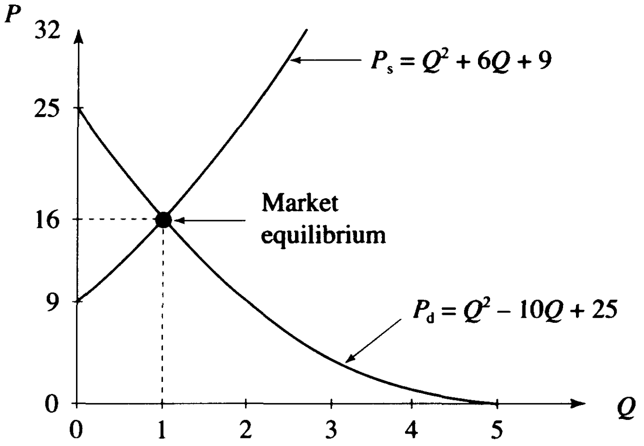


Figure 4.7 Market equilibrium with non-linear demand and supply functions

this case, it is easiest to equate the prices, so at equilibrium,

$$P_s = P_d$$

$$Q^2 + 6Q + 9 = Q^2 - 10Q + 25$$

$$Q^2 + 6Q - Q^2 + 10Q = 25 - 9$$

$$16Q = 16$$

$$Q = 1$$

$$\text{when } Q = 1, \quad P = 16 \quad \text{Check!}$$

☐ Total revenue for a profit-maximising monopolist

Total revenue functions are frequently represented by maximum type quadratics which pass through the origin.

☺ **Remember:** Equation (4.1) outlined a non-linear total revenue function as $TR = 50Q - 2Q^2$.

WORKED EXAMPLE 4.7

NON-LINEAR TOTAL REVENUE FUNCTION

The demand function for a monopolist is given by the equation $P = 50 - 2Q$.

(a) Write down the equation of the total revenue function.

(b) Graph the total revenue function for $0 \leq Q \leq 30$.

(c) Estimate the value of Q at which total revenue is a maximum and estimate the value of maximum total revenue.

Table 4.5 Points for $TR = 50Q - 2Q^2$

Q	TR
0	0
5	200
10	300
15	300
20	200
25	0
30	−300

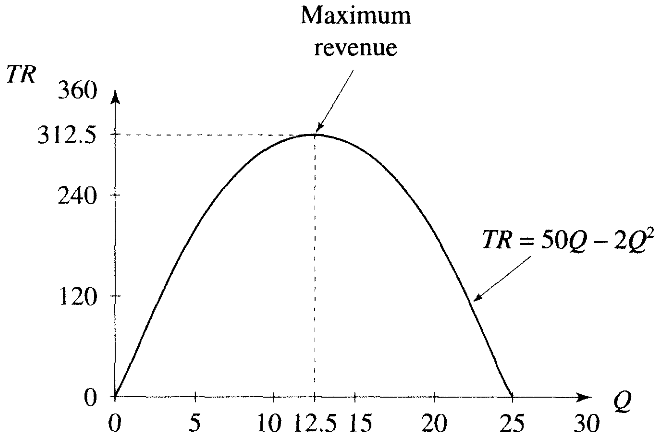


Figure 4.8 Non-linear total revenue function

Solution

(a) Since $P = 50 - 2Q$, and $TR = P \times Q$, then $TR = (50 - 2Q)Q = 50Q - 2Q^2$
(b) Calculate a table of values for $0 \leq Q \leq 30$ such as those in Table 4.5. The graph is plotted in Figure 4.8.
(c) A property of quadratic functions is that the turning point (in this case a maximum) lies halfway between the roots (solutions) of the quadratic function. The roots of the TR function are

$$
\begin{aligned}
TR &= 50Q - 2Q^2 \\
0 &= 50Q - 2Q^2 \\
0 &= Q(50 - 2Q) \\
&\downarrow \qquad \downarrow \\
Q &= 0 \qquad \downarrow \\
&\qquad 2Q = 50 \\
&\qquad Q = 25
\end{aligned}
$$

These roots are illustrated graphically as the points where the TR function intersects the x-axis. The turning point occurs halfway between these points, that is, at $Q = 12.5$. Substitute $Q = 12.5$ into the TR function and calculate maximum total revenue as

$$\begin{aligned} TR &= 50Q - 2Q^2 \\ &= 50(12.5) - 2(12.5)^2 \\ &= 625 - 312.5 = 312.5 \end{aligned}$$

WORKED EXAMPLE 4.8

CALCULATING BREAK-EVEN POINTS

The demand function for a good is given as $Q = 65 - 5P$. Fixed costs are £30 and each unit produced costs an additional £2.

(a) Write down the equations for total revenue and total costs in terms of Q.

(b) Find the break-even point(s) algebraically.

(c) Graph total revenue and total costs on the same diagram, hence, estimate the break-even point(s).

Solution

(a) $TR = P \times Q$. Therefore, if P is written in terms of Q, then TR will also be expressed in terms of Q. The expression for P in terms of Q is obtained from the equation of the demand function

$$\begin{aligned} Q &= 65 - 5P \\ 5P &= 65 - Q \\ P &= \frac{65 - Q}{5} = 13 - 0.2Q \end{aligned}$$

Substitute the expression for P (price per unit), into the equation, $TR = P \times Q$; therefore, $TR = (13 - 0.2Q)Q = 13Q - 0.2Q^2$. Total cost is given as $TC = FC + VC = 30 + 2Q$.

(b) The break-even points occur when $TR = TC$, therefore,

$$\begin{aligned} 13Q - 0.2Q^2 &= 30 + 2Q \\ 0 &= 0.2Q^2 - 11Q + 30 \end{aligned}$$

The reader is expected to solve the quadratic for Q. The solutions are $Q = 2.91$ and $Q = 52.1$.

(c) A table of values for TR and TC from $Q = 0$ to $Q = 70$ is given in Table 4.6. These points are plotted in Figure 4.9. The break-even points occur at the intersection of the two functions. The break-even points on the graph agree with those calculated in (b).

Table 4.6 Total revenue and total cost

Q	TR	TC
0	0	30
10	110	50
20	180	70
30	210	90
40	200	110
50	150	130
60	60	150
70	−70	170

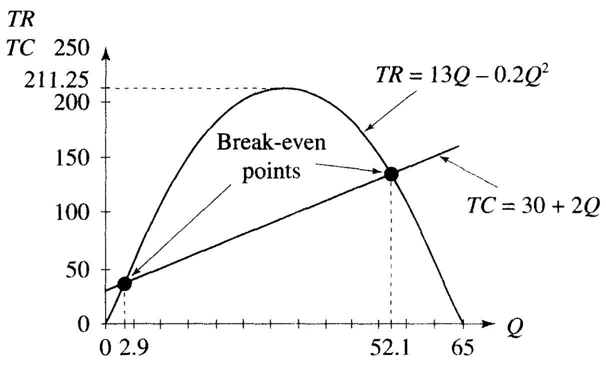


Figure 4.9 Total revenue and total cost: break-even points

Progress Exercises 4.3 Quadratic Functions: Applications

1. Find the economically meaningful equilibrium price and quantity (a) graphically and (b) algebraically when the supply and demand functions are given by the equations

$$P = Q^2 - 0.5 \quad \text{and} \quad P = -Q^2 + 4$$

2. The demand function for ringside seats at a particular boxing match is given as $P = 12 - Q$, where P is in £000s.
   (a) Write an expression for TR in terms of Q.
   (b) Graph the TR function, indicating the points of intersection with the axes and turning point. Describe, in words, the meaning of these points.
   (c) Confirm the points of intersection with the axes algebraically.

3. Sketch the quadratic total revenue function which has a maximum value of 1000 at $Q = 20$, and which is zero at $Q = 0$. Hence, locate the second point where $TR = 0$ and find the equation of the total revenue function.

4. Given the demand function of a monopolist as $P = 100 - 2Q$,
   (a) Write down the equation for TR in the form $TR = f(Q)$. Calculate TR when $Q = 10$.
   (b) Write down the equation for TR in the form $TR = f(P)$. Calculate TR when $P = 10$. Comment on your answers in (a) and (b).

5. Sketch the supply function $P_s = 0.5Q^2 + 5$ and the demand function $P_d = 155 - Q^2$ on the same diagram.
   (a) Estimate the equilibrium price and quantity graphically.
   (b) Calculate the equilibrium price and quantity algebraically.

6. A firm's total cost function is given by the equation $TC = 200 + 3Q$, while the demand function is given by the equation $P = 107 - 2Q$.
   (a) Write down the equation of the total revenue function.
   (b) Graph the total revenue function for $0 < Q < 60$. Hence, estimate the output, Q, and total revenue when total revenue is a maximum.
   (c) Plot the total cost function on the diagram in (b). Estimate the break-even point from the graph. Confirm your answer algebraically.
   (d) State the range of values for Q for which the company makes a profit.

7. (See question 6)
   Write down the equation for the profit function (Profit = $TR - TC$). Graph the profit function for $0 < Q < 100$. From the graph estimate,
   (a) The values of Q at which profit is zero
   (b) Maximum profit and the output, Q, at which profit is a maximum.

8. The demand and supply functions for a good are given by the equations

$$P_d = -(Q+4)^2 + 100, \quad P_s = (Q+2)^2$$

   (a) Sketch each function on the same diagram; hence, estimate the equilibrium price and quantity.
   (b) Confirm the equilibrium using algebra.

4.1.4 Cubic functions

A cubic function is expressed by a cubic equation which has the general form

$$ax^3 + bx^2 + cx + d = 0 \qquad a, b, c \text{ and } d \text{ are constants}$$

There is, unfortunately, no easy general formula for solving a cubic equation, but approximate solutions may be found graphically by plotting the graph and finding the points of intersection with the axes, as in Worked Example 4.9. In this section, in particular, calculating points by graph plotting is much easier and less time consuming if Excel or other spreadsheets are used.

WORKED EXAMPLE 4.9

PLOTTING CUBIC FUNCTIONS

Plot the graphs of (a) $y = x^3$ and (b) $y = -x^3$ on the same diagram. From the graph estimate the turning points and the roots of the cubic equation. Compare the graphs of (a) and (b).

Solution

The graphs are plotted in Figure 4.10 from the points calculated in Table 4.7. Figure 4.10 illustrates that the graphs of (a) and (b):

Table 4.7 Calculation of points for graphs of (a) $y = x^3$ and (b) $y = -x^3$

x	$y = x^3$	$y = -x^3$
−3	−27	27
−2	−8	8
−1	−1	1
0	0	0
1	1	−1
2	8	−8
3	27	−27

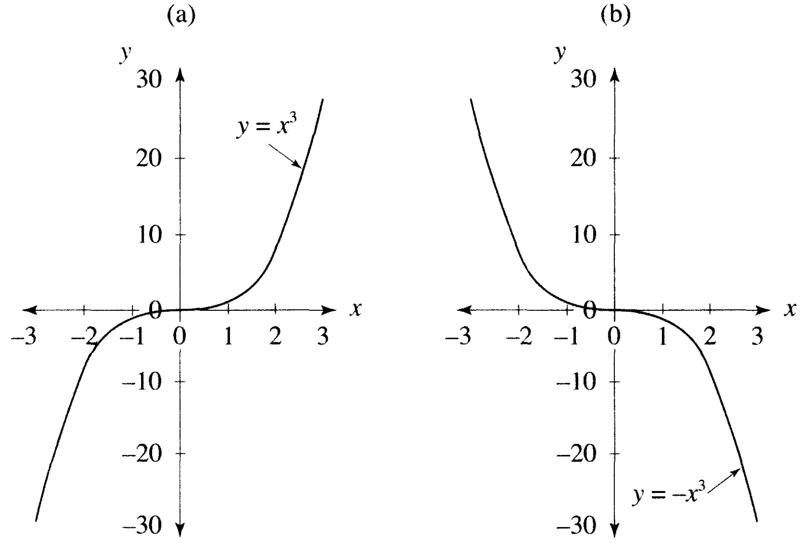


Figure 4.10 Graphs of cubics

- Have only one root at $x = 0$
- Have no turning points

Graph (a) may be viewed as a reflection of graph (b) in the x-axis or y-axis.

In Worked Example 4.9, each graph had no turning point and only one root. However, the graphs of more general cubic equations (as in Worked Example 4.10) will have:

(a) No turning points or two turning points
(b) Either one root or three roots

WORKED EXAMPLE 4.10

GRAPHS OF MORE GENERAL CUBIC FUNCTIONS

Plot the graphs of the following:

(a) $y = 0.5x^3 - 5x^2 + 8.5x + 27$ (b) $y = -0.5x^3 - 5x^2 + 8.5x + 27$.

From the graphs estimate,

(i) The roots.
(ii) The turning points.

Compare the graphs of (a) and (b).

Solution

First set up a table of values from which the graphs may be plotted. (See Table 4.8 and Figures 4.11 and 4.12.) The roots and the turning points are summarised as follows:

(a) has only one root, at $x = -1.56$ approximately.

(a) has two turning points at $x = 1.0$, $y = 31$ and at $x = 5.5$, $y = 5.6$

(b) has three roots, at $x = -11$, $x = -1.7$ and $x = 2.8$ approximately.

(b) has two turning points at $x = -7$, $y = -107$ and at $x = 0.8$, $y = 30$

Table 4.8 Points for plotting graphs in Worked Example 4.10

x	**Graph (a)**	**Graph (b)**
−12	−1659	69
−8	−617	−105
−4	−119	−55
0	27	27
4	13	−51
8	31	−481
2	273	−1455

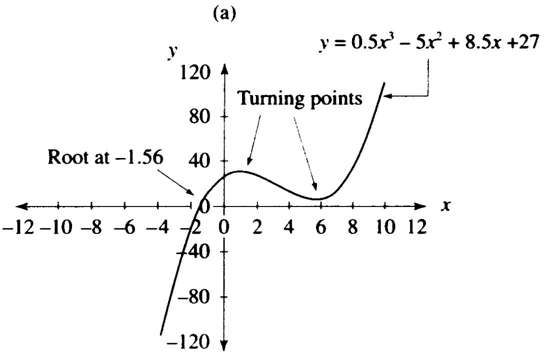


Figure 4.11 Graph for Worked Example 4.10(a)

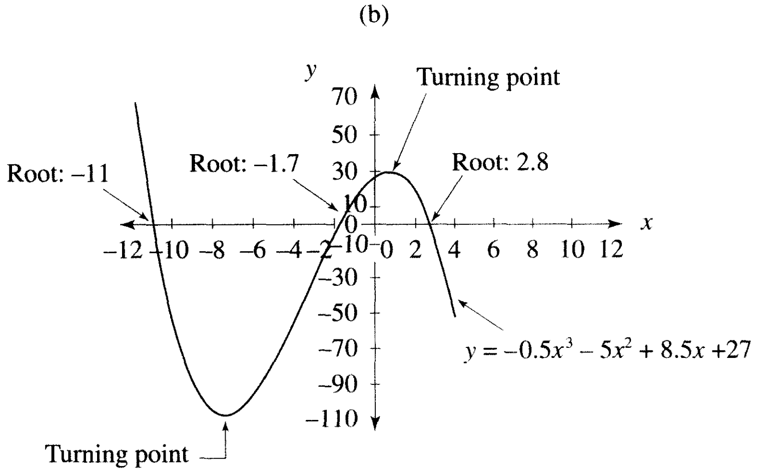


Figure 4.12 Graph for Worked Example 4.10(b)

Note: From an economic perspective, the first quadrant ($x > 0$, $y > 0$) in Figures 4.11 and 4.12 only is of interest. However, for a better understanding of mathematics, it is necessary to analyse the characteristics of the graphs in the other quadrants.

General properties of cubic equations

Cubic equations are continuous curves, which may have:

- One root or three roots
- No turning point or two turning points.

These properties are illustrated in Figures 4.10, 4.11 and 4.12.

Polynomials

Quadratic and cubic functions belong to a group of functions called **polynomials:**

$$\mathrm{f}(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_1 x + a_0 \qquad a_0, a_1, \ldots, a_n \text{ are constants}$$

The ‘degree’ of the polynomial is given by the highest power of x in the expression. Therefore, a quadratic is a polynomial of degree 2 and a cubic is a polynomial of degree 3.

We are not going to study polynomials in depth, but you do need to be aware that they are continuous graphs, with no breaks or jumps. Therefore, it is quite safe to plot these graphs by joining the points as calculated and hence use the graphs to estimate roots and turning points. Now for one final worked example in which costs are given by a cubic equation.

WORKED EXAMPLE 4.11

TR, *TC* AND PROFIT FUNCTIONS

A firm's total cost function is given by the equation, $TC = Q^3$. The demand function for the good is $P = 90 - Q$.

(a) Write down the equations for total revenue and profit. Calculate the break-even points.

(b) Graph the total revenue and total cost functions on the same diagram for $0 \leq Q \leq 12$, showing the break-even points.

(c) Estimate the total revenue and total costs at break-even.

(d) From the graph estimate the values of Q within which the firm makes (i) a profit, (ii) a loss.

(e) From the graph estimate the maximum profit and the level of output for which profit is maximized.

Solution

(a) $TR = PQ = (90 - Q)Q = 90Q - Q^2$.

Profit$(\pi) = TR - TC = 90Q - Q^2 - Q^3$

The break-even points occur when $TR = TC$, therefore, solve

$$90Q - Q^2 = Q^3$$
$$0 = Q^3 + Q^2 - 90Q$$
$$0 = Q(Q^2 + Q - 90)$$

Therefore, $Q = 0$ and solving the quadratic $Q^2 + Q - 90 = 0$ gives the solutions, $Q = 9$ and $Q = -10$.

(b) To graph TR and TC, calculate a table of values for the specified range using Excel or otherwise. (See Table 4.9 and Figure 4.13.)

Table 4.9 *TR* and *TC* for Worked Example 4.11

Q	*TR*	*TC*
0	0	0
3	261	27
6	504	216
9	729	729
12	936	1728

(c) From the graph break-even is at $Q = 0$ and $Q = 9$. (Break-even at $Q = -10$ does not lie within the range of the graph. In fact, $Q = -10$ is not economically meaningful.) At $Q = 0$, $TR = TC = 0$. At $Q = 9$, $TR = TC = 729$.

(d) The firm makes a profit when the TR curve is greater than the TC curve, that is between $Q = 0$ and $Q = 9$. When Q is greater than 9 the firm makes a loss.

(e) Maximum profit occurs at the point where the vertical distance between TR and TC is greatest. From the graph, this estimate is at $Q = 5.15$. Substitute $Q = 5.15$ into the profit function to calculate profit $= 90(5.15) - (5.15)^2 - (5.15)^3 = 300.4$.

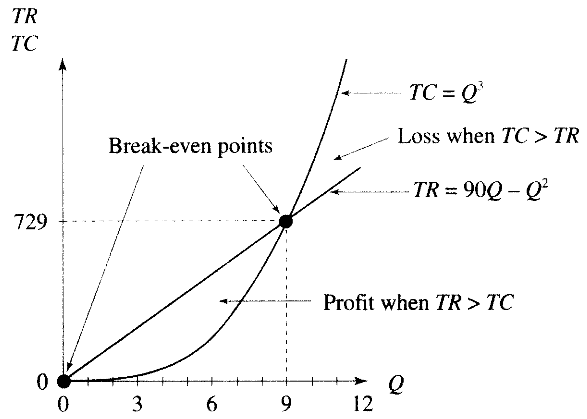


Figure 4.13 Quadratic TR and cubic TC functions

Progress Exercises 4.4 Graphical Solution of Cubic Equations

Graph the cubic functions in questions 1–4. From the graph estimate the roots of each function and estimate the coordinates of the turning point(s).

1. $TC = 0.5Q^3 - 8Q^2 + 200$ over the interval $-5 < Q < 15$
2. $Q = 3P^3 + 9P^2$ over the interval $-4 < Q < 2$
3. $P = -4Q^3 + 2Q^2$ over the interval $-0.5 < Q < 0.8$
4. $TC = 0.5Q^3 - 15Q^2 + 175Q + 1000$ over the interval $-10 < Q < 30$

4.2 Exponential Functions

At the end of this section you should be able to:

- Define and graph exponential functions
- Manipulate and solve exponential equations
- Use exponential functions in various applications

4.2.1 Definition and graphs of exponential functions

The exponential function has the general form $y = a^x$ or $f(x) = a^x$ where:

- a is a constant and is referred to as the base of the exponential function.
- x is called the index or power of the exponential function; this is the variable part of the function.

For example, assuming $a = 4$, evaluate 4^x for values of $x = 0, 1, 2, 3$. Here 4 is the constant base of the exponential while x is the variable index or power.

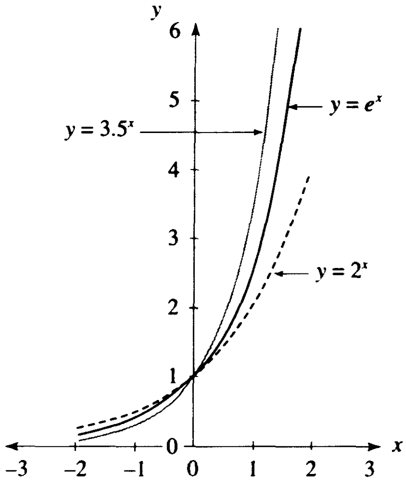
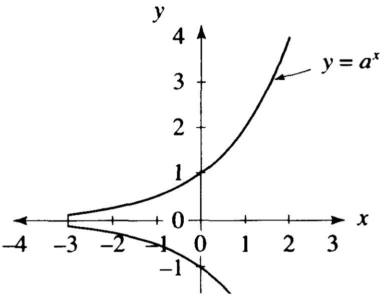


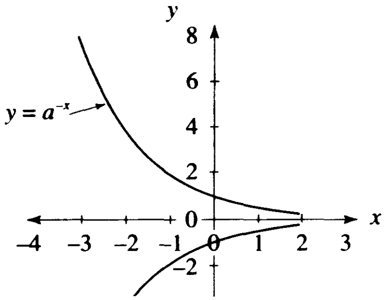
Figure 4.15 Graphs for Tables 4.10 and 4.11

❐ Properties of exponential functions

The properties of exponential functions are now summarised. Here are some features to note from Figures 4.14 and 4.15:

- All curves are continuous (no breaks), and pass through the point $x = 0$, $y = 1$.
- When the index is positive, the curves are increasing as x increases, provided $a > 1$; these are called **growth curves**.
- When the index is negative, the curves are decreasing as x increases, provided $a > 1$; these are called **decay curves**.
- Exponentials with larger bases increase more rapidly for $x > 0$ and decrease more rapidly for $x < 0$. Compare equations (4.7), (4.9) and (4.10) as graphed in Figure 4.15.





Solution

(a) The function $y = (2)^x$ has base 2 and variable power x. The function $y = (2)^{-x}$ has base 2 and variable power $-x$. To plot these functions, calculate a table of points such as Table 4.10. Using your calculator, confirm that the values in Table 4.10 are correct. These points are then plotted in Figure 4.14.

Table 4.10 Points for the functions $y = 2^x$ and $y = 2^{-x}$

x	$y = 2^x$	$y = 2^{-x}$
−2	0.25	4
−1	0.50	2
0	1	1
1	2	0.50
2	4	0.25

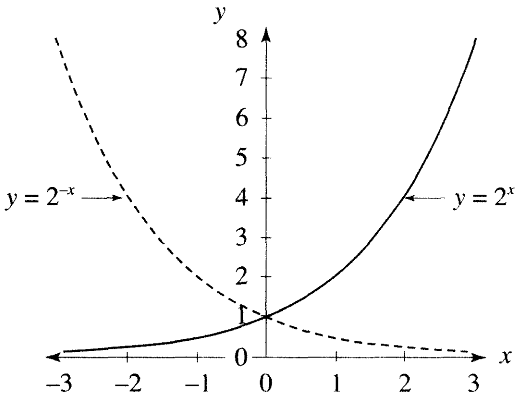


Figure 4.14 Graph for Table 4.10

(b) The function $y = (3.5)^x$ has base 3.5 and variable power x while the function $y = e^x$ has base e and variable power x. To plot these functions, calculate a table of points as in Table 4.11. These points are plotted in Figure 4.15. Also plotted in Figure 4.15 is the graph for part (a). It is now possible to compare the graphs of the three functions.

Table 4.11 Points for the functions $y = (3.5)^x$ and $y = e^x$

x	$y = (3.5)^x$	$y = e^x$
−2	0.08	0.14
−1	0.29	0.37
0	1	1
1	3.5	2.72
2	12.25	7.39

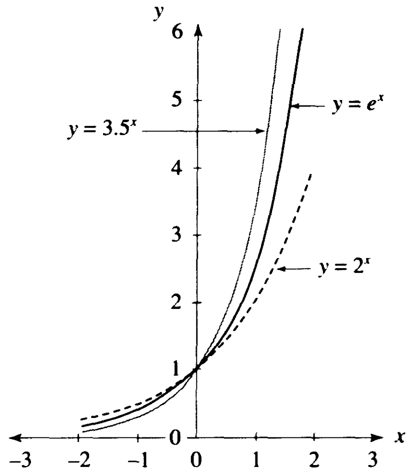


Figure 4.15 Graphs for Tables 4.10 and 4.11

❐ Properties of exponential functions

The properties of exponential functions are now summarised. Here are some features to note from Figures 4.14 and 4.15:

- All curves are continuous (no breaks), and pass through the point $x = 0$, $y = 1$.
- When the index is positive, the curves are increasing as x increases, provided $a > 1$; these are called **growth curves**.
- When the index is negative, the curves are decreasing as x increases, provided $a > 1$; these are called **decay curves**.
- Exponentials with larger bases increase more rapidly for $x > 0$ and decrease more rapidly for $x < 0$. Compare equations (4.7), (4.9) and (4.10) as graphed in Figure 4.15.

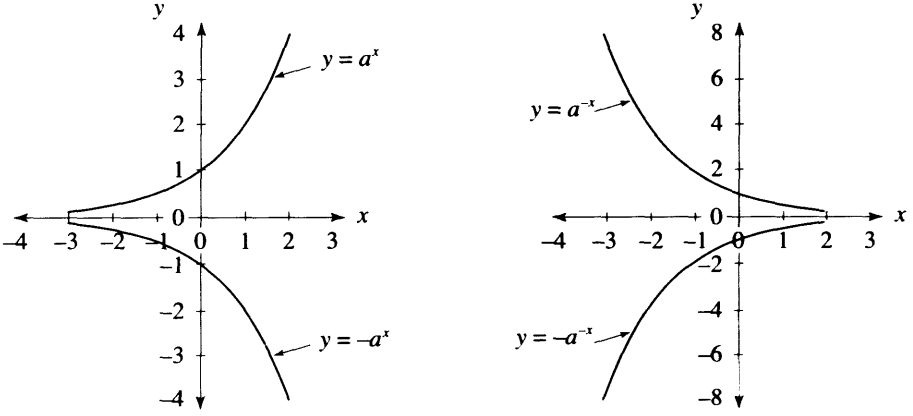


Figure 4.16 Various graphs of $y = a^x$

- The graphs of $y = a^x$ and $y = a^{-x}$ are always above the x-axis, therefore y is always positive (Figure 4.14). You can check this by evaluating several exponentials on the calculator.
- The graphs of $y = -a^x$ and $y = -a^{-x}$ are always below the x-axis, as shown in Figure 4.16.

☺ **Remember** that multiplying the RHS of $y = f(x)$ by -1 simply inverts (flips) the function through the x-axis.

- The x-axis is a horizontal asymptote. An asymptote is a straight line which the graph approaches as x increases. The graphs of the exponentials level off towards the x-axis, but in theory never touch it.

❐ Rules for using exponential functions

Exponential functions are usually referred to as indices; here is an outline of the rules for multiplying, dividing and raising an exponential function to a power.

Statement	Equation	Example illustrating rule
Rule 1: To multiply numbers with the same base, add the indices	$a^m \times a^n = a^{m+n}$	By rule $5^2 \times 5^3 = 5^{2+3} = 5^5$ Directly $(5 \times 5) \times (5 \times 5 \times 5) = 5^5$
Rule 2: To divide numbers with the same base, subtract the indices	$a^m/a^n = a^{m-n}$	By rule $3^4/3^2 = 3^{4-2} = 3^2$ Directly $(3 \times 3 \times 3 \times 3)/(3 \times 3)$ $= (3 \times 3) = 3^2$
Rule 3: To raise an exponential to a power, multiply the indices	$(a^m)^k = a^{m \times k}$	By rule $(2^3)^2 = 2^{3\times2} = 2^6$ Directly $(2 \times 2 \times 2)(2 \times 2 \times 2) = 2^6$

Notes

1. Anything to the power of zero is one: $(\text{anything})^0 = 1$.
2. If an exponential in a fraction is moved from below the line to above the line, the sign on the index changes:

$$\frac{1}{a^n} = \frac{a^0}{a^n} = a^{0-n} = a^{-n}$$

For example,

$$\frac{1}{3^2} = \frac{3^0}{3^2} = 3^{0-2} = 3^{-2}$$

3. To write roots as indices, everything under the root sign is raised to the power of 1/root: $\sqrt[n]{a} = a^{1/n}$. For example, $\sqrt[3]{9} = 9^{1/3}$.

Also

1. If any product or fraction is raised to a power, the index on each term is multiplied by the power: $(a^x b^y/c^z)^n = a^{nx} b^{ny}/c^{nz}$. For example, $(2a^2/c^5)^2 = 2^2 a^4/c^{10}$.
2. The above rules apply to products and quotients only. There are no rules for simplifying sums or differences of exponential functions. These rules cannot be used to simplify or

reduce $2^x + 2^y$. For example,

$$2^3 + 2^2 \neq 2^5$$

$$8 + 4 \neq 32$$

WORKED EXAMPLE 4.13

SIMPLIFING EXPONENTIAL EXPRESSIONS

Simplify each of the following expressions, giving your answers with positive indices only:

$$\text{(a)}\ \frac{2^5}{2^3 2^{-4}} \qquad \text{(b)}\ QQ^{1.5}P \qquad \text{(c)}\ \frac{\sqrt{3^5}}{3^{-4}3^4}$$

$$\text{(d)}\ \frac{15e^{a}e^{3}}{e^2} \qquad \text{(e)}\ \left(\frac{3L^{0.5}}{L^{-2}}\right)^2 \qquad \text{(f)}\ \left(\frac{3e^{0.5}}{e^{-2}}\right)^2$$

Solutions

$$\text{(a)}\ \frac{2^5}{2^3 2^{-4}} = \frac{2^5}{2^{3-4}}$$

use rule 1 to add the indices of the base 2 terms below the line

$$= \frac{2^5}{2^{-1}}$$

simplify

$$= 2^{5-(-1)}$$

use rule 2 and bracket the negative terms

$$= 2^{5+1}$$

$$= 2^6$$

use the calculator to evaluate exponentials

$$= 64$$

$$\text{(b)}\ QQ^{1.5}P = Q^1 Q^{1.5}P$$

since no index is given on the first Q, it is understood to be 1

$$= Q^{1+1.5}P$$

$$= Q^{2.5}P$$

$$\text{(c)}\ \frac{\sqrt{3^5}}{3^{-4}3^4} = \frac{3^{5/2}}{3^{4-4}}$$

start by writing $\sqrt{3^5} = (3^5)^{1/2} = (3^{5/1})^{1/2}$
$= 3^{5/1 \times 1/2} = 3^{5/2}$

$$= \frac{3^{5/2}}{3^0}$$

$= 3^{5/2}$ since $3^0 = 1$

$= 15.588$

(d) $\dfrac{15e^{\alpha}e^{\beta}}{e^2} = \dfrac{15e^{\alpha+\beta}}{e^2}$ multiplying numbers with same base (rule 1)

$= \dfrac{15e^{\alpha+\beta}e^{-2}}{1}$ bringing up the number from below the line (note 2)

$= 15e^{\alpha+\beta-2}$ multiplying numbers with same base (rule 1)

(e) $\left(\dfrac{3L^{0.5}}{L^{-2}}\right)^2 = \left(\dfrac{3L^{0.5-(-2)}}{1}\right)^2$ simplify the terms inside the bracket first

$$= \left(\frac{3L^{0.5+2}}{1}\right)^2$$

$$= \left(\frac{3L^{2.5}}{1}\right)^2$$

$= \left(\dfrac{3L^{2.5}}{1}\right)\left(\dfrac{3L^{2.5}}{1}\right)$ square out

$$= \left(\frac{3^2L^{2.5+2.5}}{1}\right)$$

$$= 9L^5$$

There are usually several different approaches to simplifying; for example, in part (e) we could square out the brackets first then simplify like this:

$\left(\dfrac{3^1L^{0.5}}{L^{-2}}\right)^2 = \dfrac{3^{1\times2}L^{0.5\times2}}{L^{-2\times2}}$ square first by multiplying the index on every term (including the 3) by the power 2 outside the bracket

$$= \frac{9L^1}{L^{-4}}$$

$$= \frac{9L^{1-(-4)}}{1}$$

$$= 9L^5$$

(f) $\left(\dfrac{3e^{0.5}}{e^{-2}}\right)^2 = \left(\dfrac{3e^{0.5}}{e^{-2}}\right)\left(\dfrac{3e^{0.5}}{e^{-2}}\right)$ squaring

$$= \frac{3^2(e^{0.5})^2}{(e^{-2})^2}$$

$= \dfrac{3^2 e^1}{e^{-4}}$ multiplying the indices (rule 3)

$= 9e^{1-(-4)}$ subtracting the index of the divisor (rule 2)

$= 9e^5 = 1335.7184$ simplifying, then evaluating correct to 4 decimal places

Progress Exercises 4.5 Rules for Indices

1. Use your calculator to evaluate the following:

(a) 6^2 (b) 3^3 (c) 5^1 (d) 5^3 (e) $(-3)^2$ (f) $(-4)^2$

(g) 25^0 (h) 5^{-1} (i) 6^{-2} (j) 5^{-3} (k) $(2.5)^{0.5}$ (l) $(1.5)^{-5}$

2. Simplify the following expressions, giving your answer with positive indices only:

(a) $\dfrac{2^4 2^{-3}}{2^2}$ (b) $\dfrac{(4)3^5 3^{0.8}}{3^4 3^{-2.3}}$

3. (a) $\dfrac{1}{2^3}\,\dfrac{3^2 5^{-5}}{5^2 3^{-3}}$ (b) $\dfrac{2^x 2^y}{2^x}$

4. (a) $\dfrac{a^x a^y}{a^x}$ (b) $\dfrac{5^x 2^y}{2^x}$

5. (a) $\dfrac{L^\alpha K^\beta}{L}$ (b) $\left(\dfrac{4L^2}{L^{-2}}\right)^2$

6. (a) $\dfrac{4L^2K^3}{L^{-2}}$ (b) $\dfrac{2L}{(1/L)}$ (c) $5\,\dfrac{L^2}{(5LK)^{0.5}}$

7. $\dfrac{\left(\dfrac{4(0.6)K^{0.4}L^{-0.4}}{L}\right)}{\left(\dfrac{4(0.4)K^{-0.6}L^{0.6}}{K}\right)}$

8. $\dfrac{\left(\dfrac{4L^\alpha K^{\beta-1}}{K}\right)}{\left(\dfrac{4L^{\alpha-1}K^\beta}{L}\right)}$

9. (a) $2Q\sqrt{Q}$ (b) $\dfrac{2Q}{3P\sqrt{P}}$

10. (a) $\dfrac{2L^{0.25}K^{0.75}}{4L}$ (b) $\left(\dfrac{2L^{-1}}{4L}\right)^2$

11. e^2e^x **12.** $5e^{5x}e^x$ **13.** $\dfrac{e^{2x+3}}{e^{5x-3}}$ **14.** $120e^{-0.5t}(1/e^t)$ **15.** $e^{0.75t}(1-e^t)$

16. $\left(\dfrac{e^{2t}}{5}\right)^2$ **17.** $\left(\dfrac{1}{e^{-5t}}\right)^{0.25}$ **18.** $(1+e^t)^2$ **19.** $5(e^{2t}+e^{3t})$ **20.** $e^{-5t}(e^{2t}+e^{3t})$

4.2.2 Solving equations that contain exponentials

Worked Example 4.14 demonstrates the solution of various exponential equations.

WORKED EXAMPLE 4.14

SOLVING EXPONENTIAL EQUATIONS

Solve the equations

(a) $2^x = \dfrac{1}{16}$ (b) $5^{x+3}5^x = 5^4$ (c) $2^P3^Q = \dfrac{2}{3}$ (d) $(1+K)^{0.4} = 2$ (e) $\dfrac{e^{2x}}{e^{4+x}} = 1$

Solution

The method used for solving exponential equations is straightforward. As always:

- Simplify the exponential equation by writing each side of the equation as $(\text{base})^{\text{power}}$.
- Since the bases on each side of the equation are identical, then if the LHS is to be equal to the RHS, then the indices must also be identical, so equate the indices. For example, $5^x = 5^{2.5} \rightarrow x = 2.5$.

☺ **Remember**, the solution of an equation is the value of the variable for which the LHS = RHS.

Using these ideas, the exponential equations are solved as follows:

(a)

$$2^x = \frac{1}{16}$$

$$2^x = \frac{1}{2^4}$$

$$2^x = 2^{-4}$$

equate indices

$$x = -4$$

(b)

$$5^{x+3}5^x = 5^4$$

$$5^{x+3+x} = 5^4$$

$$5^{2x+3} = 5^4$$

equate indices

$$2x + 3 = 4$$

$$2x = 4 - 3 = 1$$

$$x = \frac{1}{2} = 0.5$$

(c)

$$2^P3^Q = \frac{2}{3}$$

$$2^P3^Q = 2^13^{-1}$$

equate indices for base 2

$$P = 1$$

equate indices for base 3

$$Q = -1$$

check solutions

$$2^13^{-1} = 2^13^{-1}$$

(d)

$$(1+K)^{0.4} = 2$$

$$((1+K)^{0.4})^{1/0.4} = 2^{1/0.4}$$

raise each side to power

$$\frac{1}{0.4}$$

$$(1+K)^{0.4\times 1/0.4} = 2^{2.5}$$

$$1 + K = 5.6569$$

$$K = 4.6569$$

(e)

$$\frac{e^{2x}}{e^{4+x}} = 1$$

$e^{2x-(4+x)} = 1$ subtract the index of the divisor (rule 2)

$e^{x-4} = e^0$ simplify LHS, on RHS $1 = e^0$ (note 1)

$x - 4 = 0$ bases the same, so equate the indices

$x = 4$

OR

$$\frac{e^{2x}}{e^{4+x}} = 1$$

$e^{2x} = e^{4+x}$ multiply both sides by e^{4+x}

$2x = 4 + x$ equate the indices

$2x - x = 4$ therefore $x = 4$

Note: In part (c) other solutions exist, but finding them is beyond the scope of this book. In part (d) the power on the bracket containing K must be 1, in order to gain access to K inside the bracket. The index on the bracket is multiplied by 1/index to obtain the required 1. But as for all equations, both sides must be operated on in the same way, so each side is raised to the power 1/index.

Progress Exercises 4.6 Use the Rules for Indices to Solve Certain Equations

Solve the equations in questions 1 to 20 for the variable indicated in square brackets.

1. $2^x = \frac{1}{\sqrt{16}}$ [x]
2. $a^{0.5Q} = a^4$ [Q]
3. $\left(\frac{1}{P^3}\right)^4 = (P)^x$ [x]
4. $\frac{2^x}{4} = 2$ [x]
5. $3^{Q+2} = 9$ [Q]
6. $125 = 5^x$ [x]
7. $\left(\frac{1}{K}\right) = 8$ [K]
8. $\frac{4}{K^{0.5}} = 8$ [K]
9. $200 = \frac{10^x}{0.5}$ [x]
10. $\sqrt[5]{t} = 2$ [t]
11. $\frac{2}{5^t} = \frac{2}{1}$ [t]
12. $(1 + 2t)^{0.5} = 12$ [t]
13. $\sqrt{1 + 2t + t^2} = 4$ [t]
14. $16 - \left(\frac{1}{8}\right)^t = 0$ [t]
15. $16 - \left(\frac{1}{8}\right)^{-2t} = 0$ [t]
16. $3^x 3^{x+1} = \sqrt{9}$ [x]
17. $\sqrt{t - 3} = 2$ [t]
18. $\frac{2}{4^t} = \frac{1}{2}$ [t]
19. $\frac{2}{K} + \frac{5}{K} = \frac{1}{2}$ [K]
20. $\frac{4}{L} + L = -4$ [L]
21. $e^x = e^5 e^{2.5}$ [x]

22. $e^{5x} = \frac{1}{e^{x-5}}$ **23.** $3e^{5t+3} = \frac{3}{e^t}$ $[t]$ **24.** $5 + 2e^t = \frac{2}{e^5 e^{t+1}} + 5$ $[t]$

4.2.3 Applications of exponential functions

❐ The laws of growth

Exponential functions to base e describe growth and decay in a wide range of systems, as mentioned above. There are three main laws of growth: unlimited, limited and logistic growth. Each of these is now described.

❐ Unlimited growth

Unlimited growth is modelled by the equation $y(t) = ae^{rt}$, where a and r are constants.

Examples: Investment and some models of population growth.

WORKED EXAMPLE 4.15

UNLIMITED GROWTH: POPULATION GROWTH

The population of a village was 753 in 1980. If the population grows according to the equation

$$P = 753e^{0.03t}$$

where P is the number of persons in the population at time t.

(a) Graph the population equation for $t = 0$ (in 1980) to $t = 30$ (in 2010). From the graph, estimate the population (i) in 1990 and (ii) in 2000.
(b) Confirm your answers algebraically.
(c) In what year will the population reach 1750 persons?

Solution

(a) The general shape of the exponential is known; however, since the graph is being used for estimations, an accurate graph over the required interval, $t = 0$ to $t = 30$, is required. Using the e^x key on your calculator, calculate a table of values for different time periods. A sample of values is given in Table 4.12.

The points in Table 4.12 are used to plot the exponential function in Figure 4.17. From Figure 4.17 you can obtain the required population figures:

(i) In 1990 ($t = 10$) the population is 1016 persons.
(ii) In 2000 ($t = 20$) the population is 1372 persons.

Table 4.12 Population values for different time periods

t	$P = 753e^{0.03t}$
0	753
5	874.9
10	1016.4
15	1180.9
20	1372.1
25	1594.1
30	1852.1

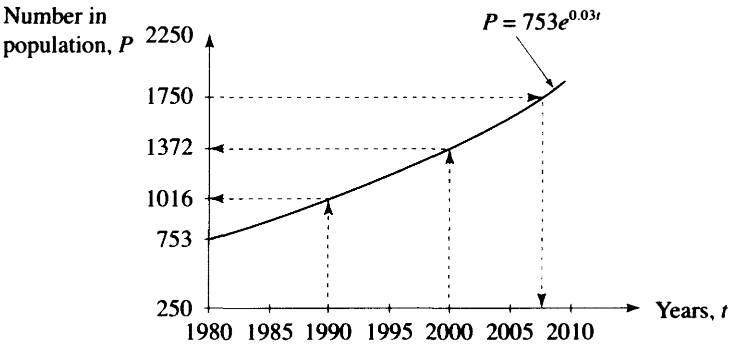


Figure 4.17 Population growth

(b) The values in (i) and (ii) can be confirmed algebraically as follows. In 1990 $t = 10$, therefore substitute $t = 10$ into equation (4.10) and evaluate P, that is, $P = 753e^{0.03t} = P = 753e^{0.03(10)} = 1016.44 = 1016$ since we cannot have 0.44 persons. In 2000 $t = 20$, therefore substitute $t = 20$ into equation (4.10) and evaluate P, that is, $P = 753e^{0.03t} = P = 753e^{0.03(20)} = 1372.05 = 1372$ persons.

(c) In Figure 4.17 draw a horizontal line across from population = 1750. When this line cuts the graph, draw a vertical line down to the horizontal axis. This line cuts the horizontal axis at $t = 28.1$. The year is $1980 + 28.1 = 2008.1$, or during 2008.

We now attempt to solve for t algebraically. As P is known, substitute $P = 1750$ into equation (4.10) and solve for t:

$$P = 753e^{0.03t}$$

$$1750 = 753e^{0.03t}$$

$$\frac{1750}{753} = e^{0.03t}$$

$$t = ?$$

We cannot solve for t as t is part of the index. To solve for a variable when the variable is part of the index, **logs** needs to be used. Logarithmic functions are generally called logs. Logs are covered in Section 4.3.

❐ Limited growth

Limited growth is modelled by the equation $y(t) = M(1 - e^{-rt})$, where M and r are constants.

Examples: consumption functions, amount of random information which can be memorised, sales with advertising (Progress Exercises 4.8, question 2), electrical and mechanical systems.

WORKED EXAMPLE 4.16

LIMITED GROWTH: CONSUMPTION AND CHANGES IN INCOME

A consumption function is modelled by the equation

$$C = 500(1 - e^{-0.3Y})$$

Graph the consumption function over the interval $0 < Y < 20$. Use the graph to describe how consumption changes as income increases.

Solution

Using your calculator, calculate C for several values of Y between 0 and 20. This is illustrated in Table 4.13. These points are used to plot the graph in Figure 4.18, which relates consumption as a function of income. The shape of the graph indicates that as income increases, consumption increases at a decreasing rate towards an upper limit of $C = 500$.

Table 4.13 Consumption values for different income levels

Y	$C = 500(1 - e^{-0.3Y})$
0	0
4	349
8	455
12	486
16	496
20	499

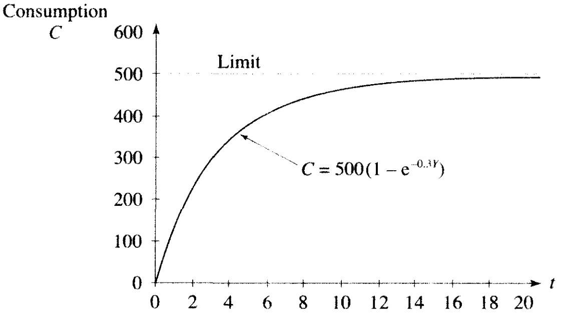


Figure 4.18 Consumption with limited growth

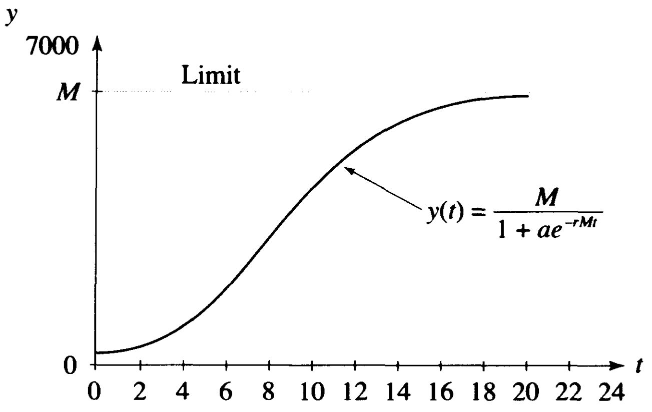


Figure 4.19 Logistic growth

☐ Logistic growth

Logistic growth is modelled by the equation

$$y(t) = \frac{M}{1 + ae^{-rMt}}$$

where M, a and r are constants. Logistic growth is modelled in Figure 4.19.

Examples: consumption functions, constrained populations, growth of epidemics, sales. (Progress Exercises 4.8, question 3).

Progress Exercises 4.7 Graphs of Exponential Functions: Limited and Unlimited Growth

Graph the functions in questions 1 to 6. For each graph from questions 1 to 4, describe how it differs from the graph for $y = e^t$. For questions 1 to 6 state whether the graph displays growth or decay and whether this is limited or unlimited.

1. 2^t, e^t, 1^t, 0.5^t plot on the same diagram, use intervals of 0.5 for $-1 < t < 2$

2. e^t, $-2e^t$, $3e^t$ plot on the same diagram, use intervals of 0.5 for $-1 < t < 2$

3. $e^{0.2t}$, e^t, e^{2t} plot on the same diagram, use intervals of 0.5 for $-1 < t < 1$

4. $e^{-0.2t}$, e^t, $-e^{-0.2t}$ plot on the same diagram, use intervals of 0.5 for $-8 < t < 8$

5. $e^{-t}, 1 + e^{-t}, 1 - e^{-t}$ plot on the same diagram, use intervals of 0.5 for $0 < t < 5$

6. $10(1 - e^{-0.2t})$ use intervals of 0.5 for $0 < t < 5$

Progress Exercises 4.8 Solve Limited, Unlimited and Logistic Growth Problems

1. In a given region, the number (000s) in the population aged 60 and over is expected to grow according to the equation

$$P = 125.5e^{0.012t}, \text{ where } t \text{ is given in years}$$

(a) If $t = 0$ at the beginning of 1995, calculate the numbers at the beginning of 1995.
(b) Calculate the numbers in the population at the end of 10, 30, 60, 90, 100 years.
(c) Graph the population for the years 1995 to 2095.
Comment on the general trend in the population.

2. Sales for a new magazine are expected to grow according to the equation

$$S = 200\,000(1 - e^{-0.05t}), \text{ where } t \text{ is given in weeks.}$$

(a) Calculate the number of magazines sold after one week.
(b) Calculate the number of magazines sold after 5, 20, 35, 45, 50, 52 weeks.
(c) Plot sales over the first 52 weeks.
Comment on the general trend in sales.

3. A virus is thought to spread through a chicken farm according to the equation

$$N = 800\frac{1}{1 + 790e^{-0.1t}}$$

where N is the number of infected chickens, t is in days.
(a) How many chickens are infected at $t = 0$?
(b) Calculate the number of chickens infected after 0, 20, 40, 60, 80, 100 days.
(c) Will all the chickens become infected eventually? Give reasons for your answer.

4.3 Logarithmic Functions

At the end of this section you should be able to:

- Define the log of a number and convert from log form to exponential form
- Recognise graphs of logarithmic functions and understand their main properties
- Use the rules for logs to simplify expressions containing logs
- Solve equations containing logs

4.3.1 How to find the log of a number

In Section 4.2 exponential equations, such as $2^x = 2^9$, were solved by equating the indices. This made sense as the bases are already identical, so if the indices are identical, then

LHS = RHS. However, in Worked Example 4.15 we attempted to solve the equation

$$\frac{1750}{753} = e^{0.03t}$$

$$2.3240 = e^{0.03t}$$

At that point, we could not solve for t. First of all, the LHS is not in the form e^{power}, so without identical bases on each side there is no point in equating indices. The problem here is trying to solve for t, when t is part of the power of the exponential. A method is required that will bring the power down onto the line, thus converting the exponential equation to an ordinary equation which can then be easily solved. Logarithms or logs provide such a method.

❐ What is the log of a number?

Logs are powers. In the following example, each number is written as 10^{power}, so the power is the log of the number:

$10 = 10^1$	$100 = 10^2$	$200 = 10^{2.30103}$	$1000 = 10^3$
$\log 10 = 1$	$\log 100 = 2$	$\log 200 = 2.30103$	$\log 1000 = 3$

But there are infinitely many numbers that may be raised to powers of 1, 2, 2.30103, 3, etc.

$2 = 2^1$	$4 = 2^2$	$4.92809 = 2^{2.30103}$	$8 = 2^3$
$\log 2 = 1$	$\log 4 = 2$	$\log 4.92809 = 2.30103$	$\log 8 = 3$

So obviously the base of the power plays a role and must be stated along with the log of the number, as follows:

$10 = 10^1$	$100 = 10^2$	$200 = 10^{2.30103}$	$1000 = 10^3$
$\log_{10} 10 = 1$	$\log_{10} 100 = 2$	$\log_{10} 200 = 2.30103$	$\log_{10} 1000 = 3$
$2 = 2^1$	$4 = 2^2$	$4.92809 = 2^{2.30103}$	$8 = 2^3$
$\log_2 2 = 1$	$\log_2 4 = 2$	$\log_2 4.92809 = 2.30103$	$\log_2 8 = 3$

Therefore, in general, the log of a number may be found as follows:

$$\boxed{\begin{array}{c} \text{Number} = \text{base}^{\text{power}} \\[1ex] \text{Number} = \text{base}^{\text{power}} \\ \uparrow\underline{\hspace{3cm}}|\quad\downarrow \\ \log_{\text{base}}(\text{Number}) = \text{power} \end{array}} \tag{4.11}$$

then

the base of the power becomes the base of the log, the power drops down

where the number and base must be real positive numbers (otherwise we may end up dealing with complex numbers).

Logs to base 10 and logs to base *e*

If logs of all numbers were tabulated for every possible base, there would be endless sets of tables, so for convenience, two sets of tables are available, one for logs to base 10 and one for logs to base e. On your calculator, these may be found on the **log key** (for log to base 10) and on the **ln key** (for log to base e).

Notation

The notation $\log(x)$ is understood to be $\log_{10}(x)$; for example, $\log(7)$ means $\log_{10}(7)$. The notation $\ln(x)$ is understood to be $\log_e(x)$; for example, $\ln(7)$ means $\log_e(7)$. Check on your calculator that $\log 12 = 1.0791812$ and $\ln 12 = 2.4849066$.

Note: Rule 4 (below) gives a simple method for taking logs to base n and converting them into logs to base 10 or logs to base e.

Progress Exercises 4.9 Logs and Transforming Equations from Index to Log Form

1. Use your calculator to evaluate the following:

(a) $\log(23)$
(b) $\log(57.98)$
(c) $\log(0.22)$
(d) $\ln(57.98)$
(e) $\dfrac{\log(57.98)}{\log(21.1)}$
(f) $\ln(0.78)\ln(9.15)$
(g) $\dfrac{\ln(57.98)}{\ln(21.1)}$
(h) $\dfrac{\ln(57.98)}{\log(21.1)}$
(i) $3\log(27) - 2\log(35)$

2. Write each of the following exponential equations in log-form:

(a) $N = a^{\text{power}}$
(b) $10 = 3^{2.0959}$
(c) $3 = 10^{0.4771}$
(d) $7.389 = e^2$
(e) $2^x = 10$
(f) $2 + 3^x = 10$
(g) $5(2^x) = 20$
(h) $2^x 2^{2x} = 15$
(i) $3(5)^Q = 12$
(j) $(5)^{Q+1} = 10$
(k) $(10)^{4x+3} = 25$
(l) $P^{0.3} = 12$
(m) $5K^{-0.5} = 28$
(n) $2^{0.3} = 25L^{0.9}$
(o) $(K + 10)^{0.8} = 2$

Solving equations containing exponentials a^x where a is any real number

Since logs are tabulated for base 10 and base e, it is now possible to solve certain equations which contain exponentials 10^x or e^x.

WORKED EXAMPLE 4.17

USE LOGS TO SOLVE CERTAIN EQUATIONS

Solve the following equations:

(a) $31 = 10^x$ (b) $25(10)^{2t} = 208$ (c) $38 + 12e^{-0.5t} = 208$

Solution

Step 1: Arrange the exponential equation into the form number = $\text{base}^{\text{power}}$, where the unknown variable is in the power.
Step 2: Convert from index form to log form, as indicated in equation (4.11), reducing the exponential equation to a linear equation.
Step 3: Solve the linear equation for the unknown.

(a)

Step 1: $31 = 10^x$

Step 2: $\log_{10}(31) = x$

Step 3: $1.4914 = x$

Solution $x = 1.4914$

(b)

$$25(10)^{2t} = 208$$

$$(10)^{2t} = \frac{208}{25} = 8.32$$

Step 1: $(10)^{2t} = 8.32$

Step 2: $2t = \log_{10}(8.32)$

Step 3: $t = \dfrac{0.9201}{2}$

$$= 0.4601$$

Solution $t = 0.4601$

(c)

$$38 + 12e^{-0.5t} = 208$$

$$12e^{-0.5t} = 208 - 38 = 170$$

Step 1: $e^{-0.5t} = \dfrac{170}{12}$

$$= 14.1667$$

Step 2:
$-0.5t = \log_e(14.1667)$

$-0.5t = 2.6509$

Step 3: $t = \dfrac{2.6509}{-0.5}$

$$= -5.0179$$

Solution $t = -5.0179$

Next return to Worked Example 4.15. This was not completed since the mathematical methods covered in the text up to that point could not solve the equation $P = 753e^{0.03t}$ for t when $P = 1750$. Now logs may be used to reduce this equation to an ordinary linear equation in t that can be solved for t.

WORKED EXAMPLE 4.18

FINDING THE TIME FOR THE GIVEN POPULATION TO GROW TO 1750

With the definition of a log function, part (c) of Worked Example 4.15 can now be finished. Given $P = 753e^{0.03t}$ find t when the population P is 1750.

Solution

Substitute $P = 1750$ into the equation for population and solve for t.

Step 1	Step 2	Step 3
$1750 = 753e^{0.03t}$ $\frac{1750}{753} = e^{0.03t}$ $2.324 = e^{0.03t}$	$\log_e(2.324) = 0.03t$ $\ln(2.324) = 0.03t$ $0.8433 = 0.03t$	$\frac{0.8433}{0.03} = t$ $28.11 = t$

So the year in which the population reaches 1750 persons is $1980 + 28.11 = 2008.11$ or during the year 2008.

Progress Exercises 4.10 Solve Equations and Applications with Logs

Solve each of the equations in questions 1–9 for the unknown variable.

1. $28 = 10^x$

2. $5 = 3(10)^x$

3. $5 = 10^{x-4}$

4. $125 = e^t$

5. $125 = 230e^t$

6. $125 = 230e^{-t}$

7. $28 = 5 + 2e^{-0.5t}$

8. $40 = 66(1 - e^{-t})$

9. $5 = 10^x$

10. If the total value of sales, S (in £000), is given by the equation $S = 500(1 - e^{-0.16t})$, where time t is given in weeks, starting with $t = 0$ when the product is first marketed.
(a) Find the total sales (i) at the beginning of week one, (ii) at the end of week one, (iii) at the end of week 5.
(b) Graph the sales for $0 < t < 10$.
From the graph estimate the time, in weeks, taken for sales to reach £400 000. Confirm your answer algebraically. What is the maximum sales attainable?

11. Scientists estimate that the maximum carrying capacity of a lake is 6000 and that the number of fish increase according to the equation

$$P(t) = \frac{6000}{1 + 29e^{-0.4t}}$$

where t is given in years and $P(t)$ is the fish population at time t.
(a) Find the fish population in the lake at $t = 0$, $t = 4$, $t = 10$.
(b) Graph the equation for $0 < t < 12$.
(c) From the graph estimate the time taken for the fish population to reach 1000, 3000, 4000. Confirm your answers algebraically.

12. The population of a country is changing according to the equation $P = 5.2e^{-0.001t}$, where t is in years, P is in millions.
(a) Calculate the numbers in the population when $t = 0$, 10, 50 years. If $t = 0$ at the beginning of 1998, state the years represented by $t = 0, 10, 50$. Is the population increasing or decreasing in time?

(b) Calculate the number of years from $t = 0$ until the population declines to 4 million. State the year in which this happens.

4.3.2 Graphs and properties of logarithmic functions

The above examples and the questions in Progress Exercises 4.10 solve exponential equations where the base is 10 or e only. To solve equations for bases other than 10 or e requires a slightly more detailed study of the log functions and the rules for manipulating logs.

The graph of any function reveals many of its characteristics and properties. It is necessary to have an overview of a function and its properties if the function is to be used in equations, and to understand that a function may not exist for all real values of the independent variable, etc.

WORKED EXAMPLE 4.19

GRAPHS OF LOGARITHMIC FUNCTIONS

Graph $\log(x)$ and $\ln(x)$ on the same diagram for $-1 < x < 2$. From the graphs summarise the properties of the log functions. How does the size of the base affect the shape of the graph?

Solution

Calculate a table of values for each function, using your calculator to evaluate the logs or, better still, use Excel. No real value exists for the logs of zero (doesn't exist) or logs of negative numbers (complex). This is indicated on the calculator by *−E−* and in Excel by *#NUM*. Table 4.14 outlines the points used to plot the graphs in Figure 4.20. From Figure 4.20, some properties of log functions can be deduced:

- there are no real logs of negative numbers
- logs of numbers less than one are negative

Table 4.14 Values of log(x) and ln(x)

x	log(x)	ln(x)
−0.2	#NUM!	#NUM!
0	#NUM!	#NUM!
0.2	−0.7	−1.61
0.4	−0.4	−0.92
0.6	−0.22	−0.51
0.8	−0.1	−0.22
1	0	0
1.2	0.08	0.18
1.4	0.15	0.34
1.6	0.20	0.47
1.8	0.26	0.59

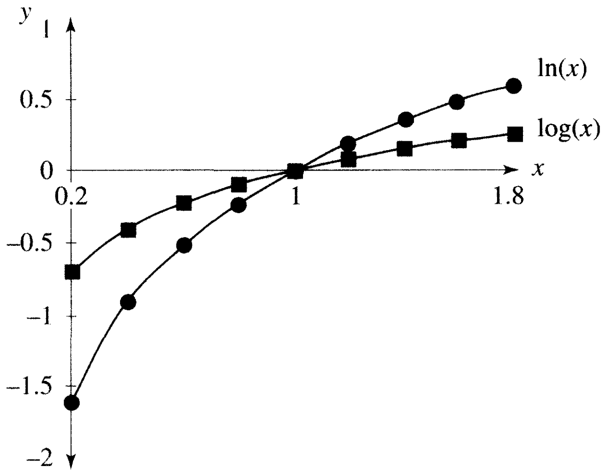


Figure 4.20 Graphs of log(x) and ln(x)

- log(1) = 0 for any base
- logs of numbers greater than one are always positive

In addition, the graph of $\log_e(x)$ is steeper, that is, it increases and decreases more rapidly than $\log_{10}(x)$.

4.3.3 Rules for logs

Finally, we need to state the rules for combining and manipulating log functions.

Operation	Notation	Example
Rule 1 Add	$\log_b(M) + \log_b(N) \Leftrightarrow \log_b(MN)$	$\ln(4) + \ln(29) = \ln(4 \times 29)$ $1.3863 + 3.3673 = \ln(116)$ $4.7536 = 4.7536$
Rule 2 Subtract	$\log_b(M) - \log_b(N) \Leftrightarrow \log_b\left(\frac{M}{N}\right)$	$\log(90) - \log(26) = \log\left(\frac{90}{26}\right)$ $1.9542 - 1.4150 = \log(3.4615)$ $0.5392 = 0.53926$
Rule 3 Log of an exponential	$\log_b(M^z) \Leftrightarrow z \log_b(M)$	$\log(5^3) = 3\log(5)$ $\log(125) = 3(0.69897)$ $2.09691 = 2.09691$

Operation	Notation	Example
Rule 4 Change of base	$\log_b(N) \Leftrightarrow \dfrac{\log_x(N)}{\log_x(b)}$ the new base, x, is usually 10 or e since both are available on the calculator	$\log_2(16) = \dfrac{\log(16)}{\log(2)} = \dfrac{1.2041}{0.3010} = 4.0$ or change to base e $\log_2(16) = \dfrac{\ln(16)}{\ln(2)} = \dfrac{2.7726}{0.6931} = 4.0$

The $\Leftrightarrow$ sign indicates that equations can be operated both ways, as in the following worked examples.

WORKED EXAMPLE 4.20

USING LOG RULES

Use the log rules to simplify the following expressions to a single term, if possible:

(a) $\log_b(25) + \log_b(70) - \log_b(55)$
(b) $4\log_x(7) - 3\log_x(0.85) + \log_x(10)$
(c) $12\log_b(12) + 3\log_x(8.25) - 2\log_b(5)$

Then evaluate each expression if $b = 3$, $x = e$.

Solution

(a) $\log_b(25) + \log_b(70) - \log_b(55)$	all logs have the same base
$= \log_b(25 \times 70) - \log_b(55)$	so use rule 1 to add the first two terms
$= \log_b\left(\dfrac{25 \times 70}{55}\right)$	use rule 2 to divide by the negative term
$= \log_b(31.818)$	this cannot be evaluated unless b is given a value

If $b = 3$ then using rule 4 to change base 3 to base 10, we have

$$\log_3(31.818) = \frac{\log(31.818)}{\log(3)} = \frac{1.50267}{0.4771} = 3.14946$$

(b) In $4\log_x(7) - 3\log_x(0.85) + \log_x(10)$ all logs have the same base, so proceed by using rule 3 in reverse, bringing the numbers in as powers before adding or subtracting. If you look at rules 1 and 2, there are no numbers outside the log terms before adding or subtracting:

$$4\log_x(7) - 3\log_x(0.85) + \log_x(10)$$
$$= \log_x(7)^4 - \log_x(0.85)^3 + \log_x(10)$$

$= \log_x\left(\frac{7^4}{0.85^3}\right) + \log_x(10)$ use rule 2 to divide by the negative ( )

$= \log_x(3909.6275) + \log_x(10)$

$= \log_x(3909.6275 \times 10)$

$= \log_x(39\,096.275)$ use rule 1 multiply by the positive ( )

If $x = e$ we obtain $\ln(39\,096.275) = 10.57378$.

(c) In this problem there are two different bases, therefore combine terms with the same base only:

$12 \log_b(12) + 3 \log_x(8.25) - 2 \log_b(5)$

$= \log_b(12)^{12} + \log_x(8.25)^3 - \log_b(5)^2$ rule 3 in reverse, bring in the constants as powers

$= \log_b\left(\frac{12^{12}}{5^2}\right) + \log_x(561.516)$ rule 2 for base b terms

$= \log_b(3.56644 \times 10^{11}) + \log_x(561.516)$ simplify the numbers

This cannot be simplified to a single term as the bases are different, but since $b = 3$ and $x = e$, the expression may be evaluated as

$$\log_3(3.56644 \times 10^{11}) + \log_e(561.516) = \frac{\ln(3.56644 \times 10^{11})}{\ln(3)} + \ln(561.516)$$

$$= 30.543$$

4.3.4 Solving equations using the log rules

There are several ways of using logs to solve equations. Worked Example 4.17 solved equations of the form number = base$^{\text{power}}$, but only for base 10 or base e. The following worked example uses rule 3 for logs (above) to solve such equations when the base is other than 10 or e. In addition, the method of solution also involves taking logs of both sides of the equation. This is called 'operating' on both sides of the equation. Given an equation

$$\text{LHS} = \text{RHS} \qquad \text{then} \qquad \log(\text{LHS}) = \log(\text{RHS})$$

Use any positive real base, preferably base 10 or e since they are available on a calculator. **Note:** You will already have 'operated' on both sides of equations, for example, multiplying both sides by constants, adding/subtracting terms to both sides.

WORKED EXAMPLE 4.21

SOLVING CERTAIN EQUATIONS WITH RULE 3 FOR LOGS

Solve the following equations:

(a) $20 = 3(1.08)^x$ (b) $20 + (2.4)^{2x} = 32.5$

Method	**(a)**	**(b)**
Step 1: Write the equation as number = base$^{\text{power}}$	$20 = 3(1.08)^x$ $20/3 = 1.08^x$	$20 + (2.4)^{2x} = 32.5$ $(2.4)^{2x} = 32.5 - 20 = 12.5$
Step 2: Take logs of both sides of number = base$^{\text{power}}$; if the base on the RHS is e then take logs to base e, else take logs to base 10	$\log(20/3) = \log(1.08)^x$	$\log(2.4^{2x}) = \log(12.5)$
Step 3: Simplify using rule 3, which says $\log(a^x) = x\log(a)$	$\log(20/3) = x\log(1.08)$	$2x\log(2.4) = \log(12.5)$
Step 4: Solve using a calculator	$\frac{\log(20/3)}{\log(1.08)} = x$ $x = \frac{0.8239}{0.0334} = 24.65$	$2x = \frac{\log(12.5)}{\log(2.4)} = \frac{0.3802}{1.0969}$ $x = \frac{1}{2} \times \frac{0.3802}{1.0969} = 0.1733$

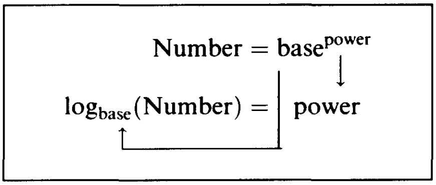
This method is useful in solving certain equations involving compound interest (Chapter 5). For immediate practice on this method, try solving questions 1 to 9 of Progress Exercises 4.10.

Solving equations that contain logs

If we attempt to solve the equation

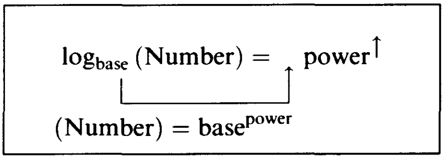
$$\log(x + 2) = 2.5$$

we first solve for $(x + 2)$. But $(x + 2)$ is contained within the expression $\log(x + 2)$. To get at $(x + 2)$ we must undo or reverse the operation by which logs were taken in the first place. This is described as taking **antilogs**. Taking logs is this process:


Number = base$^{\text{power}}$
$\log_{\text{base}}$(Number) =
power

the base of the index becomes
the base of the log, the power then drops down

Reversing the log process gives us this:


$\log_{\text{base}}$ (Number) =
power
(Number) = base$^{\text{power}}$

the base of the log becomes
the base of the power, the power goes up

For example,

$$\log_2(8) = 3$$
$$8 = 2^3$$

This is described as going from log form to index form. The following worked examples show how to solve equations which contain logs and exponentials.

WORKED EXAMPLE 4.22

SOLVE EQUATIONS CONTAINING LOGS AND EXPONENTIALS

Solve the following equations:

(a) $\log(x+2) = 2.5$ (b) $2\ln(x) - \ln(x+1) = 0$

(c) $e^{x+5} = 1.56$ (d) $2^x 2^{x+1} = 7$

Solution

(a) $\log(x+2) = 2.5$ to isolate x, it is necessary to go from log form to index form

$(x+2) = 10^{2.5}$ $10^{2.5}$ is called the antilog of 2.5; use the second function on the log key

$$x = 316.22777 - 2$$
$$= 314.22777$$

(b) $2\ln(x) - \ln(x+1) = 0$

$\ln x^2 - \ln(x+1) = 0$ rule 3, bring the 2 in as a power

$\ln\left(\dfrac{x^2}{x+1}\right) = 0$ rule 2

$\left(\dfrac{x^2}{x+1}\right) = e^0 = 1$ going from log form to index form

$x^2 = 1(x+1)$ multiplying both sides by $(x+1)$ to avoid fractions

$x^2 - x - 1 = 0$ a quadratic, so solve the quadratic using the formula

The solutions are $x = -0.618$ and $x = 1.618$.

(c) We have $e^{x+5} = 1.56$; x is in the index, so take logs to get x down. There are two possible methods, the second being the more usual. Check the solutions.

Method 1: Go from log to index

$$x + 5 = \ln(1.56)$$
$$x = 0.4447 - 5$$
$$x = -4.5553$$

Method 2: Take logs of both sides

$$\ln(e^{x+5}) = \ln(1.56)$$
$$(x+5)\ln(e) = 0.4447 \quad \ln(e) = 1$$
$$x + 5 = 0.4447$$
$$x = 0.4447 - 5 = -4.5553$$

(d)

$$
\begin{aligned}
2^x 2^{x+1} &= 7 \\
2^{2x+1} &= 7 && \text{use rule 1 for indices} \\
\log(2^{2x+1}) &= \log(7) && \text{take logs of both sides} \\
(2x+1)\log(2) &= \log(7) && \text{use rule 3 for logs} \\
(2x+1) &= \frac{\log(7)}{\log(2)} = \frac{0.8451}{0.3010} = 2.8074 && \text{evaluate and simplify} \\
2x &= 2.8074 - 1 \\
x &= 0.9037 && \text{check the solution}
\end{aligned}
$$

Progress Exercises 4.11 Transforming Equations from Log to Index Form and Use of Log Rules

Write each of the following log equations in exponential form, if possible:

1. $\log(x+5) = 1.2$

2. $3\log(x+2) = 3.5$

3. $2 + \ln(t-5) = 2.5$

4. $\dfrac{\log(x+1)}{2.5} = -0.5$

5. $3\log(2-2x) = 0$

6. $\frac{2}{3}\log(x^2+1) = \frac{1}{5}$

7. $-0.65 = \log(0.5x)$

8. $\log(x)(x+1) = 1$

9. $2 + \ln(x-4.5) = 4$

10. $\ln\left(\dfrac{x+1}{x}\right) = 3.5$

11. Solve each of the equations given in questions 1 to 10, giving your answer correct to four decimal places.

12. Solve each of the following equations:

(a) $3 = 7(10)^x$ (b) $5 = 12e^x$ (c) $5 = 12(1.05)^x$ (d) $5 = 12(1.05)^{4x}$

(e) $e^t = 7 - 2e^t$ (f) $80 = 25 + (1.5)^x$ (g) $20 = 40(1 - e^{-t})$

(h) $2.3(1.05)^{-2x} = 4$

Simplify each of the expressions in questions 13 to 17, writing your answers as the log of a single number. Confirm that your answer is correct by evaluating the original expression and the simplified expression.

13. $2\log(12) + \log(20)$

14. $\ln(125) - \ln(80) + \ln(28)$

15. $3\ln(138) - 2\ln(95)$

16. $8\ln(2) + 3\ln(8) - 2\ln(3)$

17. $\ln(4) - 2\ln(2) + \ln(80)$

In questions 18 to 23, show that the expression on the LHS simplifies to that on the RHS.

18. $\log(x) + \log(x - 5) = \log(x^2 - 5x)$

19. $2 \log(x - 2) - \log(2x) = \log\left[\dfrac{(x-2)^2}{2x}\right]$

20. $\ln\left[\dfrac{x^3}{\sqrt{x+1}}\right] = 3 \ln(x) - 0.5 \ln(x + 1)$

21. $\ln(x^5 e^x) = 5 \ln(x) + x$

22. $0.2 \ln(x) - 0.3 \ln(2x) = \ln\left(\dfrac{1}{2^{0.3}x^{0.1}}\right)$

23. $\log\left(\dfrac{x+1}{1-x}\right) + \log\left(\dfrac{1}{x+1}\right) = -\log(1 - x)$

24. Evaluate the following, giving your answer correct to four decimal places:
(a) $\log_3(27)$ (b) $\log_5(51)$ (c) $2 \log_{3.5}(30)$

25. Solve the following equations:
(a) $2 \log(5x) - \log(2x) = 2.5$ (b) $\log(Q) - \log\left(\dfrac{Q}{Q+1}\right) = 0.8$

4.4 Hyperbolic Functions of the Form *a*/(*bx* + *c*)

At the end of this section you should be able to:

- Define and graph hyperbolic functions of the form $a/(bx + c)$
- Define and solve applications which are modelled by hyperbolic functions

Note: The hyperbolic functions in this section are called rectangular hyperbolic function and are based on the rectangular hyperbola $y = 1/x$. There is another class of hyperbolic functions, such as $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$, which are not considered in this text.

4.4.1 Define and sketch rectangular hyperbolic functions

A function which is given by the equation, $y = a/(bx + c)$ is called a hyperbolic function. The simplest rectangular hyperbolic function is $y = 1/x$. The graph of $y = 1/x$ consists of two separate parts, as illustrated in Figure 4.21.

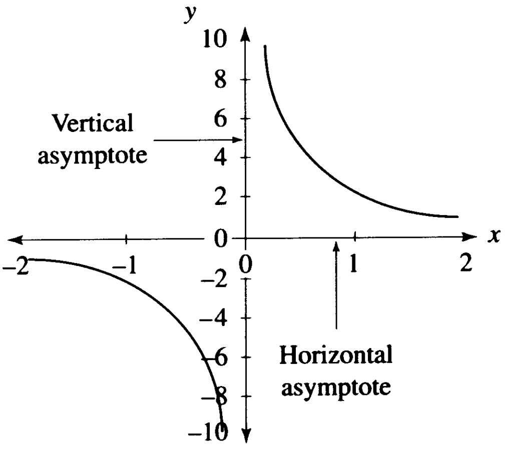


Figure 4.21 Graph of $y = 1/x$

To plot the graph, calculate a table of points such as Table 4.15. Only one point presents a problem, 1/0. There is no number defined which gives a value for division by zero; no y-value can be defined for the point $x = 0$, so we cannot plot a point ($x = 0$, $y = ?$). Not only is there a point missing from the graph, but the shape of the graph increases or decreases dramatically on either side of the undefined point. You can see this if you evaluate y for values of x close to the undefined point as in Table 4.15. In this example the y-axis is a vertical asymptote and the x-axis is a horizontal asymptote. An asymptote is a line which the curve approaches at a distance from the origin.

Table 4.15 Calculation of points for Figure 4.21

x	$y = \frac{1}{x}$
−2	−0.5
−1.5	−0.67
−1	−1
−0.5	−2
0	?
0.5	2
1	1
1.5	0.67
2	0.5

☐ Functions of the form $y = a/(bx + c)$

Functions of the form $y = a/(bx + c)$ have a similar shape to $y = 1/x$, except that division by zero no longer occurs at $x = 0$, but at $bx + c = 0$ or $x = -c/b$.

WORKED EXAMPLE 4.23

SKETCHES OF HYPERBOLIC FUNCTIONS

Sketch the functions

(i) $y = \dfrac{1}{x - 0.23}$ (ii) $y = \dfrac{1}{x} + 3$

Solution

(a)

Step 1: Calculate the value of x that gives rise to division by zero. This will determine the vertical asymptote. Division by zero occurs when the denominator is zero.

$$x - 0.23 = 0 \rightarrow x = 0.23$$

Step 2: Calculate the points of intersection with the y-axis, using the fact that $x = 0$ on the y-axis:

$$y = \frac{1}{x - 0.23} \rightarrow y = \frac{1}{0 - 0.23} = -4.35$$

Step 3: To get some idea of the curvature as the graph approaches the vertical asymptote, calculate the coordinates of some points to its left and right:

x	0	0.1	0.2	0.3	0.4
y	−4.35	−0.13	−33.33	14.29	5.88

See Figure 4.22(a).

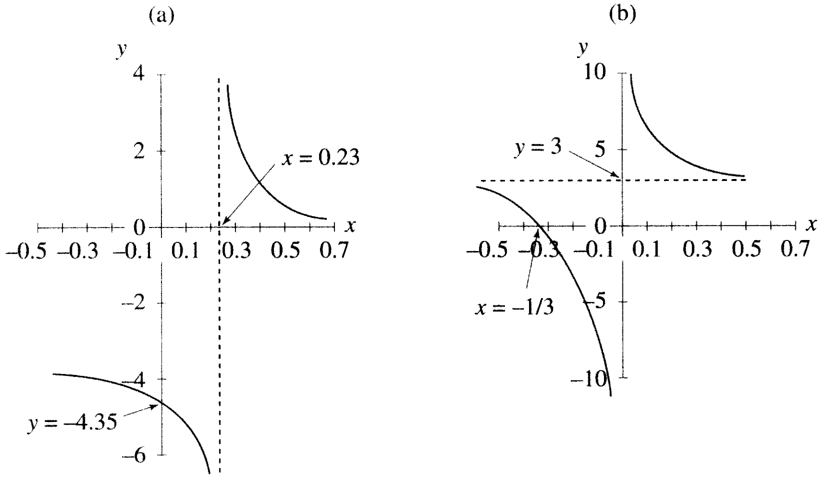


Figure 4.22 Hyperbolic functions

(b) The equation of the vertically translated function is

$$y = \frac{1}{x} + 3$$

The graph is sketched by translating $y = 1/x$ vertically up the y-axis. The point of intersection with the x-axis is easily found by solving for x when y is zero.

❒ **The main features of $y = a/(bx + c)$**

- The graph has a vertical asymptote at the value of x which results in division by zero.
- The x-axis is a horizontal asymptote.
- The graph has no maximum or minimum values.
- Otherwise the curve increases smoothly at an increasing rate or decreases at a decreasing rate.

Progress Exercises 4.12 Graphing Hyperbolic Functions

Graph the following functions, indicating asymptotes and points of intersection with the axis:

1. $y = \frac{20}{Q}$ **2.** $y = 5 + \frac{20}{Q}$ **3.** $y = \frac{4}{Q - 1}$

4. $y = \frac{4}{Q + 1}$ **5.** $y = 2 + \frac{1}{Q + 1}$ **6.** $y = \frac{4}{1 - Q}$

7. $y = 20 - \frac{4}{4 + Q}$ **8.** $y = 20 + \frac{1}{1 - 2Q}$ **9.** $y = \frac{1}{3Q - 1}$

4.4.2 Equations and applications

Functions of the form $a/(bx + c)$ model average cost, supply, demand and other functions which grow or decay at increasing rates. The solutions of equations will involve manipulating fractions (Chapter 1).

WORKED EXAMPLE 4.24

HYPERBOLIC DEMAND FUNCTION

The demand function for a good is given by the equation $Q + 1 = 200/P$ and the supply function is a linear function $P = 5 + 0.5Q$.

(a) Write the demand function in the form $P = f(Q)$.

(b) Sketch the supply and demand functions on the same diagram for $-1 \leq Q \leq 20$. From the graph estimate the equilibrium point. Confirm your answer algebraically.

Solution

(a) $$Q + 1 = \frac{200}{P}$$

$$P(Q + 1) = 200$$

$$P = \frac{200}{Q + 1}$$

(b) To sketch the graph, follow these steps:

- **General shape:** the demand function is a hyperbolic function; it has a vertical asymptote at the value of Q which results in division by zero, that is at $Q = -1$.
- **Intersection with the axis:** the graph cuts the P-axis at $Q = 0$, therefore

$$P = \frac{200}{0 + 1} = \frac{200}{1} = 200$$

The horizontal axis is an asymptote. You can show that the graph never cuts the horizontal axis as follows. Since $P = 0$ at every point on the horizontal axis (Q-axis), substitute $P = 0$ into the equation of the demand curve and find the value of Q at which the graph crosses the Q-axis, therefore

$$0 = \frac{200}{Q + 1} \quad \rightarrow \quad 0(Q + 1) = 200$$

However, there is no value of $Q + 1$ which can be multiplied by 0 and give an answer of 200. Zero multiplied by anything is zero. So there is no solution, or no point where the graph crosses the horizontal axis.

The supply function is a linear function with intercept 5 and slope 0.5. Since two points are sufficient to plot a straight line, we need one other point in addition to the intercept. Choose any value of Q then substitute this value into the equation of the supply function to calculate the corresponding P coordinate; for example, when $Q = 4$, $P = 7$. The demand and supply functions are sketched in Figure 4.23.

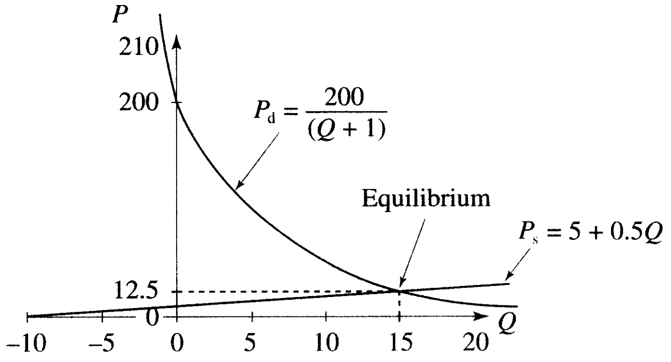


Figure 4.23 Market equilibrium

Algebraically, equilibrium exists when $P_d = P_s$ and $Q_d = Q_s$. Since each equation is in the form $P = f(Q)$, it is easiest to equate prices:

$$\frac{200}{Q+1} = 5 + 0.5Q$$

$$200 = (5 + 0.5Q)(Q + 1)$$

$$200 = 0.5Q^2 + 5.5Q + 5$$

$$0 = 0.5Q^2 + 5.5Q - 195$$

Solve this quadratic for Q. The solutions are $Q = -26$ and $Q = 15$. As the negative quantity is not economically meaningful, substitute $Q = 15$ into either original equation and solve for the corresponding equilibrium price. Therefore market equilibrium is reached when $Q = 15$ and $P = 12.5$.

Progress Exercises 4.13 Equations and Applications Based on Hyperbolic Functions

Solve the following equations in questions 1 to 6.

1. $\frac{1}{x+1} = 2$ **2.** $\frac{x}{x+4} = 3$ **3.** $\frac{10x-6}{x^2+1} = 2$

4. $\frac{5}{Q-3.5} = 10$ **5.** $\frac{Q+5}{Q-5} = Q + 1$ **6.** $\frac{1}{Q} + \frac{2}{Q} = 8$

7. The following equation relates the value of a new car (in £000) to age, in years:

$$V = 1 + \frac{84}{1+2t}$$

(a) Graph the value of the car for ages 0 to 10. Describe how the value depreciates over time.

(b) Determine how long (to the nearest year) it will take for the value to drop to £20 000.

8. The demand and supply functions for a brand of tennis shoes are

$$P_d = \frac{500}{Q+1}, \quad P_s = 16 + 2Q,$$

where P is the price per pair, Q is the quantity in thousands of pairs.

(a) Calculate the equilibrium price and quantity.

(b) Graph the supply and demand functions; hence confirm your answer graphically.

9. A new model of laptop computer is gradually replacing an earlier model. The projected sales (in thousands) for the old and new models are given by the equations

$$S_{\text{old}} = \frac{9}{t+3}, \quad S_{\text{new}} = \frac{36}{21-t}$$

where time t is in months; S is the number of laptops sold (old and new models).

(a) Calculate the number of laptops sold when $t = 0, 5, 10$. Graph the sales functions for $t = 0$ to 10. From the graph, estimate the time when the sales of both models are equal.

(b) Calculate the time when sales are equal. Did the introduction of the new model improve overall sales? Give reasons for your answer.

10. Satisfaction derived from solving maths problems is given by the utility function

$$U = 100 - \frac{100}{x+1}$$

where x is the number of problems solved.

(a) Graph the utility function for $x = 0$ to 100. Hence describe how satisfaction grows with the number of problems attempted. Does satisfaction ever decline?

(b) Calculate the number of problems that should be attempted to achieve a utility of 10, 50, 80, 100. Comment.

4.5 Excel for Non-linear Functions

In this chapter several new functions have been introduced: quadratics; cubics; exponential functions; logarithmic functions and hyperbolic functions. You should find Excel very useful for graphing these non-linear functions, even for questions set in the progress exercises. This is particularly true for the cubic functions, some of which represent typical total cost functions. Cubic equations and applications involving cubic equations were solved graphically, as the algebraic methods are beyond the scope of this text.

Excel evaluates mathematical functions such as logs and exponentials as follows: Select the function wizard key, $\boxed{f_x}$, from the Excel menu bar. A two-column menu should now appear. From the 'function category' column, select the 'math & trig' option. The second column, 'function name', is a list of names giving the format of various mathematical functions. The functions used most frequently are the following:

☺ **Remember:** Always type '=' first, to indicate that a formula is being used.

EXP:	for example, type in:	= exp(10) → means e^{10} etc.
LN:	for example, type in:	= ln(4) → means $\log_e(4)$.
LOG:	for example, type in:	= log(12, 3) → means $\log_3(12)$
LOG(10):	for example, type in:	= log 43 → means $\log_{10}(43)$

To enter any other exponential, for example, type in = 3^2.5 → means $3^{2.5}$.

The following worked examples use graphs to confirm and illustrate the solutions found by algebraic methods.

WORKED EXAMPLE 4.25

TOTAL COST FUNCTIONS WITH EXCEL

The total cost functions for two soap manufactures (Plane Soap and Round Soap) are given by equations (i) and (ii) below,

(i) $TC = 12 + 2Q$ (ii) $TC = 0.3Q^3 - 15Q^2 + 250Q$

(a) The demand function for both firms is $P = 115 - 2Q$. Determine the break-even points algebraically. Plot total revenue and TC on the same diagram for each firm.

(b) Comment on the general characteristics of the TC functions.

(c) Determine the break-even points graphically and state the range of values of Q (output) for which the firm makes a profit.

Comment on the profitability of both firms.

Solution

(a) $TR = P \times Q = 115Q - 2Q^2$

For firm (i), the break-even points are calculated algebraically by solving $TC = TR$:

$$12 + 2Q = 115Q - 2Q^2$$

$$2Q^2 - 113Q + 12 = 0$$

The solutions to this quadratic are $Q = 0.11$ and $Q = 56.4$. Therefore, when plotting the graph, choose $0 \leq Q \leq 60$.

For firm (ii), the break-even points are calculated algebraically by solving $TC = TR$:

$$0.3Q^3 - 15Q^2 + 250Q = 115Q - 2Q^2$$

$$0.3Q^3 - 13Q^2 + 135Q = 0$$

In this text, the methods for solving cubic functions are not covered, unless it is possible to factor the cubic function, thereby reducing the problem to the product of linear and quadratic functions. The above cubic is solved as follows:

$$0.3Q^3 - 13Q^2 + 135Q = 0$$

$$Q(0.3Q^2 - 13Q + 135) = 0 \qquad \text{factoring out } Q$$

Therefore, $Q = 0$ and solving the quadratic $0.3Q^2 - 13Q + 135 = 0$, gives $Q = 17.3$ and $Q = 26.1$. Therefore, when plotting the graph, choose $0 \leq Q \leq 30$.

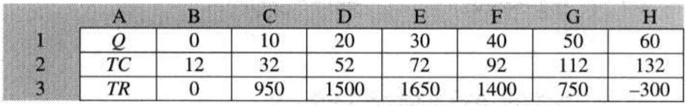
Set up tables for total revenue with each of the TC functions. Make sure that the tables contain the points of intersection and span intervals where $TC < TR$ and $TC > TR$ (unless, of course, the firm never makes a profit, in which case $TC > TR$ always). (See Figures 4.24 and 4.25.)

(b) (i) Total cost is a linear function, therefore, it increases indefinitely. See Figure 4.24. (ii) Initially, the rate of increase of TC is decreasing up to a point, after which, the rate of increase of TC is increasing. See Figure 4.25.

(c) (i) Break-even at $Q = 0.11$ and $Q = 56.4$. Between these levels of output the firm makes a profit.

(ii) Break-even at $Q = 17.3$ and $Q = 26.1$. Between these levels of output the firm makes a profit.

Comment: The Plane Soap Co. makes a much larger profit than the Round Soap Co. since the area between the TR and TC curve is substantially greater.



	A	B	C	D	E	F	G	H
1	*Q*	0	10	20	30	40	50	60
2	*TC*	12	32	52	72	92	112	132
3	*TR*	0	950	1500	1650	1400	750	-300

TC > *TR* followed by an interval where *TC* < *TR* and by a further interval where *TC* > *TR*

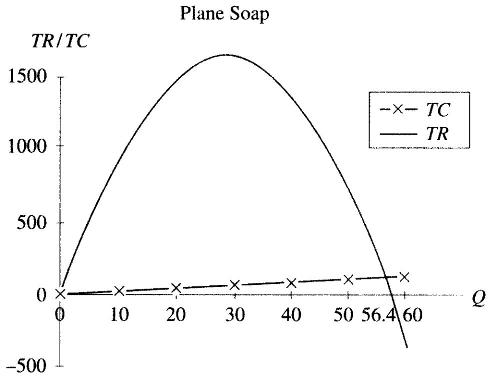
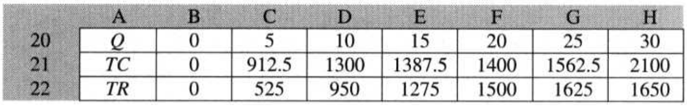


Figure 4.24 Linear *TC* and quadratic *TR* functions for Plane Soap Co.



	A	B	C	D	E	F	G	H
20	*Q*	0	5	10	15	20	25	30
21	*TC*	0	912.5	1300	1387.5	1400	1562.5	2100
22	*TR*	0	525	950	1275	1500	1625	1650

TC > *TR* followed by an interval where *TC* < *TR* and by a further interval where *TC* > *TR*

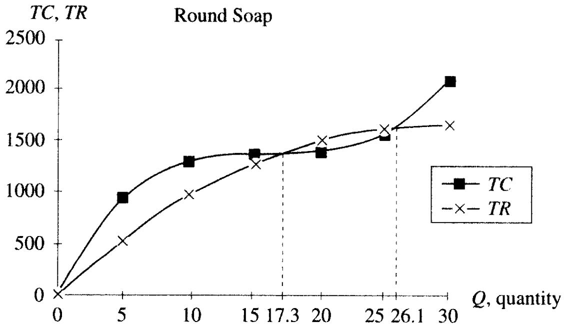


Figure 4.25 Cubic *TC* and quadratic *TR* functions for Round Soap

4.6 Summary

❐ Mathematics

- **Quadratic equations:** solution given by

$$x = \frac{-(b) \pm \sqrt{b^2 - 4ac}}{2a}$$

Also revise graphs.

- **Exponential indices:**

  Revise graphs: rules for indices.

  1. $a^m \times a^n = a^{m+n}$ **2.** $\dfrac{a^m}{a^n} = a^{m-n}$ **3.** $(a^m)^k = a^{m \times k}$

  Revise graphs.

- **Logarithms**

  (a) Convert from index to log form:

  Number = base$^{\text{power}}$

  $$\text{Number} = \text{base}^{\text{power}}$$

  the base of the index becomes the base of the log, the power drops down

  then $\log_{\text{base}}(\text{Number}) = \text{power}$

  (b) Also, convert from log form to index form.

  (c) Rules for logs:

  1. $\log_b M + \log_b N \Leftrightarrow \log_b MN$ **2.** $\log_b M - \log_b N \Leftrightarrow \log_b\left(\dfrac{M}{N}\right)$

  3. $\log_b(M^z) \Leftrightarrow z \log_b(M)$ **4.** $\log_b(N) \Leftrightarrow \dfrac{\log_x(N)}{\log_x(b)}$

- **Hyperbolic functions:** $y = \dfrac{a}{(bx + c)}$.

  Revise the graphs, note the vertical asymptote.

❐ Applications

Similar to those in Chapters 2 and 3, but applied to a wider range of functions. Other applications include the laws of growth: limited and unlimited.

- **Excel:** Particularly useful for plotting, to visualise the shape and properties of the new functions introduced in this chapter. In this chapter, certain cubic equations cannot be solved by any of the methods covered in this text, but graphical solutions are facilitated with Excel.

Test Exercises 4

1. Solve the following equations:

(a) $x^2 - 25 = 0$ (b) $x^2 + 20x = 0$ (c) $x^2 - 40x + 14 = 0$

2. A firm charges a fixed price of £80 for each shirt sold. The firm has a total cost function: $TC = Q^3 - 136Q$.

(a) Write down the equation of the total revenue function.

(b) Determine the break-even point.

3. (a) Simplify to a single term

(i) $2\log(x) - \log(2x)$ (ii) $e^{5x}e^{-x}e^{5-x}$

(b) A rural population (given in thousands) is thought to decline according to the equation $P = 15e^{-0.1t}$. If $t = 0$ at the beginning of 1998

(i) Calculate the numbers in the population at the beginning of 1990; 2000; 2016

(ii) Graph the population up to 2050

(iii) Calculate the number of years it will take for the population to decline to 10 000

4. (a) Solve the equations (i) $\dfrac{-25}{x+5} = 4$ (ii) $\dfrac{2}{Q+0.5} = 20Q$

(b) Find the equilibrium price and quantity for the demand and supply functions

$$P_{\text{d}} = \frac{50}{Q+2}, \qquad P_{\text{s}} = 10 + 2Q$$

Confirm you answer graphically

5. (a) Simplify the following expressions to a single term:

(i) $3\log(x) + \log\left(\dfrac{10}{x}\right) - 2\log(10x)$ (ii) $\dfrac{a^2\sqrt{a}}{2a^{-2}}$

(b) A new model of a popular television set is gradually replacing an earlier model. The projected sales (in thousands) for the old and new models are given by the equations

$$S_{\text{old}} = \frac{4}{t+2}, \qquad S_{\text{new}} = \frac{21}{12-t}$$

where time t is in months; S is the number of TVs sold (old and new models).

(a) Calculate the number of TVs sold when $t = 0,\ 2,\ 4,\ 8$. Graph the sales functions for $t = 0$ to 8. From the graph, estimate the time when the sales of both models are equal.

(b) Calculate the time when sales are equal.

6. (a) Simplify the following:

(i) $\dfrac{a^5\sqrt{a^5}}{2a^{0.3}}$ (i) $\dfrac{a^x\sqrt{a^x}}{3a^{1-x}}$

(b) The growth of demand for computer hardware in a college is given by the equation $Q = 200(1.025)^t$, where t is time in months.

(i) Calculate the demand when $t = 0,\ 3,\ 6,\ 10$ months.

(ii) Calculate the time taken for the demand to increase to 300.

Graph the demand for $t = 0$ to 30, hence confirm the answers to (i) and (ii).

7. (a) Solve the equations:

(i) $\ln(t+3) = 4.2$ (ii) $2\ln(t+1) - \ln(t-1) = 1.504$

(b) A firm has a total cost function $TC = \sqrt{4+6Q}$. If the firm charges a fixed price of 2 per unit sold:

(i) Write down the equation for total revenue

(ii) Determine the break-even point

CHAPTER 5

Financial Mathematics

This chapter is divided into the following sections:

5.1 Arithmetic and geometric sequences and series
5.2 Simple interest, compound interest and annual percentage rates
5.3 Depreciation
5.4 Net present value and internal rate of return
5.5 Annuities, debt repayments, sinking funds
5.6 The relationship between interest rates and the price of bonds
5.7 Excel for financial mathematics
5.8 Summary

'Households face another hike in mortgage interest rates'
'Banks make record profits'

Such statements are part of our everyday life. Without doubt, the level of interaction between the individual and financial institutions is ever-increasing. The main interaction tends to be through borrowings and savings. Individuals are concerned with the total repayments on their borrowings and the total return received on their savings.

The amount received or repaid in the future is dependent on a number of factors:

• Principal (P_0)	the amount borrowed (or invested) initially: **present value**.
• Interest rate (i)	interest rate per year, computed as a percentage of the principal.
• Time (t)	period of time over which money is borrowed (lent or invested).
• P_t	value of the principal after a period of time t: **future value**.

In this chapter, various methods are introduced for calculating the amount received or repaid in the future. The methods are based on arithmetic and geometric sequences and series.

5.1 Arithmetic and Geometric Sequences and Series

At the end of this section you should be familiar with:

- Arithmetic sequences and series
- Geometric sequences and series
- Applications of arithmetic and geometric series.

❐ Definitions

- A **sequence** is a list of numbers which follow a definite pattern or rule.

If the rule is that each term, after the first, is obtained by adding a constant, d, to the previous term, then the sequence is called an **arithmetic sequence**, such as 2, 6, 10, 14, 18, 22, … , where $d = 4$. d is known as the **common difference**.

If the rule is that each term, after the first, is obtained by multiplying the previous term by a constant, r, then the sequence is called a **geometric sequence**, such as, 4, 16, 64, 256, 1024, … , where $r = 4$. r is known as the **common ratio**.

- A **series** is the sum of the terms of a sequence.

A series is **finite** if it is the sum of a finite number of terms of a sequence.
A series is **infinite** if it is the sum of an infinite number of terms of a sequence.

❐ Arithmetic series (or arithmetic progression denoted by AP)

An arithmetic series is the sum of the terms of an arithmetic sequence, such as 2, 6, 10, 14, 18, 22, …, with $d = 4$. Denoting the first term of an arithmetic sequence with the value a and progressing by adding the value d to each previous term, the arithmetic sequence can be outlined as in Table 5.1.

Each element of a sequence can be identified by reference to its term number; for example, the first term of the sequence $T_1 = a = 2$, the second term of the sequence $T_2 = a + d = 6$. The value of any term can be calculated knowing that the nth term of the arithmetic sequence is

$$T_n = a + (n - 1)d \tag{5.1}$$

In the above example, the value of the 20th term is $a + (n - 1)d = 2 + (20 - 1)4 = 78$. The sum of the first n terms, S_n, of an arithmetic series is given by the formula

$$S_n = \frac{n}{2}[2a + (n - 1)d] \tag{5.2}$$

Formula 5.2 is derived in the appendix to this chapter.

Table 5.1 Arithmetic sequence

Sequence	a	$a + d$	$a + 2d$	⋯	$a + (n - 1)d$
Term number	T_1	T_2	T_3	⋯	T_n

WORKED EXAMPLE 5.1

SUM OF AN ARITHMETIC SERIES

Find the sum of the first 15 terms of the series: $20 + 18 + 16 + 14 + \ldots$

Solution

This is an arithmetic series since the difference $d = -2$. Therefore, $a = 20$, $d = -2$, $n = 15$, and hence, using formula (5.2),

$$S_{15} = \frac{15}{2}[2(20) + (15 - 1)(-2)]$$

$$= 90$$

☐ Geometric series (or geometric progression denoted by GP)

A geometric series is the sum of the terms of a geometric sequence, such as 4, 16, 64, 256, 1024, ..., with $r = 4$. Denoting the first term of a geometric sequence with the value a and progressing by multiplying the previous term by a common ratio r, the geometric sequence can be outlined as in Table 5.2. In this pattern it can be seen that when any term is divided by the previous term, the result is a common ratio, r.

The nth term of a geometric series is

$$T_n = ar^{n-1} \tag{5.3}$$

The sum of the first n terms of a geometric series is given by the formula,

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \tag{5.4}$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \tag{5.5}$$

It is more convenient to use formula (5.4) when $r < 1$ and formula (5.5) when $r > 1$. Formula (5.5) is obtained by multiplying formula (5.4) by $-1/-1$. Formula (5.4) is derived in the appendix to this chapter.

Table 5.2 Geometric sequence

Sequence	a	ar	ar^2	$\cdots$	ar^{n-1}
Term number	T_1	T_2	T_3	$\cdots$	T_n

WORKED EXAMPLE 5.2

SUM OF A GEOMETRIC SERIES

Find the sum of the first 12 terms of the series: $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} \ldots$

Solution

This is a geometric series since the ratio, $r = 1/2$. Therefore, $r = 1/2$, $a = 4$, $n = 12$, and hence, using formula (5.4),

$$S_{12} = \frac{4\left(1 - \left(\frac{1}{2}\right)^{12}\right)}{\left(1 - \frac{1}{2}\right)} = \frac{4(1 - 2.44 \times 10^{-4})}{\frac{1}{2}} = 8(0.9997558) = 7.9980$$

❒ **Infinite number of terms of a GP**

In Worked Example 5.2 the calculation of r^n gave a very small value,

$$\left(\frac{1}{2}\right)^{12} = 0.0002441$$

If $r < 1$, then as n becomes larger $r^n \to 0$. For example,

$$\left(\frac{1}{2}\right)^{5} = 0.03125, \quad \left(\frac{1}{2}\right)^{10} = 0.00098, \quad \left(\frac{1}{2}\right)^{15} = 0.000031$$

If the series has an infinite number of terms, r^n will become so small that in practice it will be zero. Formula (5.4) reduces to

$$S_\infty = \frac{a(1 - 0)}{1 - r} = \frac{a}{1 - r} \qquad \textbf{(5.6)}$$

Progress Exercises 5.1 APs and GPs

1. Find the sum of the first eight terms of the series: $26 + 22 + 18 + 14 + \cdots$
2. Find the sum of the first 10 terms of the series: $\frac{1}{4} + \frac{3}{4} + \frac{5}{4} + \frac{7}{4} + \cdots$
3. Find the sum of the first eight terms of the series: $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$
4. Find the value of the 50th term of the series: $180 + 162 + 145.8 + 131.22 + \cdots$.

In Questions 5 to 7, (i) write down the values of a and d, (ii) use the formula for S_n to determine how many terms of the series must be added to give the sum indicated:

5. $1 + 2 + 3 + \cdots = 210$

6. $2 + 4 + 6 + \cdots = 420$
7. $90 + 87 + 84 + \cdots = 420$
8. The sum of the first 12 terms of an AP is 222, the sum of the first 5 terms is 40. Write out the first four terms of the series.
9. Show that the sum of the first n terms of the series: $1 + 3 + 5 + \ldots$ is given by the formula, $S_n = n^2$.
10. The fourth term of a GP is 56 while the sixth term is $\frac{7}{8}$. Find the GP and hence S_∞.
11. The first term of a GP represents the number of tins of beans sold in a supermarket the week before an advertising campaign is launched. Sales increase by 10% each week for 8 weeks during the campaign.
    (a) Write out the weekly sales as a geometric sequence. What is the value of r?
    (b) Calculate the increased number of tins sold over the eight weeks of the campaign in terms of a.
    (c) Is it worth while running this campaign if $a = 800$, the profit on beans is 9p per tin, but the cost of the campaign is £200 per week?

WORKED EXAMPLE 5.3

APPLICATION OF ARITHMETIC AND GEOMETRIC SERIES

A manufacturer produces 1200 computers each week. After week 1, he increases production by:

Scheme I: 80 computers each week
Scheme II: 5% each week

(a) (i) Find the output in week 20 under each scheme.
(ii) Find the total output over the first 20 weeks under each scheme.
(b) Calculate for schemes I and II the week in which production first exceeds 8000.

Solution

Scheme I forms an arithmetic series: $a = 1200$ and $d = 80$.
Scheme II is a geometric series: $a = 1200$, $r = 1.05$. How is $r = \mathbf{1.05}$ calculated? If output increases by 5%, then output in any one week is the output in the previous week plus 5% of the output in the previous week, that is,

$$Q_{t+1} = Q_t + \frac{5}{100} Q_t$$
$$= Q_t\left(1 + \frac{5}{100}\right) = Q_t(1.05) = Q_t(r)$$

where Q_t is the output in week t and Q_{t+1} is the output in week $t + 1$.

Note: r is *not* 5, $r = 5\% = \frac{5}{100} = 0.05$

(a) (i) The output in week 20 is the value of term number 20.

(ii) The total output over 20 weeks is the sum of the first 20 weekly outputs. Under scheme I, use the AP formulas (5.1) and (5.2) to calculate term number 20 and the sum of the first 20 terms respectively while under scheme II use the GP formulas (5.3) and (5.5) to calculate term 20 and the sum of the first 20 terms respectively. The calculations are as follows:

Scheme I

(i) Output in week n is $[a + (n - 1)d]$, output in week 20

$$= [1200 + (20 - 1)80]$$

$$= 2720$$

(ii) Total output in 20 weeks is

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_{20} = \frac{20}{2}[2(1200) + (20 - 1)80]$$

$$= 10[2400 + (19)(80)]$$

$$= 10[2400 + 1520]$$

$$= 39\,200$$

Scheme II

(i) Output in week n is ar^{n-1}, output in week 20

$$= 1200(1.05)^{19}$$

$$= 3032.34 = 3032$$

(ii) Total output in 20 weeks is

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{20} = \frac{1200(1.05^{20} - 1)}{1.05 - 1}$$

$$= \frac{1200(1.6533)}{(0.05)}$$

$$= \frac{1983.96}{0.05} = 39\,679.2$$

(b) Scheme I forms an arithmetic series with $a = 1200$ and $d = 80$. Each term number in the series represents the corresponding week. The value of each term represents the output for that week, that is,

Week	T_1 = week 1	T_2 = week 2	T_3 = week 3	...	T_n = week n
Output	1200	1200 + 80 = 1280	1200 + 2(80) = 1360	...	1200 + (n − 1)(80) = 8000

So, solve for the value of n (week n) for which $T_n = 8000$:

$$T_n = a + (n - 1)d$$

$$8000 = 1200 + (n - 1)(80)$$

$$8000 - 1200 = (n - 1)(80)$$

$$\frac{6800}{80} = n - 1$$

$$86 = n$$

The output for week 86 is 8000.

Under scheme II, the series is geometric with $a = 1200$ and $r = 1.05$; therefore, each weekly output corresponds to the terms of the GP:

Week	T_1 = week 1	T_2 = week 2	T_3 = week 3	...	T_n = week n
Output	1200	1200(1.05)	$1200(1.05)^2$	...	$1200(1.05)^{n-1}$

So, solve for the value of n (week n) for which $T_n = 8000$:

$$T_n = ar^{n-1}$$

$$8000 = 1200(1.05)^{n-1}$$

$$\frac{8000}{1200} = (1.05)^{n-1} \qquad \text{dividing both sides by 1200}$$

$$\log(6.6667) = \log(1.05)^{n-1} \qquad \text{take logs of both sides}$$

$$\log(6.6667) = (n-1)\log(1.05) \qquad \text{use rule 3 for logs}$$

$$\frac{\log(6.6667)}{\log(1.05)} = n - 1 \qquad \text{divide both sides by } \log(1.05)$$

$$38.8833 = n - 1 \qquad \text{evaluate the logs on the calculator,}$$

$$39.8833 = n \qquad \text{and divide}$$

During week 39, output reaches 8000 units.

❐ Summary

Characteristics of arithmetic sequences

a is the first term of the sequence
d is the common difference
nth term, T_n, is given by $[a + (n-1)d]$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Characteristics of geometric sequences

a is the first term of the sequence
r is the common ratio
nth term T_n, is given by ar^{n-1}

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Progress Exercises 5.2 Applications of APs and GPs

1. A TV manufacturer plans to increase his output by 5% each month. If he is now producing 300 TVs per month, calculate, using series,
   (a) His monthly output in 15 months from now.
   (b) His total output in 15 months, starting with the present month.
   (c) The month in which his output reaches 500.
2. At a certain time each year, the scaling down of weekly production of sun hats starts.
   (a) If weekly production is 190 000 hats, and is reduced by 15 200 hats each week, use series to calculate
      (i) The weekly output after 10 weeks of scaling down
      (ii) The total production during that 10 weeks

(b) Calculate the weekly reduction if the production should be scaled down from 190 000 to zero in 12 weeks.

3. A company manufactures deckchairs for a ten-week period each year. Production starts with 600 chairs in week 1 and increases by 50 for each subsequent week. Use series to calculate

   (a) The number of chairs manufactured in week 7
   (b) The total number of chairs manufactured during the ten weeks.

4. Two competing companies produce mobile phones. Company A starts production at 1000 phones per week and plans to increase output by 200 each week. Company B starts production with 500 phones per week and plans to increase output by 20% each week.

   (a) Calculate the weekly production in weeks 1; 5; 10; 15. Plot the graph of weekly output for each producer on the same diagram (use Excel, if available). Estimate the week in which the output is the same for both firms.
   (b) Calculate the total production during the first 15 weeks for each firm.

5. Heating a large building requires 50 000 gallons of oil in 1997. As the building is extended, two forecasts for growth in demand for oil are made.
   Forecast 1: increases of 5000 gallons per year.
   Forecast 2: increases of 6% each year.
   Use series to calculate for each forecast

   (a) The number of gallons required in 1999, 2004.
   (b) The total consumption of oil from 1997 to 2004 inclusive.

5.2 Simple Interest, Compound Interest and Annual Percentage Rates

At the end of this section, you should be able to:

- Calculate simple interest
- Calculate compound interest and derive the compound interest formula
- Solve for unknown variables using the compound interest formula
- Calculate compound interest when compounded several times per year
- Calculate compound interest when compounded continuously
- Calculate the annual percentage rate.

Arithmetic and geometric progressions have many applications in financial mathematics.

❐ Simple interest

Simple interest is a fixed percentage of the principal, P_0, that is paid to an investor each year, irrespective of the number of years the principal has been left on deposit; that is, money invested at simple interest will increase in value by the same amount each year. So, if the investor is paid a fixed annual amount, i% of P_0, then the amount of simple interest, I, received over t years is given by the formula

$$I = P_0 \times i \times t \tag{5.7}$$

Therefore, the total value after t years, P_t, is the principal plus interest and is given by

$$P_t = P_0 + (P_0 \times i \times t)$$

$$P_t = P_0(1 + it) \tag{5.8}$$

When the total value (future value), the interest rate and time are known, the principal (present value) may be calculated by rewriting formula (5.8) as

$$P_0(1 + it) = P_t$$

$$P_0 = \frac{P_t}{1 + it} \tag{5.9}$$

Formula (5.9) is often referred to as the 'present value' formula.

Note: The interest rate is always quoted as a percentage, $i\%$. In calculations, you write $i\%$ as the fraction $i/100$ (Section 1.6).

Go to the web site www.wiley.co.uk/bradley2ed for Worked Example 5.4.

❑ Compound interest

In the modern business environment, the interest on money borrowed (lent or invested) is usually compounded. Compound interest pays interest on the principal plus on any interest accumulated in previous years. The total value, P_t, of a principal, P_0, when interest is compounded at $i\%$ per annum is

$$P_t = P_0(1 + i)^t \tag{5.10}$$

Formula (5.10) is derived as follows.

❑ Deriving the compound interest formula

Before proceeding with worked examples on compound interest, the derivation of the compound interest formula is explained briefly:

Principal at start of year	**Interest paid each year**	**Total at end of year = principal + interest**	**Total at end of year = principal at start of next year**
P_0	$i \times P_0$	$P_0 + iP_0 = P_0(1 + i)$	$P_1 = P_0(1 + i)$
P_1	$i \times P_1$	$P_1 + iP_1 = [\mathbf{P_1}] \cdot (1 + i) =$ $[\mathbf{P_0(1 + i)}] \cdot (1 + i) = P_0(1 + i)^2$	$P_2 = P_0(1 + i)^2$
P_2	$i \times P_2$	$P_2 + iP_2 = [\mathbf{P_2}] \cdot (1 + i) =$ $[\mathbf{P_0(1 + i)^2}] \cdot (1 + i) = P_0(1 + i)^3$	$P_3 = P_0(1 + i)^3$
...	...	...	...
P_{t-1}	$i \times P_{t-1}$	$P_{t-1} + iP_{t-1} = [\mathbf{P_{t-1}}] \cdot (1 + i) =$ $[\mathbf{P_0(1 + i)^{t-1}}](1 + i)$ $= P_0(1 + i)^t$	$P_t = P_0(1 + i)^t$

Each year, the interest earned is added to the total amount on deposit (principal, P_0, plus any accumulated interest), at the beginning of that year.

Notice that the amounts due at the end of each year form a geometric progression where $a = P_0$, and r, the common ratio, is $(1 + i)$, giving the sequence

$$P_0, \quad P_0(1+i)^1, \quad P_0(1+i)^2, \ldots, P_0(1+i)^t$$

WORKED EXAMPLE 5.5

COMPOUND INTEREST CALCULATIONS

Calculate the amount owed on a loan of £1000 over three years at an interest rate of 8% compounded annually.

Solution

Substitute $P_0 = 1000$, $i = \frac{8}{100} = 0.08$, $t = 3$ years into formula (5.10),

$$P_3 = P_0(1+i)^t = 1000(1+0.08)^3 = 1000(1.08)^3$$
$$= 1000(1.2597120) = 1259.712$$

❐ Present value at compound interest

The present value of a future sum, P_t, is the amount which, when put on deposit at $t = 0$, at i% rate of interest, will grow to the value of P_t after t years. The present value, P_0, is calculated by rearranging the compound interest formula (5.10):

$$P_t = P_0(1+i)^t \quad \rightarrow \quad \frac{P_t}{(1+i)^t} = P_0$$

$$P_0 = \frac{P_t}{(1+i)^t} \tag{5.11}$$

WORKED EXAMPLE 5.6

FUTURE AND PRESENT VALUES WITH COMPOUND INTEREST

£5000 is invested at 8% compound interest per annum for three years.

(a) Calculate the value of the investment at the end of three years.
(b) Compute the present value of receiving (i) £6298.50, (ii) £15 000 in three years' time when the discount rate is 8%.

Solution

(a) $P_0 = £5000$, $i = 8/100$, $t = 3$. Using formula (5.10), the total amount received after three years is

$$\begin{aligned} P_3 &= 5000(1+0.08)^3 \\ &= 5000(1.2597) \\ &= 6298.5 \end{aligned}$$

(b) The present value of receiving £6298.50 and £15 000 in three years at a discount rate of 8% is calculated by using formula (5.11):

Present value of £6298.5	Present value of £15 000
$P_0 = \dfrac{P_3}{(1+i)^3}$	$P_0 = \dfrac{P_3}{(1+i)^3}$
$= \dfrac{6298.50}{(1.08)^3} = 5000$	$= \dfrac{15\,000}{(1.08)^3} = 11\,907.48$

☐ Other applications of the compound interest formula

In the compound interest formula, there are four variables, P_0, P_t, i, and t. If any three of these variables are given, the fourth may be determined. In some cases, you will require the rules for indices and logs. For example, a general expression for i may be derived as

$$P_t = P_0(1+i)^t \quad \rightarrow \quad (1+i)^t = \frac{P_t}{P_0} \quad \rightarrow \quad 1+i = \left(\frac{P_t}{P_0}\right)^{1/t}$$

$$\text{or} \quad i = \sqrt[t]{\frac{P_t}{P_0}} - 1 \qquad \textbf{(5.12)}$$

However, in the long term, you will find it difficult to remember all these formulae. Therefore, to solve for any one unknown variable, substitute the given values of the other variables into the compound interest formula and solve for the unknown.

WORKED EXAMPLE 5.7

CALCULATING THE COMPOUND INTEREST RATE AND TIME PERIOD

(a) Find the compound interest rate required for £10 000 to grow to £20 000 in six years.
(b) A bank pays 7.5% interest, compounded annually. How long will it take for £10 000 to grow to £20 000?

Solution

(a) Calculate i when given $P_0 = £10\,000$, $P_t = £20\,000$ and $t = 6$.

Using formula (5.12)

$$1+i=\left(\frac{P_t}{P_0}\right)^{1/t}$$

$$1+i=\left(\frac{20\,000}{10\,000}\right)^{1/6}$$

$$1+i=1.12$$

$$i=0.12$$

Directly from the compound interest formula, (5.10)

$$20\,000=10\,000(1+i)^6$$

$$\frac{20\,000}{10\,000}=(1+i)^6 \quad\rightarrow\quad 2=(1+i)^6$$

$$2^{1/6}=(1+i)$$

$$1.12=1+i$$

$$0.12=i$$

$$i=0.12$$

An interest rate of 12% is required if this investment is to double in value over a period of six years.

(b) Calculate t when given $P_0=£10\,000$, $P_t=£20\,000$ and $i=7.5/100=0.075$. Substitute the given values into formula (5.10),

$20\,000=10\,000(1+0.075)^t$	
$\frac{20\,000}{10\,000}=(1+0.075)^t$	divide both sides by 10 000
$2=(1.075)^t$	
$\log(2)=\log(1.075)^t=t\log(1.075)$	take logs of each side of the equation
$\frac{\log 2}{\log(1.075)}=t$	divide both sides by log(1.075)
$9.5844=t$	evaluate the logs on the calculator and divide

So, at 7.5% interest it will take over nine years for the investment to double in value.

Progress Exercises 5.3 Simple and Compound Interest

1. Suppose £5000 is invested for five years. Calculate the amount accumulated at the end of five years if interest is compounded annually at a nominal rate of (a) 5%, (b) 7%, (c) 10%.
2. A savings account of £10 000 earns simple interest at 5% per annum. Calculate the value of the account (future value) after six years.
3. £2500 is invested at a nominal rate of intereat of 5% per annum. Calculate the amount accumulated at the end of (a) 1 year, (b) 4.5 years, (c) 10 years, (d) 20 years.
4. Calculate the present value of £6000 that is expected to be received in three years' time with simple interest of 7.5% per annum.

5. How much is a sum £3500 worth at the end of five years if deposited at (a) 11% simple interest and (b) 11% compound interest, each calculated annually?
6. Calculate the present value of £6500 to be received in two years' time when the interest rate is 8% per annum, compounded annually.
7. Calculate the present value of £10 000 due in five years if interest is compounded annually at 4.5%.
8. Calculate the annual rate of interest required for an investment to double in value in (a) 12 years, (b) 8 years, (c) 5 years. Assume annual compounding.
9. How long should it take for an investment of £3700 to grow to £5000 when interest is compounded annually at 6.5%?
10. How long should it take for an investment of £56 000 to grow to £74 000 when interest is compounded annually at (a) 6% and (b) 3%?
11. How long should it take for an investment to double in value when interest is compounded annually at (a) 6% and (b) 3%?
12. Calculate the compound interest rate required for £5000 to grow to £9000 in four years.
13. Calculate the number of years it will take for a sum of £5000 to grow to £20 000 when invested at 5.5% interest compounded annually.
14. Set up a table of values comparing the growth of an investment of £1 at (a) an annual interest rate of 10% simple interest, (b) an annual interest rate of 10% compounded annually for 0, 5, 10, 15 and 20 years.
15. Suppose £2500 is invested for eight years and grows to £4525 when the annual rate of interest is i%. Calculate i.

☐ When interest is compounded several times per year

So far, it has been assumed that compound interest is compounded once a year. In reality, interest may be compounded several times per year, for example it may be compounded daily, weekly, monthly, quarterly, semi-annually or continuously. Each time period is known as a conversion period or interest period. The number of conversion periods per year is denoted by the symbol, m; the interest rate applied at each conversion is i/m. For example, an investment compounded twelve times per year will have twelve conversion periods; therefore if a five-year investment was compounded twelve times annually, then the investment would have sixty conversion periods; that is,

$$n = m \times t$$

where n = total number of conversion periods
m = conversion periods per year
t = number of years

The value of the investment at the end of n conversion periods is

$$P_t = P_0\left(1 + \frac{i}{m}\right)^n = P_0\left(1 + \frac{i}{m}\right)^{m \times t} \qquad \textbf{(5.13)}$$

WORKED EXAMPLE 5.8

COMPOUNDING DAILY, MONTHLY AND SEMI-ANNUALLY

£5000 is invested for three years at 8% per annum compounded semi-annually.

(a) Calculate the total value of the investment.

(b) Compare the return on the investment when interest is compounded annually to that when compounded semi-annually.

(c) Calculate the total value of the investment when compounded, (i) monthly, (ii) daily. Assume all months consist of 365/12 days.

Solution

(a) $P_0 = £5000$, $i = 8/100 = 0.08$, $t = 3$ and $m = 2$. Using formula (5.13), the total value after three years with $n = m \times t = 6$ conversion periods is calculated as

$$P_3 = 5000\left(1 + \frac{0.08}{2}\right)^{(2)(3)}$$

$$= 5000(1 + 0.04)^6$$

$$= 6326.59$$

(b) In Worked Example 5.6, the total value of £5000 after three years compounded annually was £6298.50. When the same investment is compounded semi-annually, the total value is £6326.59, a gain of £28.09 over the value when compounded annually.

(c) The value of the investment at the end of three years for monthly and daily compounding is calculated as

(i) **Monthly compounding**

$m = 12 \rightarrow n = m \times t = 12(3) = 36$

$$P_3 = P_0\left(1 + \frac{i}{m}\right)^{(m)(t)}$$

$$= 5000\left(1 + \frac{0.08}{12}\right)^{(12)(3)}$$

$$= 5000(1.2702)$$

$$= 6351$$

This earns $6351 - 6298.5 = £52.5$ more than when interest is compounded annually.

(ii) **Daily compounding**

$m = 365 \rightarrow n = m \times t = 365(3)$
$= 1095$

$$P_3 = P_0\left(1 + \frac{i}{m}\right)^{(m)(t)}$$

$$= 5000\left(1 + \frac{0.08}{365}\right)^{(365)(3)}$$

$$= 5000(1.2712)$$

$$= 6356$$

This earns $6356 - 6298.5 = £57.5$ more than when interest is compounded annually.

❒ Continuous compounding

In the previous example, compounding over several time intervals was illustrated: semi-annually, monthly and daily. In each case, the return on the investment increases as m, the number of compoundings per year, increases. One could also compound continuously, for example every minute, second or microsecond. With continuous compounding, the total value is given by the formula

$$P_t = P_0 e^{it} \tag{5.14}$$

where $e = 2.718281828$.

This formula is derived from formula (5.13), by letting the number of compoundings per year, m, become very large so that

$$P_t = P_0\left(1 + \frac{i}{m}\right)^{(m)(t)} = P_0\left[\left(1 + \frac{i}{m}\right)^{m}\right]^{t} = P_0[e^i]^t = P_0 e^{it}$$

since $\left(1 + \frac{i}{m}\right)^m$ approaches e^i as m gets larger. The reader should check these by evaluating $\left(1 + \frac{i}{m}\right)^m$ for one value of i and various values of m, such as $i = 5/100$, $m = 10, 20, 30, 40, 50, 365$. You should get, 1.05114, 1.0512055, 1.051227, 1.0512383, 1.0512448, 1.0512675 while $e^i = e^{0.05} = 1.0512711$.

WORKED EXAMPLE 5.9

CONTINUOUS COMPOUNDING

A financial consultant advises you to invest £5000 at 8% compounded continuously for three years. Find the total value of your investment.

Solution

$P_0 = £5000$, $i = 8/100 = 0.08$ and $t = 3$. Therefore, the total value is given by

$$\begin{aligned} P_3 &= 5000e^{(0.08)(3)} \\ &= 5000(1.27) \\ &= 6356.24 \end{aligned}$$

When interest was compounded annually in Worked Example 5.6, the value of the investment after three years was £6298.5, which is £57.74 less compared to continuous compounding.

❒ Annual percentage rate (*APR*)

Interest rates are usually cited as nominal rates of interest expressed as per annum figures. However, as compounding may occur several times during the year with the nominal rate, the amount owed or accumulated will be different from that calculated by compounding once a year. So, a standard measure is needed to compare the amount earned (or owed) at quoted nominal rates of interest when compounding is carried out several times per year. This standard measure is called the annual percentage rate (*APR*) or effective annual rate. If interest is compounded once a year at the *APR* rate, the investment would yield exactly the same return, P_t, at the end of t years, as it would if interest were compounded m times per year at the nominal rate.

For example, consider an investment of £100 at a nominal interest rate of 10% compounded semi-annually, that is, at 5% per six months. The investment at the end of the first six months is worth £[100 + 0.05(100)] = £105. Then invest this £105 for a further six months and it grows to £[105 + 0.05(105)] = £110.25. At the end of the year, the initial investment of £100 has grown to £110.25, giving an annual percentage rate, *APR*, of 10.25% which is greater than the nominal rate of 10%.

The formula for the annual percentage rate (*APR*) is derived as follows,

$$P_t = P_0\left(1+\frac{i}{m}\right)^{mt} \qquad \text{nominal rate compounded } m \text{ times per year}$$

$$P_t = P_0(1+APR)^t \qquad APR \text{ rate compounded annually}$$

$$P_0(1+APR)^t = P_0\left(1+\frac{i}{m}\right)^{mt} \qquad \text{since the yield is the same}$$

$$(1+APR)^t = \left(1+\frac{i}{m}\right)^{mt} \qquad \text{dividing both sides by } P_0$$

$$1+APR = \left(1+\frac{i}{m}\right)^{m} \qquad \text{taking the } t\text{th root of each side}$$

$$APR = \left(1+\frac{i}{m}\right)^{m} - 1$$

The reader is asked to explain each step in the derivation of the formula for the *APR* when interest is compounded continuously as outlined below:

$$P_t = P_0e^{it} \quad \text{and} \quad P_t = P_0(1+APR)^t$$

$$P_0(1+APR)^t = P_0e^{it}$$

$$(1+APR)^t = e^{it}$$

$$1+APR = e^i$$

$$APR = e^i - 1$$

The *APR*, therefore, may be calculated by the following formulae:

- When the nominal rate is compounded m times per year,

$$APR = \left(1+\frac{i}{m}\right)^{m} - 1 \tag{5.15}$$

- When the nominal rate is compounded continuously,

$$APR = e^i - 1 \tag{5.16}$$

Again, when calculating the *APR*, it will be easier, and it will make more sense, to work through the method rather than memorising formulae such as (5.15) and (5.16).

WORKED EXAMPLE 5.10

ANNUAL PERCENTAGE RATES

(a) Find the annual percentage rate on a loan corresponding to 6.0% compounded monthly.

(b) Two banks in a local town quote the following nominal interest rates: bank A pays interest on a savings account at 6.60% compounded monthly and bank B pays 6.65% on a savings account compounded semi-annually. Which bank pays its savers the most interest?

Solution

(a)

$$APR = \left(1+\frac{i}{m}\right)^m - 1 = \left(1+\frac{0.06}{12}\right)^{12} - 1 = 1.06167 - 1 = 0.06167 = 6.17\%$$

The annual percentage rate payable on the loan is 6.17% which is greater than the quoted nominal rate of 6.0%.

(b) Bank A pays 6.60% compounded monthly. The annual percentage rate is given by

$$APR = \left(1+\frac{i}{m}\right)^m - 1 = \left(1+\frac{0.066}{12}\right)^{12} - 1 = 1.0680 - 1 = 0.0680 = 6.8\%$$

Bank B pays 6.65% compounded semi-annually. The annual percentage rate is given by

$$APR = \left(1+\frac{i}{m}\right)^m - 1 = \left(1+\frac{0.0665}{2}\right)^{2} - 1 = 1.0676 - 1 = 0.0676 = 6.76\%$$

Bank A offers the greater annual percentage rate and thus pays its savers more interest despite its lower nominal interest rate.

Progress Exercises 5.4 Interest Compounded at Various Intervals: APR

1. Suppose £50 000 is invested at a nominal interest rate of 5.5% per annum. Interest is calculated (a) annually as simple interest, (b) annually as compound interest, (c) four times annually as compound interest, (d) continuously.
   (i) Calculate the value of the investment at the end of each year for the first five years.
   (ii) Calculate the number of years it will take for the investment to double in value.

2. Suppose £5000 is invested for five years. Calculate the amount accumulated at the end of five years if interest is compounded continuously at a nominal annual rate of (a) 5%, (b) 7%, (c) 10%.
3. Calculate the *APR* for a 6% per annum nominal rate of interest that is compounded (a) 4 times per year, (b) 12 times per year, (c) continuously.
4. Suppose £2500 is invested at a nominal rate of interest of 5% per annum. Calculate the amount accumulated at the end of 10 years if interest is calculated (a) annually, (b) 4 times annually, (c) 52 times per year, (d) continuously.
5. Calculate the *APR* for a 6% per annum nominal rate of interest compounded 20 times annually.
6. Calculate the APR for an 8% per annum nominal rate of interest compounded four times annually.
7. Calculate the APR for an 8% per annum nominal rate of interest compounded continuously.
8. Calculate the APR for a 3% per annum nominal rate of interest compounded continuously.
9. Calculate i if £2500 grows to £5162 when invested at an annual rate of interest of i% for 7.5 years.
10. How long should any sum of money be left on deposit at 4.5% *APR* if the value of the deposit is to increase by 70%?
11. Suppose £5500 is deposited at a nominal rate of interest of 6% per annum. Calculate the value of the investment in three years' time if interest is compounded (a) semi-annually, (b) monthly, (c) daily, (d) continuously.
12. Two banks in a local town quote the following nominal interest rates: bank A charges interest at 8.80% per annum compounded semi-annually and bank B charges 8.75% per annum compounded quarterly. Which bank charges the most interest?

5.3 Depreciation

At the end of this section, you should be able to:

- Calculate straight-line depreciation
- Calculate reducing-balance depreciation

Depreciation is an allowance made for the wear and tear of equipment during the production process. It involves the deduction of money from the original asset value, A, each year. Two depreciation techniques are analysed, straight-line and reducing-balance depreciation.

❐ Straight-line depreciation

Straight-line depreciation is the converse of simple interest with equal amounts being subtracted from the original asset value each year. For example, if the original value of a machine was £20 000 and after four years its value is estimated to be £8000, then the amount of straight-line depreciation subtracted each year is (£20 000− £8000) ÷ 4 = £3000.

☐ Reducing-balance depreciation

Reducing-balance depreciation is the converse of compound interest with larger amounts being subtracted from the original asset value each year. The formula for reducing balance depreciation is given as

$$A_t = A_0(1 - i)^t \tag{5.17}$$

where A_t = value of the asset after t years taking account of depreciation
A_0 = original value of the asset
i = depreciation rate expressed as a decimal (e.g. 5% is 0.05)
t = number of time periods, usually years.

WORKED EXAMPLE 5.11

FUTURE VALUE OF ASSET AND REDUCING-BALANCE DEPRECIATION

A machine costing £30 000 depreciates by 15% each year. Calculate the value of the machine after five years. What is the total amount of depreciation?

Solution

Write down any information given: $A_0 = £30000$, $i = 15/100 = 0.15$, $t = 5$. Substitute these values into formula (5.17) to calculate the book value of the machine after five years:

$$A_t = A_0(1 - i)^t = 30000(1 - 0.15)^5 = £13311.16$$

After five years the book value of the machine is £13 311.16. Therefore, the total amount of depreciation is $(30000 - 13311.16) = £16688.84$.

WORKED EXAMPLE 5.12

PRESENT VALUE OF ASSET AND REDUCING-BALANCE DEPRECIATION

Five years after purchase, a computer has a scrap value of £300. The depreciation rate is 26%. Calculate the value of the computer when it was bought five years ago.

Solution

In this example, you are required to calculate the original value of the computer when first bought. Write down any information given: $A_t = 300$.

$i = 26/100 = 0.26$, $t = 5$. Substitute these values into formula (5.17):

$$A_t = A_0(1 - i)^t$$

$$A_0 = \frac{A_t}{(1 - i)^t} \quad \text{rearranging}$$

$$= \frac{300}{(1 - 0.26)^5} = \frac{300}{0.2219} = 1351.96$$

The original value of the computer five years ago was £1351.96.

5.4 Net Present Value and Internal Rate of Return

At the end of this section you should be able to:

- Use the net present value as an investment appraisal technique
- Use the internal rate of return as an investment appraisal technique
- Compare the two techniques.

Net present value and internal rate of return are two techniques used to appraise investment projects. Each technique is analysed in turn.

❐ Net present value (*NPV*)

In Section 5.2 the present value of a sum due to be paid in t years' time is calculated by equation (5.11):

$$P_0 = \frac{P_t}{(1 + i)^t}$$

The net present value is the present value of several future sums discounted back to the present.

The net present value technique uses present values to appraise the profitability of investments undertaken by firms. For example, the costs (indicated with a minus sign), and the returns of a project, as estimated by the project team, are given in Table 5.3.

The problem confronting the project manager is whether the investment is worth while. The decision is made even more difficult since the returns associated with the project will be received in the future while the costs usually occur at present. Therefore, to make an informed decision, the project manager needs to compare costs and returns, all brought back to present values. These present returns are then compared to present costs to estimate a net present value calculation on the project where

$$NPV = \text{present value of cash inflows} - \text{the present value of cash outflows} \quad \textbf{(5.18)}$$

A decision rule is

$$NPV > 0 \quad \text{Invest in the project}$$

$$NPV < 0 \quad \text{Don't invest in the project}$$

Table 5.3 Cash flows of an investment project

Year	Cash flows (£000)
0	−400
1	120
2	130
3	140
4	150

WORKED EXAMPLE 5.13

CALCULATING NET PRESENT VALUE

Calculate the net present value of the investment project as outlined in Table 5.3 assuming 8% discount rate compounded annually.

Note: When future values are brought back to present values at a given rate of interest, the interest rate is referred to as the discount rate.

Solution

The present value of the cash outflow is the initial outlay of £400 000. The present value of cash inflows is calculated by summing the present value of each cash inflow as follows:

$$\text{Present value of return in year 1: } P_0 = \frac{120\,000}{(1+0.08)^1} = 111\,111$$

$$\text{Present value of return in year 2: } P_0 = \frac{130\,000}{(1+0.08)^2} = 111\,454$$

$$\text{Present value of return in year 3: } P_0 = \frac{140\,000}{(1+0.08)^3} = 111\,137$$

$$\text{Present value of return in year 4: } P_0 = \frac{150\,000}{(1+0.08)^4} = 110\,254$$

$$\text{Total present value over the four years: } \quad 443\,956$$

The net present value is then

$$NPV = 443\,956 - 400\,000 = 43\,956$$

As the NPV is positive, the firm will invest in this project. Thus, the investment is cost effective at 8% discount rate compounded annually.

The size of the NPV will become smaller when

(i) The initial cost outlay increases

(ii) The discount rate at which NPV is computed increases.

Table 5.4 Calculation of *NPV*s for various interest rates

t years	Net cash flow	Dis. factor $i = 8\%$ $\frac{1}{(1+0.08)^t}$	Present value (8%)	Dis. factor 12.6555% $\frac{1}{(1+0.126555)^t}$	Present value (12.6555%)	Dis. factor $i = 15\%$ $\frac{1}{(1+0.15)^t}$	Present value (15%)
0	−400	1	−400	1	−400	1	−400
1	120	0.9259	111	0.8877	107	0.8696	104
2	130	0.8573	111	0.7879	102	0.7561	98
3	140	0.7938	111	0.6994	98	0.6575	92
4	150	0.7350	110	0.6209	93	0.5718	86
			43		0		−20

Note: The present value of each future cash flow above was calculated by multiplying the cash due in t years by a factor $1/(1+i)^t$, where i is the discount rate. In fact, tables of discount factors are available for various rates. However, in the following problems, the discounting factors are readily calculated with the use of the calculator, so we shall not avail ourselves of the tables.

❒ Internal rate of return (*IRR*)

In the previous section, the profitability of a project was assessed on the basis of the *NPV* calculated at a given rate of interest of 8%. The profitability changes as the interest rate changes. If the interest rate was 12.6555%, the *NPV* would be zero, while further increases in the interest rate, such as $r = 15\%$, would result in negative *NPV*s. Table 5.4 shows the calculation of the *NPV* for these three interest rates.

The *IRR* is the interest rate for which the *NPV* is zero. A project is viable if the prevailing interest rate is less than the *IRR*, but not profitable if the prevailing interest rate is greater than the *IRR*. The decision rule is

> Accept the project if the *IRR* is greater than the market rate of interest.

❒ To calculate the *IRR* for a given project: (a) graphically, (b) by calculation

(a) The graphical method: The *NPV* for several rates of interest is plotted against the interest rates, such that the *NPV*s range from positive to negative. Then, according to the definition given above, the *IRR* is the interest rate at which the $NPV = 0$. This is the point at which the curve crosses the horizontal axis.

(b) By calculation: As a follow-on from the graphical method described above, the point may be calculated at which the curve of *NPV* plotted against i crosses the horizontal axis. If two points on the curve are known, one above the axis (i_1, NPV_1) where the $NPV_1 > 0$ and the other below the axis (i_2, NPV_2) where the $NPV_2 < 0$, the *IRR*, which is the point of intersection with the horizontal axis, is given by the formula

$$IRR = \frac{(i_1 \times NPV_2) - (i_2 \times NPV_1)}{NPV_2 - NPV_1} \tag{5.19}$$

Formula (5.19) is proved in the appendix to this chapter.

Note: The IRR is determined here by linear interpolation, that is, the straight line joining the points (i_1, NPV_1) to (i_2, NPV_2). This line is an approximation to a curve. Therefore, the IRR may have slightly different values when calculated using different pairs of points. The best estimate is obtained by taking points on opposite sides, but as close as possible to the horizontal axis.

WORKED EXAMPLE 5.14

IRR DETERMINED GRAPHICALLY (EXCEL) AND BY CALCULATION

A project involves an initial outlay of £550 000. The expected cash flow at the end of the next five years is given as

Year	1	2	3	4	5
Cash flow	108 300	130 500	170 900	200 000	160 000

Determine the IRR
(a) Graphically by plotting the NPV against i for $i = 0.05$; 0.07; 0.09; 0.11; 0.13; 0.15; 0.17
(b) By calculation. Show that the value of the IRR is slightly different when calculated from different pairs of points.

Solution

(a) Graphically: Excel is ideal at this point, since the calculation of NPVs requires the repeated use of formulae. For part (a), several NPVs are required to plot the curve of NPV against i. The formula which calculates the present value of the cash flow at discount rate i, expressed as a decimal, is entered into the appropriate cell as follows:

$$= (\text{cash flow})*(1/(1+i)\hat{}\,t)$$

Since the values of t are always in column A and cash flow in column B (Table 5.5), the addresses of these columns may be entered in the following formula at cell C2:

$$= (\$B2)*(1/(1+i)\hat{}\,\$A2)$$

- Enter this formula in cell C2, with $(1 + i) = 1.05$. Then copy the formula down column C to calculate the present values for years 1 to 5. Point to cell C8. Then use the Σ button to sum the contents of cells C2 to C7 to find the NPV.
- To calculate the NPV for each of the other discount rates, copy the formula in cell C2 across row 2. The \$ sign in front of the A and B means that these references do not change. However, since the discount rate is different for

Table 5.5 Excel sheet for calculating *NPV*s at different interest rates

	A	B	C	D	E	F	G	H	I
1	t	cash flow	$i = 0.05$	$i = 0.07$	$i = 0.09$	$i = 0.11$	$i = 0.13$	$i = 0.15$	$i = 0.17$
2	0	−550000	−550000	−550000	−550000	−550000	−550000	−550000	−550000
3	1	108300	103143	101215	99358	97568	95841	94174	92564
4	2	130500	118367	113984	109839	105917	102201	98677	95332
5	3	170900	147630	139505	131966	124961	118442	112370	106705
6	4	200000	164540	152579	141685	131746	122664	114351	106730
7	5	160000	125364	114078	103989	94952	86842	79548	72978
8		*NPV* =	109045	71361	36837	5143	−24011	−50881	−75691

each column, this part of the formula must be changed. For example, in the 11% column, replace $(1 + i)$ by 1.11.

- Repeat the 'copy down and sum' procedure described above for columns D, E, F, G, H and I in Table 5.5.

The *NPV* for each discount rate is plotted in Figure 5.1. The *IRR* is the value of i at which this graph crosses the horizontal axis. In Figure 5.1, this point is between $i = 11\%$ and 13%, but considerably closer to 11%. $IRR \cong 11.45\%$.

(b) By calculation: In Table 5.5, there are several positive and negative *NPV*s. Therefore, the *IRR* given by formula (5.19) is calculated from any such pair.

Summarising the points already calculated,

Points	C	D	E	F	G	H	I
i	0.05	0.07	0.09	0.11	0.13	0.15	0.17
NPV	109 045	71 361	36 837	5143	−24 011	−50 881	−75 691

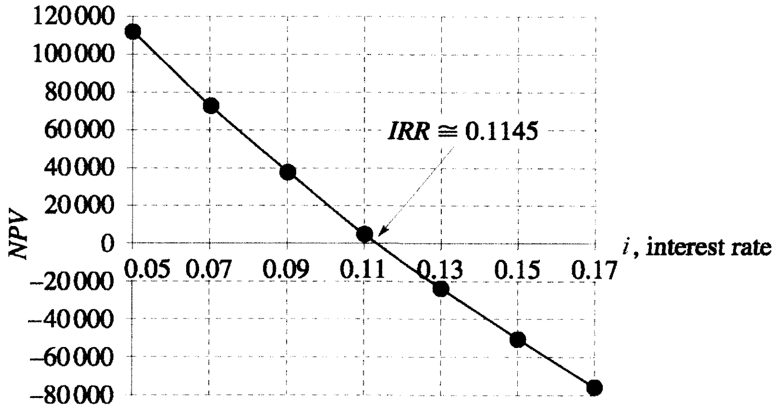


Figure 5.1 Graphical determination of *IRR*

Note: For convenience, the points are labelled according to the column headings in Table 5.5.

(i) From points C and G,

$$IRR = \frac{(i_1 \times NPV_2) - (i_2 \times NPV_1)}{NPV_2 - NPV_1} = \frac{(0.05)(-24011) - (0.13)(109045)}{-24011 - 109045}$$

$$= \frac{-15376.4}{-133056} = 0.11556$$

Therefore, $IRR = 11.556\%$

(ii) From points F and I,

$$IRR = \frac{(i_1 \times NPV_2) - (i_2 \times NPV_1)}{NPV_2 - NPV_1} = \frac{(0.11)(-75691) - (0.17)(5143)}{-75691 - 5143}$$

$$= \frac{-9200.32}{-80834} = 0.1138175$$

Therefore, $IRR = 11.382\%$

The two results (i) and (ii) demonstrate that slightly different estimates are calculated from different pairs of points.

☐ Comparison of appraisal techniques: *NPV*, *IRR*

When comparing the profitability of two or more projects, the most profitable project would be (a) the project with the largest *NPV*, (b) the project with the largest *IRR*.

The advantage of the *NPV* method is that it gives results in cash terms; it is practical as it discounts net cash flows. A disadvantage is that it relies on the choice of one discount rate; it is possible that a change in the discount rate could lead to a change in the choice of project. The *IRR* method of appraisal does not depend on any external rates of interest, but a major disadvantage of this method is that it does not differentiate between the scale of projects; for example, one project might involve a cash flow in units of £100 000 while another might involve units of £1.

Progress Exercises 5.5 Depreciation, NPV, IRR

1. £150 000 is invested in machinery which depreciates at 8% per annum. How much will the machinery be worth in 10 years by the reducing-balance method?
2. (a) Derive the formula for the net present value factor.
   (b) The net cash flow for two projects, A and B, is as follows,

Year	0	1	2	3	4
Project A	−10 000	−3000	4000	6000	8000
Project B	−5000	−2000	1000	3000	5000

(i) Use the net present value criterion to decide which project is the most profitable if a discount rate of 6% is used.

(ii) Calculate the *IRR*. State the conditions for each project to be profitable.

3. The net cash flow for two projects, A (fast food) and B (amusements), is as follows:

Year	0	1	2	3	4	5
Project A	−420 000	−5 000	122 000	130 000	148 000	150 000
Project B	−95 000	−10 000	−120 000	200 000	110 000	−50 000

(a) Use the net present value criterion to decide which project is the most profitable if a discount rate of (i) 6% and (ii) 8% is used.
(b) Estimate the *IRR* of each project. Which project would now be considered more profitable?

5.5 Annuities, Debt Repayments, Sinking Funds

In this section the compound interest problem is extended to annuities, debt repayment and sinking funds. The relevant formulae are derived on the assumption of annual compounding at a rate of i% per annum, but these formulae are easily adapted to compounding over any interval using equations (5.13) and (5.14) respectively.

❒ Compound interest for fixed deposits at regular intervals of time

In Section 5.2 the standard compound interest formula $P_t = P_0(1+i)^t$ was derived to calculate the value of an investment of P_0 after t years when interest is compounded annually at i% per annum. We now consider the situation where, in addition to the initial investment of P_0, a fixed amount A_0 is deposited at the end of each year for a period of t years. The following table sets out the calculation of the value of such an investment, year by year.

At the end of	**Value of deposit**
Year 1	$P_0(1+i)+A_0$
Year 2	$P_0(1+i)^2+A_0(1+i)+A_0$
Year 3	$P_0(1+i)^3+A_0(1+i)^2+A_0(1+i)+A_0$
$\vdots$	$\vdots$
Year t	$P_0(1+i)^t+A_0(1+i)^{t-1}+A_0(1+i)^{t-2}+\cdots+A_0(1+i)+A_0$
Value at the end of t years	$\underbrace{P_0(1+i)^t}_{P_0 \text{ at } i\% \text{ per annum for } t \text{ years}} + \underbrace{A_0(1+i)^{t-1}+A_0(1+i)^{t-2}+\cdots+A_0(1+i)+A_0]}_{A_0 \text{ at } i\% \text{ per annum for years } (t-1),(t-2),\ldots,2,1,0}$ **(5.20)**

The first term, $P_0(1+i)^t$ is simply the value of P_0 after t years with an interest rate of i% per annum compounded annually. The series in the square brackets is the sum of the sequence of deposits, A_0, made at the end of years $(t-1), (t-2), \ldots, 2, 1, 0$ with interest rate i% per annum compounded annually. This series is actually a geometric series, the sum of which may be calculated according to equation (5.4) or (5.5). To show how this is accomplished, rewrite the series

$$A_0(1+i)^{t-1}+A_0(1+i)^{t-2}+\cdots+A_0(1+i)+A_0$$

in reverse order and compare this series with the standard geometric series.

$$\underbrace{A_0}_{a} + \underbrace{A_0(1+i)}_{ar} + \cdots + \underbrace{A_0(1+i)^{t-2}}_{ar^{t-2}} + \underbrace{A_0(1+i)^{t-1}}_{ar^{t-1}} \tag{5.21}$$

The series (5.21) is a geometric series where $a \equiv A_0$ and $r \equiv (1+i)$. The sum of the first n terms of the standard geometric series is given by (5.5):

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

hence the sum of the first t terms of the series (5.21) is

$$S_t = \frac{A_0[(1+i)^t - 1]}{(1+i) - 1} = A_0 \frac{(1+i)^t - 1}{i} \tag{5.22}$$

Finally the value (V_t) of the investment at the end of t years is equal to initial investment (P_0) plus the annual investments (A_0) all compounded annually:

$$V_t = P_0(1+i)^t + A_0 \frac{(1+i)^t - 1}{i} \tag{5.23}$$

This equation forms the basis for a range of applications, such as annuities, debt repayments and sinking funds. In the remainder of this section, formulae are given for annual compounding. It is a simple matter to adjust these formulae for compounding at other intervals, as demonstrated in worked examples.

WORKED EXAMPLE 5.15

COMPOUND INTEREST FOR FIXED PERIODIC DEPOSITS

New members of a club are admitted at the start of each year and pay a joining fee of £2000. Henceforth members pay the annual subscription of £400, which falls due at the end of each year. How much does the club earn from a new member over the first 10 years, assuming an annual interest rate of 5.5%.

Solution

$$P_0 = 2000,\ A_0 = 400,\ i = 5.5/100 = 0.055,\ t = 10$$

Hence

$$\begin{aligned} V &= P_0(1+i)^t + A_0 \frac{(1+i)^t - 1}{i} \\ &= 2000(1+0.055)^{10} + 400 \frac{(1+0.055)^{10} - 1}{0.055} \\ &= 3416.2889 + 400 \frac{0.7081}{0.055} \\ &= 3416.2889 + 400(12.8745) \\ &= 8566.0889 \end{aligned}$$

❒ Annuities

An annuity is a series of equal deposits (or withdrawals) made at equal intervals of time; for example, a deposit of £2500 made each year for 20 years towards a pension fund. If the deposit is made at the time of compounding, then the annuity is called an **ordinary annuity**. This is the only type of annuity considered here. To derive the formula for calculating the value of an annuity at the end of t years, consider its value at the end of each year, as follows.

At the end of	Value of annuity
Year 1	A_0
Year 2	$A_0(1+i)+A_0$
Year 3	$A_0(1+i)^2+A_0(1+i)+A_0$
Year 4	$A_0(1+i)^3+A_0(1+i)^2+A_0(1+i)+A_0$
$\vdots$	$\vdots$
Year t	$A_0(1+i)^{t-1}+A_0(1+i)^{t-2}+A_0(1+i)^{t-3}+\cdots+A_0(1+i)+A_0$
Value of annuity after t years	$A_0(1+i)^{t-1}+A_0(1+i)^{t-2}+A_0(1+i)^{t-3}+\cdots+A_0(1+i)+A_0$

This series is identical to the geometric series in equation (5.21). Hence the total amount of the annuity at the end of t years is

$$V_{ANU,t}=A_0\frac{(1+i)^t-1}{i} \tag{5.24}$$

Note: This result may be deduced as a special case of (5.23) where $P_0=0$.

WORKED EXAMPLE 5.16

ANNUITIES

To provide for future education costs, a family considers various methods of saving. Assume saving will continue for a period of 10 years at an interest rate of 7.5% per annum.

(a) Calculate the value of the fund at the end of 10 years when a single deposit of £2000 is made annually.

(b) How much should be deposited each year if the final value of the fund is £40 000?

(c) How much should be deposited each month if the final value of the fund is £40 000?

Solution

(a) In this case $A_0 = 2000$, $i = 0.075$, $t = 10$ payments. Substitute these values into equation (5.24):

$$V_{ANU,t} = A_0 \frac{(1+i)^t - 1}{i}$$

$$V_{ANU,10} = 2000 \frac{(1+0.075)^{10} - 1}{0.075}$$

$$V_{ANU,10} = 2000 \frac{2.0610 - 1}{0.075} = 2000(14.1467)$$

$$V_{ANU,10} = 28\,293.4$$

The fund will be worth £28 293.4 at the end of 10 years.

(b) $V_{ANU,10} = 40\,000$, $i = 0.075$, $t = 10$. Substitute these values into equation (5.24):

$$40\,000 = A_0 \frac{(1+0.075)^{10} - 1}{0.075}$$

The fraction has been evaluated in (a), so we have

$$40\,000 = 14.1467A_0$$

$$A_0 = \frac{40\,000}{14.1467} = 2827.5145$$

To end up with £40 000, the annual deposit should be £2827.5145.

(c) $V_{ANU} = 40\,000, = \frac{i}{12} = \frac{0.075}{12}, t \times 12 = 10 \times 12 = 120$. Substitute these values into equation (5.24):

$$40\,000 = A_0 \frac{(1+0.075/12)^{12\times 10} - 1}{0.075/12}$$

$$40\,000 = A_0 \frac{1.1120646}{0.00625} = 177.9303A_0$$

$$A_0 = \frac{40\,000}{177.9303} = 224.8071$$

Hence saving £224.8071 per month will also provide a fund of £40 000 at the end of 10 years.

Annuities may also be viewed as a series of equal payments (or withdrawals) to be made at equal intervals of time in the future. In such situations we may be interested in the present value of the annuity. In other words, how much should be invested now (V_0) for t years at a given rate of interest (i% per annum) to cover a series of t equal annual payments. If V_0 is invested for t years at i% per annum, its value (V_t) at the end of t years is calculated by the equation $V_t = V_0(1+i)^t$.

The value of a series of payments, each of A_0, at the end of t years is

$$V_{ANU,t} = A_0 \frac{(1+i)^t - 1}{i}$$

And if the amount invested is sufficient to provide for the series of payments then

$$V_t = V_{ANU,t}$$

$$V_0(1+i)^t = A_0 \frac{(1+i)^t - 1}{i}$$

Solving for V_0 will give the required present value:

$$V_0(1+i)^t = A_0 \frac{(1+i)^t - 1}{i}$$

$$V_0 = \frac{1}{(1+i)^t} A_0 \frac{(1+i)^t - 1}{i} \quad \text{dividing both sides by } (1+i)^t$$

This equation has been used in parts (b) and (c) of Worked Example 5.16. However, for convenience, it is further simplified as follows:

$$V_0 = \frac{A_0}{i} \frac{(1+i)^t - 1}{(1+i)^t} = \frac{A_0}{i}\left(\frac{(1+i)^t}{(1+i)^t} - \frac{1}{(1+i)^t}\right) = \frac{A_0}{i}\left(1 - \frac{1}{(1+i)^t}\right) = \frac{A_0}{i} \frac{1 - (1-i)^{-t}}{1}$$

Hence

$$V_0 = A_0 \underbrace{\frac{1 - (1+i)^{-t}}{i}}_{\text{annuity factor}} \tag{5.25}$$

where $\frac{1 - (1+i)^{-t}}{i}$ is called the **annuity factor** or **cumulative present value factor**.

WORKED EXAMPLE 5.17

PRESENT VALUE OF ANNUITIES

(a) How much should you pay for an annuity of £1000 a year payable in arrears for 20 years, assuming an interest rate of 6%, if you are to break even?
(b) How much should you pay for an annuity of £1000 a year payable in arrears quarterly for 20 years, assuming an interest rate of 6% per annum.

Solution

Questions (a) and (b) are asking for the present value of the amount of the annuity: How much should be invested now in order to cover the required regular payments? The answers may be calculated by direct application of equation (5.25).

(a) $A_0 = 1000$, $i = 0.06$, $t = 20$, hence

$$V_0 = A_0 \underbrace{\frac{1-(1+i)^{-t}}{i}}_{\text{annuity factor}}$$

$$V_0 = 1000 \times \frac{1-(1+0.06)^{-20}}{0.06}$$

$$= 1000 \times 11.4699 = 11\,469.92$$

Alternatively, calculate the amount of the annuity first, then its present value:

$$V_{ANU,t} = A_0 \frac{(1+i)^t - 1}{i}$$

$$V_{ANU,20} = 1000 \frac{(1+0.06)^{20} - 1}{0.06} = 36\,785.5912$$

The present value of $V_{ANU,20}$ is given by

$$V_0 = \frac{V_{ANU,20}}{(1+i)^{20}} = \frac{36\,785.5912}{(1.06)^{20}} = 11\,469.9212$$

(b) For quarterly payments,

$$A_0 = \frac{1000}{4} = 250, \ \frac{i}{4} = \frac{0.06}{4} = 0.015, \ t \times 4 = 20 \times 4 = 80$$

And with these adjustments to equation (5.25) for quarterly payments,

$$V_0 = A_0 \underbrace{\frac{1-(1+i/4)^{-t\times 4}}{i/4}}_{\text{annuity factor}}$$

$$V_0 = 250 \times \frac{1-(1+0.015)^{-80}}{0.015}$$

$$= 250 \times 46.4073 = 11\,601.8309$$

❐ Debt repayments

A loan is said to be **amortised** if both principal and interest are to be repaid by a series of equal payments made at equal intervals of time, assuming a fixed rate of interest throughout. For such a repayment scheme, the value of the loan (L) and interest rate are usually known but the amount to be repaid at each interval must be calculated. To deduce the formula for such calculations, return to equation (5.23) and make the following substitutions:

$$\underbrace{V_t}_{\substack{\text{At the end of } t \text{ years.} \\ \text{loan is repaid.} V_t=0}} = \underbrace{P_0(1+i)^t}_{\substack{\text{Debt at end of } t \text{ years assuming} \\ \text{no repayments where } -L=P_0}} + \underbrace{A_0 \frac{(1+i)^t - 1}{i}}_{\substack{\text{Value of a series of regular} \\ \text{payments at the end of } t \text{ years}}} \qquad \textbf{(5.23)}$$

Since the loan L is a debt, it is negative, hence

$$0 = -L(1+i)^t + A_0 \frac{(1+i)^t - 1}{i} \qquad \textbf{(5.26)}$$

(i) Equation (5.26) may be solved for the size of the loan, L, given A_0, i and t:

$$L = \frac{A_0}{i}\left(1 - \frac{1}{(1+i)^t}\right) \qquad \textbf{(5.27)}$$

This equation is identical to (5.25).

(ii) Equation (5.26) may be solved for the size of the repayments, A_0, given L, i and t:

$$A_0 = \frac{Li}{1 - \dfrac{1}{(1+i)^t}} = L \underbrace{\frac{i}{1-(1+i)^{-t}}}_{\text{capital recovery factor}} \qquad \textbf{(5.28)}$$

where $\dfrac{i}{1-(1+i)^{-t}}$ is called the capital recovery factor.

Compare the capital recovery factor to the annuity factor (5.25) and you will observe that

$$\text{capital recovery factor} = \frac{1}{\text{annuity factor}}$$

Note: Tables of net present values, capital recovery factors, annuity or cumulative present value factors are published. However, these factors are easily evaluated with basic calculators. Tables will not be used in this text.

WORKED EXAMPLE 5.18

MORTGAGE REPAYMENTS

A mortgage of £200 000 is to be repaid over a 25 year period at a fixed interest rate of 4.5%. Calculate the monthly repayments.

Solution

Here we are given that $L = 200\,000$, $\frac{i}{12} = \frac{0.0045}{12} = 0.00375$ per month, $t = 25 \times 12 = 300$ payments. Substitute these values into equation (5.29), adjusted for monthly repayments, then solve for the monthly repayment, A_0:

$$A_0 = L \underbrace{\frac{i/12}{1-(1+i/12)^{-t\times 12}}}_{\text{capital recovery factor}}$$

$$= 200\,000 \times \frac{0.00375}{1-(1.00375)^{-300}}$$

$$= 200\,000 \times 0.0055583$$

$$= 1111.6650$$

Repayments per month = £1111.6650

☐ How much of the repayment is interest?

When a loan is repaid by a series of equal instalments over equal intervals of time, the proportion of interest being repaid decreases steadily. The following worked example illustrates this behaviour.

WORKED EXAMPLE 5.19

HOW MUCH OF THE REPAYMENT IS INTEREST?

A loan of £5000 is to be paid off in four equal payments at the end of each quarter. Assuming an interest rate of 20%, calculate (a) the amount of each payment, (b) the present value of the loan, (c) the total amount of interest paid, (d) the amount of interest paid each quarter.

Solution

$$L = 5000,\ \frac{i}{4} = \frac{0.20}{4} = 0.05 \text{ per quarter, } t = 1 \times 4 \text{ payment intervals}$$

(a)

$$A_0 = L \times \underbrace{\frac{i/4}{1-(1+i/4)^{-t\times 4}}}_{\text{capital recovery factor}}$$

$$A_0 = 5000 \times \frac{0.05}{1-(1.05)^{-4}}$$

$$= 5000 \times 0.2820 = 1410.06$$

(b) The present value of the loan is $V_0 = 5000$. Check by calculation:

$$V_0 = A_0 \underbrace{\frac{1-(1+i/4)^{-t\times 4}}{i/4}}_{\text{annuity factor}} = 1410.06 \frac{1-(1.05)^{-4}}{0.05} = 5000$$

(c) Total payments = 1410.06 × 4 = 5640.24

Total interest = total payments − value of loan = 5640.24 − 5000 = 640.24

(d) At the start of quarter 1, the borrower has the loan of 5000 in his account. Calculate the value of the account at the end of each quarter, where

$$\begin{pmatrix}\text{value at start}\\ \text{of quarter}\end{pmatrix} + \begin{pmatrix}\text{interest for}\\ \text{quarter}\end{pmatrix} - \begin{pmatrix}\text{quarterly}\\ \text{payment}\end{pmatrix} = \begin{pmatrix}\text{value at end}\\ \text{of quarter}\end{pmatrix}$$

Quarterly calculations are set out as follows.

Value at start (VSQ)	+ Interest (VSQ × 0.05)	−Payment (−1410.06)	Value at end (VSQ + VSQ × 0.05 − 1410.06)
5000.000	250.000	−1410.06	3839.940
3839.940	191.997	−1410.06	2621.877
2621.877	131.103	−1410.06	1342.920
1342.920	67.146	−1410.06	0.006
	640.246	−5640.24	

❐ Sinking funds

A sinking fund is created by putting aside a fixed sum each year for the purpose of paying debts, replacing equipment, etc. In other words, an annuity is set up to repay the debt. If a fixed sum, A_0, is set aside at the start of each year and interest is compounded annually at i%, the fund will grow year by year as follows.

At the end of	Value of sinking fund
Year 1	$A_0(1+i)+A_0$
Year 2	$A_0(1+i)^2+A_0(1+i)+A_0$
Year 3	$A_0(1+i)^3+A_0(1+i)^2+A_0(1+i)+A_0$
$\vdots$	$\vdots$
Year t	$A_0(1+i)^t+A_0(1+i)^{t-1}+A_0(1+i)^{t-2}+\cdots+A_0(1+i)+0$
Value of sinking fund after t years	$A_0(1+i)^t+A_0(1+i)^{t-1}+A_0(1+i)^{t-2}+\cdots+A_0(1+i)$

Note: No deposit is made at the end of year t. The fund has matured and is sufficient to repay the debt. The value of the fund is

$$V_{SF,t}=A_0(1+i)^t+A_0(1+i)^{t-1}+A_0(1+i)^{t-2}+\cdots+A_0(1+i)$$

This series is a simple geometric series whose sum is readily calculated by equation (5.5), after some slight simplifications. Start by writing the series in reverse order:

$$V_{SK,t}=A_0(1+i)+\cdots+A_0(1+i)^{t-2}+A_0(1+i)^{t-1}+A_0(1+i)^t$$

Factor out $A_0(1+i)$, which is common to all terms:

$$V_{SK,t}=A_0(1+i)\underbrace{\left[1+(1+i)+(1+i)^2+\cdots+(1+i)^{t-1}\right]}_{\text{geometric series with } a=1,\, r=(1+i)} \qquad \textbf{(5.29)}$$

The series in the square bracket is geometric, $a=1$, $r=(1+i)$. The sum of the series is calculated by (5.5) with these values for a and r:

$$1+(1+i)+(1+i)^2+\cdots+(1+i)^{t-1}=\frac{(1+i)^t-1}{(1+i)-1}$$

Hence equation (5.29) simplifies to

$$V_{SK,t}=A_0(1+i)\,\frac{(1+i)^t-1}{(1+i)-1}=A_0(1+i)\,\frac{(1+i)^t-1}{i} \qquad \textbf{(5.30)}$$

It is left as an exercise to show that equation (5.30) may also be deduced from equation (5.23) by substituting $P_0 = A_0$ and subtracting A_0 from the series, since there is no payment at the end of year t. Note that a sinking fund is a form of annuity where deposits are made at the beginning of each interval.

WORKED EXAMPLE 5.20

SINKING FUNDS

A taxi service must replace cars every 5 years at a cost of £450 000. At an 8% rate of interest, calculate

(a) the size of the fund if £4000 is deposited at the beginning of each month

(b) the size of each quarterly payment necessary to meet this target

Solution

(a) Given $A_0 = 4000$, $\dfrac{i}{12} = \dfrac{0.08}{12} = 0.0067$, $t = 5 \times 12 = 60$ instalments

Hence by

$$V_{SK.t} = A_0(1 + i/12)\,\frac{(1 + i/12)^t - 1}{i/12} \qquad \textbf{(5.29)}$$

$$\begin{aligned} V_{SK.60} &= 4000(1 + 0.0067)\,\frac{(1.0067)^{60} - 1}{0.0067} \\ &= 4000(1.0067)\,\frac{0.4928}{0.0067} \\ &= 4000(1.0067)(73.5522) \\ &= 296\,179.9989 \end{aligned}$$

Not sufficient to meet target of £450 000.

(b) Substitute the information given into equation (5.29):

$V_{SK} = 450\,000$, $\dfrac{i}{4} = \dfrac{0.08}{4} = 0.02$, $t \times 4 = 5 \times 4 = 20$ payment intervals

$$\begin{aligned} V_{SK.t} &= A_0(1 + i/4)\,\frac{(1 + i/4)^{t\times 4} - 1}{i/4} \\ 450\,000 &= A_0(1 + 0.02)\,\frac{(1 + 0.02)^{20} - 1}{0.02} \\ 450\,000 &= A_0(1.02)(24.2974) \\ 450\,000 &= A_0(24.7833) \\ A_0 &= \frac{450\,000}{24.7833} = 18\,157.3882 \end{aligned}$$

Quarterly payments of £18 157.3882 are required to meet the target.

Progress Exercises 5.6 Annuities, Debt Repayments, Sinking Funds

1. A company operates a savings scheme (interest rate 4% per annum) for its employees. (a) Calculate the amount saved at the end of one year when £200 is deposited at the end of each month. (b) How much should be saved from a monthly salary in order to have £5000 at the end of the year?
2. A child is given a savings account with £2000. A further £500 is deposited in the account at the end of each year. Assuming an annual interest rate of 6%, calculate the value of the account (a) at the end of 10 years, (b) at the end of 20 years.
3. How much should you pay for an annuity of quarterly payments of £1500 for five years, assuming interest rates of 5% per annum?
4. A loan of £100 000 is to be repaid in annual payments over 10 years. Assuming a fixed interest rate of 10% per annum, calculate (a) the amount of each annual repayment, (b) the total interest paid, (c) the amount of interest paid in the first and second repayment.
5. A loan of £50 000 is to be repaid in equal quarterly instalments over a period of five years. If interest is 7.5% per annum, calculate (a) the amount of each quarterly payment, (b) the total amount of interest paid, (c) the interest paid each quarter in the first year of repayments.
6. A loan is repaid at the rate of £200 per week for 15 years. If interest is 6.65% per annum, calculate (a) the total amount repaid, (b) the size of the loan, (c) the total amount of interest paid.
7. A company must replace machinery every two years at a cost of £100 000. It is decided to set aside equal amounts at the beginning of each quarter. If interest is 5.75% per annum, calculate the size of the quarterly deposits.
8. Starting on her 35th birthday, a woman saves £200 at the end of each month with a view to retiring on her 50th birthday.
   (a) What is the value of her savings if the interest rate is 8% per annum?
   (b) From the age of 50, how much should she withdraw each month if the fund is to last for the next 15 years, assuming the interest rate is 9% per annum?
9. An education fund of £40 000 is set up for a child on his 4th birthday. The fund is to mature on his 18th birthday. Assuming an interest rate of 10% per annum, calculate (a) the present value of the fund assuming annual compounding, (b) the size of monthly deposits (in arrears) required to achieve the fund of £40 000, (c) the size of quarterly deposits required to achieve the fund of £40 000.
10. A student registers for a four-year course. She has a fund of £40 000 to cover expenses over the next four years. Calculate the amount of periodic withdrawals (in arrears) if she withdraws money (a) weekly and (b) monthly. Assume the interest rate is 7.5% per annum.
11. A company has the option of leasing equipment for an annual fee of £3500 over four years or buying it for £12 500. If the equipment is worthless at the end of 4 years, which option is preferable if the annual rate of interest is (a) 5% and (b) 4%?

5.6 The Relationship between Interest Rates and the Price of Bonds

This section analyses the relationship between interest rates, the speculative demand for money and government bonds. It involves everything from simple interest, present values to *NPV*s.

A **bond** is a cash investment made to the government, usually in units of £1000 for an agreed number of years. In return, the government pays the investor a fixed sum at the end of each year; in addition, the government repays the original value (face value) of the bond to the investor with the final payment.

To make bonds attractive to investors, the size of the fixed annual payments (sometimes referred to as the coupon) must be based on the prevailing rate of interest (r) at the time of purchase. The fixed annual payment is calculated as follows:

$$\text{Annual payment} = r \times (\text{price of bond}) \qquad \textbf{(5.31)}$$

For example, a £1000 bond is bought when the prevailing interest rate is 8%, then

$$\text{Annual payment} = r \times (\text{price of bond}) = 0.08 \times 1000 = 80$$

The annual payment, £80, is fixed for the duration of the life of the bond. On maturity, the face value of £1000 is repaid to the investor. (In effect, this is an application of simple interest.)

So why are bonds attractive as an investment? If interest rates do not change, it certainly does not make sense to invest when the growth of the investment is based on simple interest. Compound interest is much more attractive! However, interest rates *do* change. To assess the attractiveness of the future fixed payments (cash flow) as interest rates increase or decrease, we return to *NPV*s. The future cash flows are discounted to the present, hence, comparisons may be easily made; in addition, the value of the bond itself, due with the final payment is also discounted to the present, to determine whether a capital gain or loss is made on the cash investment. These calculations are carried out in Worked Example 5.21.

WORKED EXAMPLE 5.21

THE INTEREST RATE AND THE PRICE OF BONDS

A 5-year government bond valued at £1000 is purchased when the market rate of interest is 8%.

(a) Calculate the annual repayment (value of the coupon) made to the investor at the end of each year.

(b) Calculate the *NPV* of the agreed cash flow when interest rates change immediately to each of the following (the original 8% is included for comparison): 6.5%, 7%, 7.5%, 8%, 8.5%, 9%, 9.5%, 10%.

(c) Calculate the present value of the bond which is due to be repaid at the end of the 5-year lifetime.

Comment on

(i) The relationship between interest rates and the attractiveness of bonds as an investment

(ii) The return from bonds when the interest does not change compared to investing the same amount at interest rates compounded annually.

Solution

(a) When the interest rate is 8%, the annual payment (or coupon) is $(8/100) \times 1000 = 80$, that is, £80 is received by the investor at the end of each year.

(b) Table 5.6 outlines the calculation of the *NPV*s for the future cash flows at the given interest rates. (Excel is very useful for these calculations.)

Table 5.6 Interest rates and the *NPV* for the cash flow for a £1000 bond

End of year	**Cash flow**	**0.065**	**0.07**	**0.075**	**0.08**	**0.085**	**0.09**	**0.095**	**0.1**
1	80	75.12	74.77	74.42	74.07	73.73	73.39	73.06	72.73
2	80	70.53	69.88	69.23	68.59	67.96	67.33	66.72	66.12
3	80	66.23	65.30	64.40	63.51	62.63	61.77	60.93	60.11
4	80	62.19	61.03	59.90	58.80	57.73	56.67	55.65	54.64
5	80	58.39	57.04	55.72	54.45	53.20	51.99	50.82	49.67
	NPV	332.45	328.02	323.67	319.42	315.25	311.17	307.18	303.26
P_0 of bond		729.88	712.99	696.56	680.58	665.05	649.93	635.23	620.92
(P_0 of bond+*NPV*)		1062.34	1041.00	1020.23	1000.00	980.30	961.10	942.40	924.18

(c) The present value of the redeemed bond at the end of 5 years is calculated by equation (5.11), $P_0 = P_t/(1+i)^t$, where $P_t = £1000$

$$i = 6.5\% = 0.065 \quad \rightarrow \quad P_0 = \frac{1000}{(1+0.065)^5} = £729.88$$

$$i = 7.0\% = 0.07 \quad \rightarrow \quad P_0 = \frac{1000}{(1+0.07)^5} = £712.99$$

and similarly for each of the other interest rates. The results are summarised in Table 5.6.

Comment

(i) From Table 5.6 it is evident from a comparison of the present value of the annual payments together with the present value of the returned bond that when interest rates:

- Decrease, the present value of the cash flow and the present value of the bond increases.
- Increase, the present value of the cash flow and the present value of the bond decreases.

Hence, the inverse relationship between the attractiveness of bonds as an investment and the interest rate.

(ii) If the interest rate does not change, the investment holds its value (see Table 5.6). The outcome would be exactly the same if £1000 were put on deposit at 8% interest compounded annually when £80 is withdrawn at the end of each year. Comparisons for the returns from bonds and investments in which interest is compounded annually for rates above and below the purchase rate is left as an exercise for the student in Progress Exercises 5.7, question 4.

With the above information, we are now able to explain in more detail the relationship between the interest rate and the speculative demand for money which was introduced in Chapter 3.

If the prevailing interest rate is higher than the accepted normal rate for an economy, one expects the rate to fall towards normal in the future. Therefore, at present, investors would buy bonds hoping to make a gain (from the fixed cash flow and a capital price gain on the present value of the bond) in the future. Therefore, speculative balances are low as investors put their money into bonds.

If the prevailing interest rate is lower than the accepted normal rate for an economy, one expects the rate to rise towards normal in the future. Therefore, at present, investors would not buy bonds since they are likely to incur a loss (from the fixed cash flow and a capital price loss on the present value of the bond). Therefore, speculative balances are high as investors do not put their money into bonds.

Progress Exercises 5.7 Bonds and Interest Rates

1. A five-year government bond valued at £5000 is purchased when the market rate of interest is 20%.

   (a) Calculate the annual repayment (value of the coupon) made to the investor at the end of each year.

   (b) Calculate the *NPV* of the agreed cash flow when interest rates change immediately to each of the following (the original 20% is included for comparison):
   5%, 10%, 15%, 20%, 25%, 30%.

   (c) Calculate the present value of the bond which is due to be repaid at the end of the five-year lifetime when interest rates are 5%, 10%, 15%, 20%, 25% or 30%.

   Comment on the relationship between interest rates and the attractiveness of bonds as an investment.

2. A 10-year government bond valued at £10 000 was purchased when the market rate of interest was 9%.

   (a) Calculate the annual repayment (value of the coupon) made to the investor at the end of each year.

   (b) Calculate the *NPV* of the agreed cash flow when interest rates change immediately to each of the following (include the original 9% for comparison):
   6.5%, 9%, 12%.

   (c) After a period of 5 years, the bond is sold to another investor for £10 000.

   Determine whether the purchase is profitable for the remaining five years, given interest rates of (i) 6.5%, (ii) 12%.

3. A bond is purchased for £5000 when the interest rate is 8%.
   (a) Calculate the fixed annual payments.
   (b) Calculate the capital gain if the interest rate immediately
   (i) falls to 6.25%, (ii) increases to 10%.
4. A £5000 bond is purchased for a 10-year period when interest rates are 20%. At the same time £5000 is put on deposit at a bank and £1000 withdrawn from the account at the end of each year. Compare the return from the investment in the bond to that from the bank if the bank offers
   (a) A 20% rate of interest compounded annually
   (b) A 5% rate of interest compounded annually
   (c) A 30% rate of interest compounded annually

5.7 Excel for Financial Mathematics

In this chapter, Excel is particularly useful for calculating tables of values, such as in Worked Example 5.14, where several *NPV*s were calculated at different discount rates. Worked Example 5.22 compares the growth of an investment for the same nominal rate of interest, but using different methods of compounding.

WORKED EXAMPLE 5.22

GROWTH OF AN INVESTMENT USING DIFFERENT METHODS OF COMPOUNDING (EXCEL)

£1 is invested at a nominal rate of interest of 50%.

(a) Set up a table comparing the growth of the investment over a 20-year period if interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) continuously.
(b) Graph the growth of the investment by all four methods given in (a) on the same diagram for the last five years.

Solution

Set up a table in Excel similar to Table 5.7. The formula for each method of compounding is entered into cells B4, C4, D4 and E4 respectively as follows:

(i) Annual compounding: $P_t = P_0(1 + i)^t$ for $t = 1$ to 20.

With the cursor (or pointer) in cell B4, type = B$3*1.5^A4.
This formula states: principal × 1.5 raised to the power of time, when rate = 0.5.
The $ sign in front of the number 3 is required since we want to reference cell B3 (the same principal), when this formula is copied down column B. If the $ sign is not used, then as the formula is copied down, B3 would become B4, B4 becomes B5, etc.
(Of course, with a £1 deposit, it is not necessary to reference any cell, but once the table is set up, the table will recalculate itself if the initial value of the investment is changed. Try it!)

Table 5.7 Different methods of compounding

	A	B	C	D	E
1			Method of compounding		
2	Year	Annually	Quarterly	Monthly	Continuous
3	0.00	1.00	1.00	1.00	1.00
4	1.00	1.50	1.60	1.63	1.65
5	2.00	2.25	2.57	2.66	2.72
6	3.00	3.38	4.11	4.35	4.48
7	4.00	5.06	6.58	7.10	7.39
8	5.00	7.59	10.55	11.58	12.18
9	6.00	11.39	16.89	18.90	20.09
10	7.00	17.09	27.06	30.85	33.12
11	8.00	25.63	43.34	50.35	54.60
12	9.00	38.44	69.42	82.17	90.02
13	10.00	57.67	111.20	134.11	148.41
14	11.00	86.50	178.12	218.88	244.69
15	12.00	129.75	285.31	357.23	403.43
16	13.00	194.62	457.02	583.03	665.14
17	14.00	291.93	732.05	951.56	1 096.63
18	15.00	437.89	1 172.60	1 553.03	1 808.04
19	16.00	656.84	1 878.28	2 534.70	2 980.96
20	17.00	985.26	3 008.65	4 136.86	4 914.77
21	18.00	1 477.89	4 819.27	6 751.75	8 103.08
22	19.00	2 216.84	7 719.55	11 019.50	13 359.73
23	20.00	3 325.26	12 365.22	17 984.87	22 026.47

(ii) Quarterly compounding:

$$P_t = P_0\left(1 + \frac{i}{4}\right)^{4t} \quad \text{for } t = 1 \text{ to } 20$$

With the cursor in cell C4, type = C$3*1.125^(A4*4), then copy this formula down column C. This formula states: principal × (1 + rate/4) raised to the power of (time × 4), when rate = 0.5.

(iii) Monthly compounding:

$$P_t = P_0\left(1 + \frac{i}{12}\right)^{12t} \quad \text{for } t = 1 \text{ to } 20$$

With the cursor in cell D4, type = D$3*1.041667^(A4*12), then copy this formula down column D.

This formula states: principal × (1 + rate/12) raised to the power of (time × 12), when rate = 0.5

(iv) Continuous compounding: $P_t = P_0 e^{it}$.

With the cursor in cell E4, type = E$3*exp(0.5*A4), then copy this formula down column E. This formula states: principal × *e* raised to the power of (time × rate), when rate = 0.5

The results of the calculations are given in Table 5.7.

(b) Since we require the graph for the last 5 years only, copy the row of column titles to a new area on the sheet. Immediately below the row of titles copy the last five rows of Table 5.7. The new table is given as Table 5.8. Now go through the usual steps of plotting the graph. The final graph should look like Figure 5.2.

From the graphs, the continuous method of compounding leads to the greatest cumulative total after 20 years, while the annual method gives the smallest sum. Furthermore, the longer the investment is left on deposit, the wider the differences in its value when compounded by methods (i), (ii), (iii) and (iv).

Table 5.8 Data for plotting Figure 5.2

Year	Annually	Quarterly	Monthly	Continuous
16.00	656.84	1 878.28	2 534.70	2 980.96
17.00	985.26	3 008.65	4 136.86	4 914.77
18.00	1 477.89	4 819.27	6 751.75	8 103.08
19.00	2 216.84	7 719.55	11 019.50	13 359.73
20.00	3 325.26	12 365.22	17 984.87	22 026.47

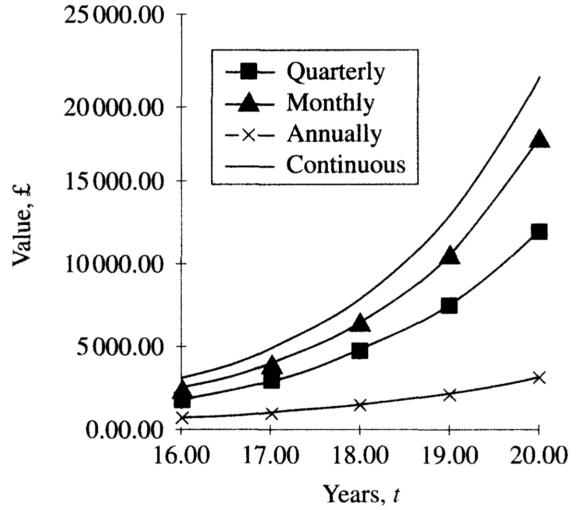


Figure 5.2 Growth of £1 at $i = 50\%$ using different compounding methods

5.8 Summary

☐ Series

Characteristics of arithmetic sequences

a is the first term of the sequence
d is the common difference
nth term, T_n, is given by $[a + (n - 1)d]$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Characteristics of geometric sequences

a is the first term of the sequence
r is the common ratio
nth term, T_n, is given by ar^{n-1}

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

❒ Financial mathematics

- **Amount due after *t* years (future value)**—bringing forward a single payment.

Simple interest: $P_t = P_0(1 + it)$

Compound interest (annual): $P_t = P_0(1 + i)^t$

Compound m times annually: $P_t = P_0\left(1 + \frac{i}{m}\right)^{mt}$

Continuous compounding: $P_t = P_0 e^{it}$

- **Present value**—of a single payment due in t years from now.

Simple discounting: $P_0 = \frac{P_t}{1 + it} = P_t(1 + it)^{-1}$

Compound discounting: $P_0 = \frac{P_t}{(1 + i)^t} = P_t(1 + i)^{-t}$

Continuous discounting: $P_0 = P_t e^{-it}$

- Annual percentage rate

(a) When the nominal rate is compounded m times per year,

$$APR = \left(1 + \frac{i}{m}\right)^m - 1$$

(b) When the nominal rate is compounded continuously,

$$APR = e^i - 1$$

- **Depreciation**

Straight-line depreciation
Reducing-balance depreciation, $A_t = A_0(1 - i)^t$

Present value: $A_0 = \frac{A_t}{(1 - i)^t}$

- **Net present value and *IRR***

NPV: present value of a future cash flow, discounted at a given discount rate r.
IRR: the discount rate for which $NPV = 0$. The *IRR* may be estimated graphically or by the formula:

$$IRR = \frac{(i_1 \times NPV_2) - (i_2 \times NPV_1)}{NPV_2 - NPV_1}$$

- **Annuities, debt repayments, sinking funds**

In the text, the basic time interval was assumed to be one year. Calculations for other time intervals are adjusted accordingly. The following assume t time periods, interest compounded at i% per time interval.
The value V_t for an initial investment of P_0 and t periodic investments of A_0 is

$$V_t = P_0(1 + i)^t + A_0\,\frac{(1 + i)^t - 1}{i}$$

The value of an annuity is

$$V_{ANU,t} = A_0 \frac{(1+i)^t - 1}{i}$$

The present value of an annuity is

$$V_0 = A_0 \underbrace{\frac{1-(1+i)^{-t}}{i}}_{\text{annuity factor}}$$

The amount of periodic repayments on a loan L is

$$A_0 = \frac{Li}{1 - \dfrac{1}{(1+i)^t}} = L \times \underbrace{\frac{i}{1-(1+i)^{-t}}}_{\text{capital recovery factor}}$$

- The value of a sinking fund, payments A_0 made at the start of each year, is

$$V_{SK,t} = A_0(1+i)\,\frac{(1+i)^t - 1}{i}$$

☐ Excel

Excel is particularly useful for calculating tables of values where the calculations are based on formulae for graphing cash flows. Besides that, Excel has a variety of financial functions.

Appendix

- **Formula (5.2): Sum of terms of an AP**

$$\begin{aligned}
S_n &= a + (a+d) + (a+2d) + \cdots + [a+(n-2)d] + [a+(n-1)d] \\
S_n &= a + (a+d) + (a+2d) + \cdots + (a+nd-2d) + (a+nd-d) \\
\underline{S_n} &= \underline{(a+nd-d) + (a+nd-2d) + \cdots + (a+2d) + (a+d) + a} \\
2S_n &= a + (2a+nd) + (2a+2d) + \cdots + (2a+nd) + (2a+nd) + a
\end{aligned}$$

There are (n) $2a$ terms $\rightarrow$ $2a \times n = 2an$;
There are $(n-1)$ nd terms $\rightarrow$ $nd \times (n-1) = nd(n-1)$; therefore,

$$\begin{aligned}
2S_n &= 2an + nd(n-1) \\
2S_n &= n[2a + (n-1)d] \\
S_n &= \frac{n}{2}[2a + (n-1)d]
\end{aligned}$$

- **Formula (5.4): Sum of terms of a GP**

$$\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \\
rS_n &= \underline{ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n} \\
S_n - rS_n &= a + 0 + 0 \qquad\qquad\qquad - ar^n \\
S_n(1-r) &= a(1-r^n) \\
S_n &= \frac{a(1-r^n)}{1-r}
\end{aligned}$$

- **Formula (5.19): Estimating the *IRR***

Equation of line joining (x_1, y_1) to (x_2, y_2) is:

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This line cuts the horizontal axis at $y = 0$; hence, solve for x. Therefore

$$\begin{aligned}
-y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \\
-y_1(x_2 - x_1) &= (y_2 - y_1)(x - x_1) \\
-x_2y_1 + x_1y_1 &= x(y_2 - y_1) - x_1y_2 + x_1y_1 \\
\frac{x_1y_2 - x_2y_1}{y_2 - y_1} &= x \\
\frac{i_1NPV_2 - i_2NPV_1}{NPV_2 - NPV_1} &= IRR
\end{aligned}$$

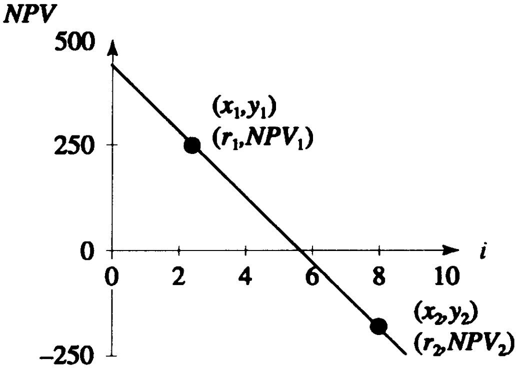


Figure 5.3 Estimating the internal rate of return

Text Exercises 5

1. (a) Find the sum of the first seven terms of the series:

(i) $3 + 9 + 27 + 81 + \cdots$ (ii) $2 + 1 + \frac{1}{2} +$

(b) A food-processing company gradually closes down the production of potato chips when the strawberry season begins. If the quantity of chips produced is 10 500 bags per week and reduction is at the rate of 20% each week and 200 kg of strawberries are processed in the first week of the season, increasing by 100 kg each for 8 weeks, calculate

(i) The number of bags of chips and kg of strawberries produced in week 5 of the strawberry season

(ii) The number of weeks taken for production of chips to drop to 500 bags per week

(iii) The total weight of strawberries processed in 8 weeks

2. Deduce the compound interest formula. If £20 000 is invested at 6.5% compound interest compounded annually calculate

(a) The value of the investment at the end of 10 years

(b) The number of years that the investment should be left on deposit if it is to double in value.

3. An investor is quoted an interest rate of 7.5%. Determine the *APR* if interest is compounded

(a) six times annually, (b) 50 times annually, (c) continuously

4. Projects A and B involve the following net cash flows:

	Initial cost	Net cash flows			
		Year 1	Year 2	Year 3	Year 4
Project A	600 000	−300	650 000	885 000	−2000
Project B	4045	2000	2500	3000	3500

Decide which project is the most profitable by

(a) Determining the *NPV* at a discount rate of 6.5%

(b) The *IRR*.

Go to the web site www.wiley.co.uk/bradley2ed for questions 5, 6 and 7.

CHAPTER 6

Differentiation and Applications

At the end of this chapter, you should understand the term differentiation and its applications in economics and business. The chapter is divided into the following sections:

6.1 Slope of a curve and differentiation
6.2 Applications of differentiation, marginal functions, average functions
6.3 Optimisation for functions of one variable
6.4 Economic applications of maximum and minimum
6.5 Curvature and other applications
6.6 Further differentiation and applications
6.7 Elasticity and the derivative
6.8 Summary

This chapter on differentiation is extremely important as it:

- Lays the foundation for economic analysis contained in subsequent chapters
- Provides tools for further analysis of economic models: utility, production, revenue, costs, profits, etc. It also allows the analysis of marginal functions, rates of increase or decrease and problems of optimisation.

It is strongly recommended that the reader work through the chapter sequentially and attempts the progress exercises as they arise.

Some of the tables and graphs, although not difficult, are very time consuming to calculate and plot by hand. The use of Excel or other spreadsheets will simplify and speed up this task.

6.1 Slope of a Curve and Differentiation

At the end of this section you should be able to:

- Show how to measure the average slope of a curve between two points
- Show how to calculate the slope of a curve at a point approximately
- Use differentiation to determine the equation for slope from the equation of the curve
- Differentiate functions containing x^n using the power rule
- Use rules for differentiating sums and differences of several functions
- Calculate higher derivatives
- Use differentiation to determine whether a function is increasing or decreasing

6.1.1 The slope of a curve is variable

In Section 2.1 the slope of a line was defined as the change in height (Δy) per unit increase in the horizontal distance (Δx). It was also noted that the slope of a straight line is the same when measured between any two points on the line. The situation is, however, different for a curve (smooth continuous curve). To see how the slope varies along a curve, the curve in Figure 6.1 is approximated by a series of straight lines or chords. A chord is a line joining any two points on a curve, such as A to B to give chord AB, B to C to give chord BC. The slope of the curve along any interval is given approximately by the slope of the chord along that same interval. It is obvious that the slope of the curve is changing as the slope of each chord is different.

The slope of a curve varies at every point on the curve

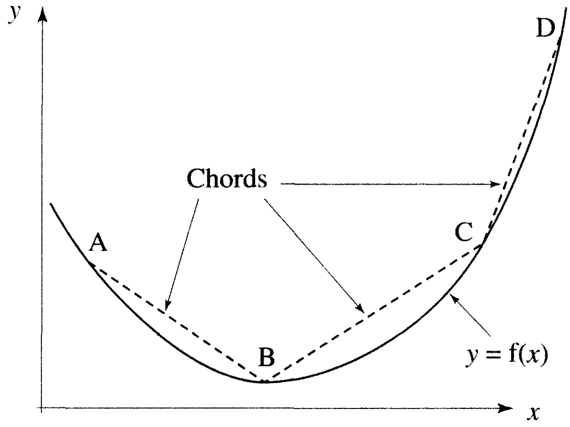


Figure 6.1 The slope of a curve is variable

6.1.2 Slope of a curve and turning points

Consider the chord BC in Figure 6.2(a) as a magnification of a very small chord, where the point B has coordinates (x, y), very close to the point C $(x + \Delta x, y + \Delta y)$, whose coordinates are only slightly different from those of B by small amounts Δx and Δy, respectively. The slope of the chord BC is a reasonable approximation to the slope of the curve over this interval:

$$\text{Slope of chord } BC = \frac{\text{change in vertical height}}{\text{change in horizontal distance}} = \frac{\Delta y}{\Delta x}$$

However, if the chord is made progressively shorter (Figure 6.2(b)) by moving the point C closer to the point B, the slopes of the chord and curve become closer as Δx and Δy become smaller. Ultimately, when the point C reaches the point B, the slopes of chord and curve are the same and the straight line that touches the curve B is called the tangent to the curve at B. At this stage the horizontal distance Δx is almost zero but not exactly zero (otherwise we have division by zero, which is not defined). We describe this by saying Δx tends to zero' or mathematically $\Delta x \rightarrow 0$. The slope at a point is then described as 'the limit of Δy over Δx as Δx tends to zero', written $\lim_{\Delta x \to 0} \Delta y/\Delta x$. A more concise way of writing this fraction is dx/dy:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

The area of mathematics which deals with infinitely small changes is called **calculus**. The process of finding dx/dy for a given curve, $y = f(x)$, by first determining $\Delta y/\Delta x$ and then allowing $\Delta x \rightarrow 0$, i.e. determining $\lim_{\Delta x \to 0} \Delta y/\Delta x$, is called differentiation from first principles. This process yields a general equation for the slope of the curve $y = f(x)$. When the equation for the slope is available, the value of the slope at any point is obtained by substituting the x-coordinate of that point. The following worked example should clarify these concepts.

WORKED EXAMPLE 6.1

EQUATION FOR THE SLOPE OF $y = x^2$ FROM FIRST PRINCIPLES

(a) Derive the equation for the slope of $y = x^2$ from first principles, i.e. use $dx/dy = \lim_{\Delta x \to 0} \Delta y/\Delta x$.

(b) Calculate (i) the y-coordinate at $x = 1.5$ and $x = -1$, and (ii) the slope at $x = 1.5$ and $x = -1$. Illustrate the answers to (i) and (ii) graphically by drawing tangents to the curve at $x = 1.5$ and $x = -1$.

Solution

(a) To determine $\lim_{\Delta x \to 0} \Delta y/\Delta x$ we need to find an expression for $\Delta y/\Delta x$. Consider any point B with coordinates (x, y) on the curve $y = x^2$ and a point C with coordinates $(x + \Delta x,\ y + \Delta y)$, very close to B. Δx and Δy are very small distances (Figure 6.2).

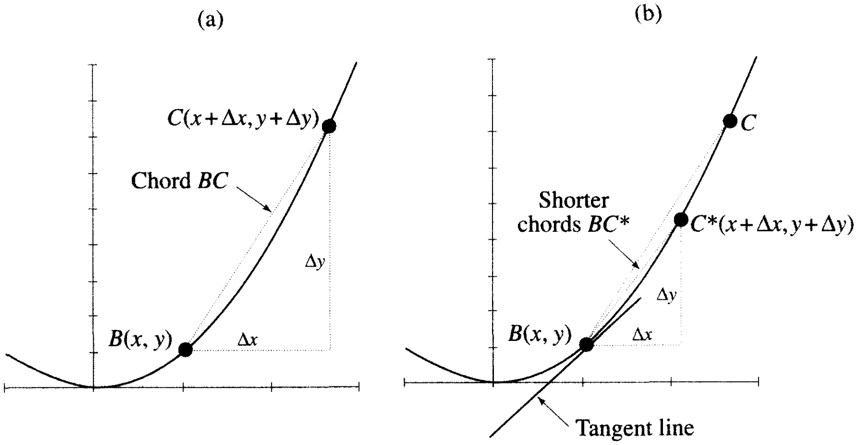


Figure 6.2 (a) Slope of chord approximates slope of curve. (b) Point $C \rightarrow C^*$ as C moves towards B. Slope of tangent at B = slope of curve at B

$y = x^2$	equation of curve satisfied by coordinates of B
$(y + \Delta y) = (x + \Delta x)^2$	equation of curve satisfied by coordinates of C
$(y + \Delta y) = x^2 + 2x\,\Delta x + (\Delta x)^2$	squaring the RHS
$\Delta y = (x^2 - y) + 2x\,\Delta x + (\Delta x)^2$	bring y over to RHS
$\Delta y = (x^2 - x^2) + 2x\,\Delta x + (\Delta x)^2$	since $y = x^2$
$\Delta y = 2x\,\Delta x + (\Delta x)^2$	since $(x^2 - x^2) = 0$
$\dfrac{\Delta y}{\Delta x} = \dfrac{2x\,\Delta x}{\Delta x} + \dfrac{(\Delta x)^2}{\Delta x}$	dividing everything on both sides by Δx
$\dfrac{\Delta y}{\Delta x} = 2x + \Delta x$	simplifying

Now let $\Delta x \rightarrow 0$, i.e. we are getting the limit as $\Delta x \rightarrow 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x$$

So the slope of the curve $y = x^2$ has the equation $\mathrm{d}y/\mathrm{d}x = 2x$.

(b) (i) To find y for a given value of x, substitute the value of the x-coordinate of the point into the equation of the curve, $y = \mathrm{f}(x)$.

(ii) To find the value of the slope $\mathrm{d}y/\mathrm{d}x$ at a given point, substitute the value of the x-coordinate of the point into the equation for the slope, $\mathrm{d}y/\mathrm{d}x = \mathrm{f}'(x)$. The values of y and $\mathrm{d}y/\mathrm{d}x$ at $x = 1.5$ and $x = -1$ are calculated below:

x-coordinate	equation of curve, $y = x^2$	equation of slope, $\mathrm{d}y/\mathrm{d}x = 2x$
$x = 1.5$	$y = x^2 = (1.5)^2 = 2.25$	$\mathrm{d}y/\mathrm{d}x = 2x = 2(1.5) = 3$
$x = -1$	$y = x^2 = (-1)^2 = 1$	$\mathrm{d}y/\mathrm{d}x = 2x = 2(-1) = -2$

The slope of the curve at $x = 1.5$ and $x = -1$ is the same as the slope of the tangents at these points (Figure 6.3).

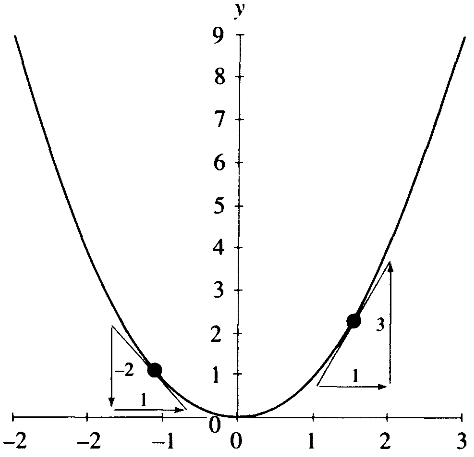


Figure 6.3 Slope of curve $y = x^2$ at $x = -1$ and $x = 1.5$

6.1.3 The derivative

Here is a brief summary of the ideas and terminology introduced in the previous section:

- dy/dx is called the derivative of y with respect to x
- The process of finding dy/dx is called differentiation
- dy/dx is the equation for the slope of the curve at any point (x, y) on the curve
- $\Delta y/\Delta x$ is the slope of the chord over a small interval Δx
- For very small intervals Δx, the slope of the curve is approximately equal to the slope of the chord, i.e. $dy/dx \approx \Delta y/\Delta x$ (the symbol $\approx$ means approximately equal to)

☐ The power rule for differentiation

- Worked Example 6.1 demonstrated how calculus was used to derive the equation for the slope of the curve $y = x^2$ from first principles.
- First principles may be used to derive the derivatives for a wide variety of other functions, such as x^n and e^x.
- From first principles, the derivative of $y = x^n$ is $dy/dx = nx^{n-1}$, where n is any real number.
- From first principles, the derivative of $y = e^x$ is $dy/dx = e^x$.

Once the general formula for the derivative of a given function is deduced, the formula may be applied to this and related functions, for example:

$$\boxed{y = x^n \rightarrow \frac{dy}{dx} = nx^{n-1} \text{ is called the power rule}} \qquad \textbf{(6.1)}$$

Worked Example 6.1 was a particular case of the power rule with $n = 2$:

$$y = x^2 \rightarrow \frac{dy}{dx} = 2x^{2-1} = 2x$$

6.1.4 How to use the power rule for differentiation

The power rule is stated again:

$$\boxed{\text{If } y = x^n \quad \text{then} \quad \frac{dy}{dx} = nx^{n-1}}$$

The use of the power rule could be described as

$$\frac{dy}{dx} = \overset{\overset{n \text{ drops down}}{\downarrow}}{n}\, x^{(n-1)\leftarrow\leftarrow\text{power loses 1}}$$

The power, n, may be any real number: positive, negative, decimal or a fraction. For example, if $y = x^5$, then the power is, $n = 5$. According to the power rule, the slope of the curve is given by the equation:

$$\text{Slope} = \frac{dy}{dx} = 5x^{5-1} = 5x^4.$$

Worked Example 6.2 illustrates how this rule is used for various powers, and also for variables other than x and y.

WORKED EXAMPLE 6.2

USING THE POWER RULE

(a) $y = x^3,\ n = 3 \quad \rightarrow \quad \text{then } \frac{dy}{dx} = 3x^{3-1} = 3x^2$

(b) $y = x^4,\ n = 4 \quad \rightarrow \quad \text{then } \frac{dy}{dx} = 4x^{4-1} = 4x^3$

(c) $P = Q^{2.6},\ n = 2.6 \quad \rightarrow \quad \text{then } \frac{dP}{dQ} = 2.6Q^{2.6-1} = 2.6Q^{1.6}$

(d) $C = Q^2,\ n = 2 \quad \rightarrow \quad \text{then } \frac{dC}{dQ} = 2Q^{2-1} = 2Q$

(e) $C = Q^{-1},\ n = -1 \quad \rightarrow \quad \text{then } \frac{dC}{dQ} = -1Q^{-1-1} = -\frac{1}{Q^2}$

❒ Some important points to note before using the power rule

- Differentiation is with respect to the variable on the RHS of each equation. In Worked Example 6.2, the variable is x in the first two cases and Q in the other three.
- Always simplify the expression to be differentiated before starting **and** whenever possible, at any stage, while working through a problem. The problem may simplify to a much easier one, which should save you a good deal of time, and mistakes are less likely. For example, differentiate C, where $C = Q^6/Q^{-4.5}$. When simplified, the expression for C is much easier to differentiate:

$$C = \frac{Q^6}{Q^{-4.5}} = \frac{Q^{6-(-4.5)}}{1} = \frac{Q^{10.5}}{1} \Rightarrow \frac{dC}{dQ} = 10.5Q^{9.5}$$

- Before using the power rule, write all base^n terms above the line, then state the value of n. For example,

$$P = \frac{1}{Q^5} = Q^{-5} \qquad n \text{ has value } -5$$

$$\frac{dP}{dQ} = -5Q^{-5-1} = -5Q^{-6}$$

- Write root signs as indices; for example,

$$y = \frac{1}{\sqrt[5]{x}} = \frac{1}{(x)^{1/5}} = \frac{1}{x^{0.2}} = x^{-0.2}$$

$$\frac{dy}{dx} = (-0.2x^{-0.2-1}) = -0.2x^{-1.2}$$

❒ The slope of a curve at a point

To find the slope at a given point, simply substitute the x-coordinate of the point into the equation for dy/dx. This is also the slope of the tangent at the given point.

The following are other notations that may be used for the derivative of $y = f(x)$:

$$\frac{dy}{dx}, \quad \frac{df(x)}{dx}, \quad \frac{df}{dx}, \quad f'(x), \quad y', \quad Dy, \quad D(f(x))$$

❒ Practical problems

In most practical problems, functions such as total cost are the sum or difference of several terms. For example, a total cost function is given by

$$TC = 0.2Q^3 - 15Q^2 + 175Q + 1000$$

where the Q terms are multiplied by constants and there is a constant term, 1000. So how do we differentiate terms such as these? The working rules listed in Section 6.1.5 deal with these problems.

6.1.5 Working rules for differentiating sums and differences of several functions

Working Rule	**Notation**	**Example**
Rule 1: The derivative of a **constant term** is zero.	$y = 10 \quad \longrightarrow \quad y = 10x^0$ $\therefore \ \frac{dy}{dx} = 10(0x^{0-1})$ $= 0$	The derivative is the rate of change and the rate of change of a constant term is zero. 10 is a constant term but $10x$ is not a constant term; it varies with x.
Rule 2: The derivative of $Kf(x)$, where K is any constant (so $Kf(x)$ is *not* a constant term), is K multiplied by the derivative of $f(x)$.	$y = Kx^n$ $\frac{dy}{dx} = K\frac{d(x^n)}{dx}$ $= K(nx^{n-1})$	$y = 25x^{0.85}$ $\frac{dy}{dx} = 25(0.85x^{0.85-1})$ $= 21.25x^{-0.15}$
Rule 3: To differentiate the sum or difference of several terms, simply differentiate each term separately.	$y = K_1x^m + K_2x^n$ $\frac{dy}{dx} = K_1(mx^{m-1})$ $+ K_2(nx^{n-1})$ $= K_1mx^{m-1}$ $+ K_2nx^{n-1}$	$TC = 0.5Q^3 - 12Q^2 + 300$ $\frac{d(TC)}{dQ} = 0.5(3Q^2) - 12(2Q) + 0$ $= 1.5Q^2 - 24Q$

Note: Sometimes functions may be written in such a way that it may be more convenient to initially find dx/dy rather than dy/dx. In order to find dy/dx, simply invert dx/dy, that is,

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

For example,

$$x = 0.75y^{-1.8}$$

$$\frac{dx}{dy} = 0.75(-1.8y^{-1.8-1}) = -1.35y^{-2.8} = -\frac{1.35}{y^{2.8}}$$

$$\frac{dy}{dx} = -\frac{y^{2.8}}{1.35} \qquad \text{inverting both sides}$$

WORKED EXAMPLE 6.3

MORE DIFFERENTIATION USING THE POWER RULE

Find the derivatives of each of the following using the power rule.

(a) $y = 5x^7$

(b) $Q = \dfrac{12}{L^{2.5}}$

(c) $y = \dfrac{2}{3\sqrt{x}}$

(d) $AC = 5 + \dfrac{2}{Q}$

Solution

(a) $y = 5x^7$

☺ $\dfrac{dy}{dx} = 5(7x^{7-1}) = 5(7x^6) = 35x^6$ the derivative of x is multiplied by the constant 5

(b) $Q = \dfrac{12}{L^{2.5}}$

$Q = 12(L^{-2.5})$ bringing the L-term above the line

$$\frac{dQ}{dL} = 12(-2.5L^{-2.5-1}) = 12(-2.5L^{-3.5}) = -30L^{-3.5}$$

☺ $\dfrac{dQ}{dL} = \dfrac{-30}{L^{3.5}}$

(c) $y = \dfrac{2}{3\sqrt{x}}$

$y = \dfrac{2}{3}(x)^{-0.5}$ arrange the x term above the line, use rules for indices

$\dfrac{dy}{dx} = \dfrac{2}{3}(-0.5x^{-0.5-1})$ differentiating the x term

$= -\dfrac{2}{3}\dfrac{(0.5x^{-1.5})}{1}$ simplifying by writing signs first, then numbers, followed by variables

☺ $= \dfrac{-1}{3x^{1.5}}$

(d) $AC = 5 + \dfrac{2}{Q}$

$AC = 5 + 2Q^{-1}$ rules for indices

$\dfrac{d(AC)}{dQ} = 0 + 2(-1Q^{-1-1})$ the derivative of a constant term is zero

$= -2Q^{-2}$

☺ $= \dfrac{-2}{Q^2}$

6.1.6 Higher derivatives

Differentiating the first derivative gives the second derivative, differentiating the second derivative gives the third, and so on. The same rules for differentiation apply always, whether the required derivative is first, second or higher.

Notation: The second derivative is written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=y'' \tag{6.2}$$

Second and third derivatives will be used later to find and confirm maximum and minimum points, and points of inflection.

WORKED EXAMPLE 6.4

CALCULATING HIGHER DERIVATIVES

Determine the first, second and third derivatives of the following demand function:

$$P = 25Q^4 - 10Q^2 + 200$$

Solution:

$$\frac{\mathrm{d}P}{\mathrm{d}Q}=25(4Q^3)-10(2Q)+0=100Q^3-20Q$$

$$\frac{\mathrm{d}^2P}{\mathrm{d}Q^2}=100(3Q^2)-20(Q^0)=300Q^2-20 \qquad \text{since} \quad Q^0=1$$

$$\frac{\mathrm{d}^3P}{\mathrm{d}Q^3}=300(2Q)-0=600Q$$

Progress Exercises 6.1 Introduction to Differentiation: the Power Rule

1. (a) Plot the curve, $y = x^2$, for $x = -2.5$; -2; -1.5; -1; -0.5; 0; 0.5; 1; 1.5; 2 and 2.5.
(b) Draw tangents to the curve at $x = -1.5$, $x = 0$ and $x = 2$; then, measure the slope of the tangents at each point.
(c) Find the slope of the curve at $x = -1.5$, $x = 0$ and $x = 2$ using differentiation. Compare your answers with those obtained in (b).

2. Differentiate each of the following, giving your answer with positive indices only:

(a) $y = x^5$ (b) $y = Q^{5.5}$ (c) $y = 10Q^8$ (d) $y = 20$

(e) $y = 20 + 3Q^2$ (f) $P = 15.2Q^{0.2}$ (g) $C = 10 - 2Y^{0.7}$

3. Differentiate each of the following, giving your answer with positive indices only:

   (a) $y = \dfrac{1}{x^2}$ (b) $y = \dfrac{5}{x^2}$ (c) $y = 10 + 5x + \dfrac{1}{x^2}$

   (d) $y = 10 + \dfrac{5}{x^2} + \dfrac{1}{x^2}$ (e) $P = \dfrac{Q^3}{3} + 70Q - 15Q^2$

4. Differentiate each of the following, giving your answer with positive indices only:

   (a) $y = 5\sqrt{x}$ (b) $y = \dfrac{1}{\sqrt{x}}$ (c) $y = \dfrac{1}{\sqrt{Q}}$ (d) $y = 7 + \dfrac{7}{\sqrt{Q}}$ (e) $y = 3\sqrt{Q^5}$

   (f) $y = \dfrac{1}{3}\sqrt[3]{Q}$ (g) $P = \dfrac{5Q+2}{5}$ (h) $P = \dfrac{5Q+2}{Q}$

5. Simplify each of the following so that differentiation may be carried out using the power rule:

   (a) $\dfrac{Q^{10}}{Q^2}$ (b) $4\dfrac{\sqrt{Q}}{Q^2}$ (c) $\dfrac{x^2+x}{x^3}$ (d) $\sqrt{5Q^3}$ (e) $3\left(\dfrac{Q^2+2Q}{Q}\right)+2$

6. Differentiate each of the functions given in question 5.

7. Find the first and second derivatives of the following functions:

   (a) x^5 (b) $3Q^2$ (c) $10Q + Q^{0.5}$ (d) $Q^{1/3}$ (e) $\dfrac{1}{x}$ (f) $Y^4 - \dfrac{1}{Y^4}$

8. Find the first, second and third derivatives of the following total cost function:

$$TC = \frac{Q^3}{5} - 8Q^2 + \frac{5Q}{2} + 180$$

9. Find dP/dQ and dQ/dP when $30P + 3Q - 81 = 0$.

10. Find dy/dx and dx/dy in terms of x, when $x^2 = 25y$.

11. Find dS/di when $480i^2 + 40i^5 - 48i = S$.

12. Find dx/dt and dt/dx in terms of x, when $x^3 6t = 18$.

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 6.

6.1.7 Use differentiation to determine whether a function or its slope is increasing or decreasing

☐ Use the first derivative to determine whether a function (y) is increasing or decreasing

☐ Use the second derivative to determine whether slope is increasing or decreasing

☐ What is an increasing or decreasing slope or rate?

Table 6.1 The coordinates and slope at selected points on the curve $y = x^2$

Figure 6.4 Graph of $y = x^2$ and slope $dy/dx = 2x$

Table 6.2 The coordinates and slope at selected points on the curve $y = 6x - x^2$

Figure 6.5 Graph of $y = 6x - x^2$ and slope $dy/dx = 6 - 2x$

Worked Example 6.5 Determining the rate of change of the slope of a curve

Progress Exercises 6.2 First and Second Derivatives: Applications

Solutions to Progress Exercises 6.2

6.2 Applications of Differentiation, Marginal Functions, Average Functions

At the end of this section you should be familiar with the following concepts:

- Marginal functions: an introduction, cost and revenue
- Average functions: an introduction, cost and revenue
- Marginal and average product of labour
- Marginal and average labour cost
- Marginal propensity to consume and marginal propensity to save
- Marginal utility and the law of diminishing marginal utility.

6.2.1 Marginal functions: an introduction

In Section 6.1 the derivative of a function was described as the equation for the rate of change of that function. For example, given the function, $y = x^3$, the equation $\mathrm{d}y/\mathrm{d}x = 3x^2$ describes the rate of change in y per unit increase in x. The derivative of certain economic variables such as *TR*, *TC*, profit, etc. is called the **marginal function**:

$$\text{Marginal revenue: } MR = \frac{\mathrm{d}(TR)}{\mathrm{d}Q} \qquad \textbf{(6.5)}$$

The marginal revenue is the rate of change in total revenue per unit increase in output, Q:

$$\text{Marginal cost: } MC = \frac{\mathrm{d}(TC)}{\mathrm{d}Q} \qquad \textbf{(6.6)}$$

The marginal cost is the rate of change in total cost per unit increase in output, Q.

To determine an expression for a marginal function the steps are as follows:

Step 1: Determine an expression for the total function.

Step 2: Differentiate the total function.

❐ Marginal revenue

To determine the equation for marginal revenue one must first obtain the equation for total revenue: $TR = P \times Q$, where P, the price, is expressed in terms of Q through the equation of the demand function.

WORKED EXAMPLE 6.6

CALCULATING MARGINAL REVENUE GIVEN THE DEMAND FUNCTION

Given the demand function $P = 6 - 0.5Q$ find the value of MR for $Q = 1, 2, 3, 4, 5, 6, 7$.

Solution

Step 1: Determine an expression for total revenue:

$$TR = P \times Q = (6 - 0.5Q)Q = 6Q - 0.5Q^2$$

Step 2: Differentiate TR to deduce the equation for MR:

$$TR = 6Q - 0.5Q^2$$

$$MR = \frac{\mathrm{d}(TR)}{\mathrm{d}Q} = 6(1) - 0.5(2Q) \quad \rightarrow \quad MR = 6 - Q$$

Column 4 of Table 6.3 shows the values of MR calculated at points $Q = 1$ to $Q = 7$.

Table 6.3 Total revenue and marginal revenue

(1)	**(2)**	**(3)**	**(4)**
Q	$TR = 6Q - 0.5Q^2$	$MR = \dfrac{\Delta TR}{\Delta Q}$	$MR = 6 - Q$
1	5.5		5
		4.5	
2	10		4
		3.5	
3	13.5		3
		2.5	
4	16		2
		1.5	
5	17.5		1
		0.5	
6	18		0
		−0.5	
7	17.5		−1

Columns (2) and (3) are calculated in Worked Example 6.8.

Note: In introductory economics textbooks, MR is often defined as the change in TR per unit change in output:

$$\boxed{MR = \frac{\Delta TR}{\Delta Q}} \qquad \textbf{(6.7)}$$

Equation (6.7) gives an expression for MR over a small interval, ΔQ, not the MR at a point. This is illustrated in Figure 6.7 where MR is the change in TR over the interval of

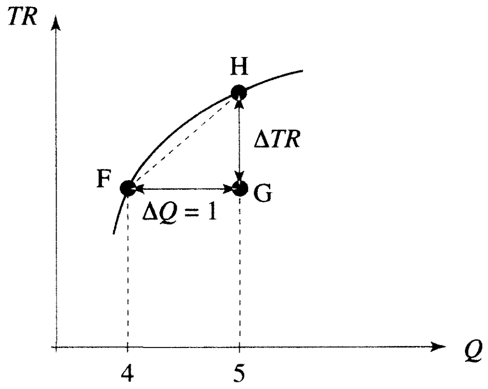


Figure 6.7 Marginal revenue measured along a chord

length ΔQ between points F and H, and not the rate of change in TR at either point F or at point H.

When the TR function is non-linear, equation (6.7) provides a reasonable approximation to equation (6.5), provided the interval is small, hence,

$$\frac{\Delta TR}{\Delta Q} \cong \frac{\mathrm{d}(TR)}{\mathrm{d}Q} \quad \rightarrow \quad \frac{\Delta TR}{\Delta Q} \cong MR$$

Therefore

$$\Delta TR \cong MR \times \Delta Q \tag{6.8}$$

That is, the change in total revenue is approximately equal to marginal revenue multiplied by the change in output. Similarly, for marginal cost (see next page):

$$\frac{\Delta TC}{\Delta Q} \cong MC \quad \rightarrow \quad \Delta TC \cong MC \times \Delta Q \tag{6.9}$$

WORKED EXAMPLE 6.7

CALCULATING MARGINAL REVENUE OVER AN INTERVAL

Given the demand function $P = 6 - 0.5Q$ find the value of MR for $1 \leq Q \leq 7$ using equation (6.7) with intervals $\Delta Q = 1$.

Solution

In this case, MR is calculated as the change in total revenue per unit change in output,

$$MR = \frac{\Delta TR}{\Delta Q}$$

The total revenue function is $TR = P \times Q = (6 - 0.5Q)Q = 6Q - 0.5Q^2$. **Note:** the price, as always, is defined by the demand function.

Column 2, in Table 6.3 shows the TR for $1 \leq Q \leq 7$. In column 3, marginal revenue is calculated for consecutive values of Q. For example, the marginal revenue over the interval $Q = 4$ to $Q = 5$ where $\Delta TR = 1.5$ and $\Delta Q = 1$ is

$$MR = \frac{\Delta TR}{\Delta Q} = \frac{1.5}{1} = 1.5$$

Comment: Compare columns 3 and 4 in Table 6.3. The MR over an interval (column 3) is the average of the point MR at the start and end of the interval (column 4). For example, the MR over the interval $Q = 4$ to $Q = 5$ is 1.5. The average of MR at points $Q = 4$ and at $Q = 5$ is

$$\frac{2+1}{2} = 1.5$$

☐ Marginal cost

Marginal cost, MC, is defined as the derivative of total cost with respect to output. In turn, total cost, TC, is the sum of fixed cost, FC, and variable cost, VC. Since FC are constant, MC may be shown to equal marginal variable costs, MVC as follows:

$$MC = \frac{\mathrm{d}(TC)}{\mathrm{d}Q} = \frac{\mathrm{d}(FC + VC)}{\mathrm{d}Q} = \frac{\mathrm{d}(VC)}{\mathrm{d}Q} = MVC \qquad \textbf{(6.10)}$$

since the derivative of fixed costs (a constant) is zero.

WORKED EXAMPLE 6.8

DERIVE MARGINAL COST EQUATION FROM TOTAL COST FUNCTION

(a) Given the total cost function, $TC = 10 + 4Q$:
  (i) Derive an equation for MC. Does MC vary with output?
  (ii) Show that the derivative of total costs and variable costs with respect to output provide the same answer.

(b) Given the total cost function, $TC = \frac{1}{3}Q^3 - 8Q^2 + 120Q$:
  (i) Deduce the equation for MC. Does MC vary with output?
  (ii) Estimate the approximate change in TC as output, Q, increases from 15 to 16 units using equation (6.9).

Solution

(a) (i) Since the TC function is given, differentiate TC with respect to output to obtain the equation for MC:

$$TC = 10 + 4Q$$

$$MC = \frac{\mathrm{d}(TC)}{\mathrm{d}Q} = 0 + 4(1) = 4$$

Comment: MC is a constant. It does not vary with output, that is, if output increases, MC remains at £4.

(ii) From the total cost function, variable cost $VC = 4Q$. The derivative of VC with respect to output is

$$VC = 4Q$$

$$MVC = \frac{\mathrm{d}(VC)}{\mathrm{d}Q} = 4$$

Comment: MC and MVC are the same.

(b) (i) The MC for this cubic TC function is derived as

Step 1: $$TC = \frac{1}{3}Q^3 - 8Q^2 + 120Q$$

Step 2: $$MC = \frac{\mathrm{d}(TC)}{\mathrm{d}Q} = Q^2 - 16Q + 120$$

Comment: MC does vary with output, for example,
At $Q = 10$, $MC = Q^2 - 16Q + 120 = (10)^2 - 16(10) + 120 = 60$.
At $Q = 20$, $MC = Q^2 - 16Q + 120 = (20)^2 - 16(20) + 120 = 200$.

(ii) As Q increases from 15 to 16 units ($\Delta Q = 1$), the approximate change in TC is

$$\Delta TC \approx MC \cdot \Delta Q \qquad \text{(see equation (6.19))}$$

$$= [(15)^2 - 16(15) + 120](1) = 105$$

where MC is evaluated at $Q = 15$. So, when output increases from 15 to 16 units, TC increases by approximately £105. That is, when the level of production is at 15 units, the total cost of producing another unit is approximately £105.

Note: The exact change in $TC = £112.33$. This is calculated by evaluating TC at $Q = 15$ and $Q = 16$ and subtracting.

6.2.2 Average functions: an introduction

From the above discussion, the equations for marginal functions, such as MR, may be used to determine the rate at which TR changes per unit increase in Q at any point. The **average function** gives an expression for the average value of an economic variable throughout an interval. The average function is defined as

$$\frac{\text{Total function}}{Q}$$

❒ **Average revenue**, AR, is defined as average revenue per unit for the first Q successive units sold. The AR is determined by dividing total revenue by the quantity sold, Q:

$$\boxed{\text{Average revenue: } AR = \frac{TR}{Q}} \qquad \textbf{(6.11)}$$

Multiplying each side of equation (6.11) by Q gives another equation for total revenue:

$$TR = AR \times Q \tag{6.12}$$

❐ The relationship between *AR* and price

Total revenue has already been defined as

$$TR = P \times Q$$

but $TR = AR \times Q$, therefore

$$AR \times Q = P \times Q$$

$$P = AR \tag{6.13}$$

The AR function is equal to price, P, where P is given by the demand function. $P = AR$ for any demand function, whether it is horizontal or downward sloping.

❐ **Average cost:** AC is total cost divided by the level of output produced, that is,

$$\text{Average cost: } AC = \frac{TC}{Q} \tag{6.14}$$

Average cost is also the sum of average fixed cost, AFC, and average variable cost, AVC:

$$\text{Average cost: } AC = \frac{TC}{Q} = \frac{FC + VC}{Q} = \frac{FC}{Q} + \frac{VC}{Q} = AFC + AVC \tag{6.15}$$

where AFC and AVC are defined as

$$AFC = \frac{FC}{Q} \text{ and } AVC = \frac{VC}{Q}$$

In Worked Example 6.9, marginal and average functions are derived. A comparison is then made between the marginal and average functions of a perfectly competitive firm and a monopolist.

WORKED EXAMPLE 6.9

MR, *AR* FOR A PERFECTLY COMPETITIVE FIRM AND A MONOPOLIST

(a) Given a perfectly competitive firm's demand function, $P = 20$, find expressions for the marginal and average revenue functions.

(b) A monopolist is faced with a linear demand function $P = 50 - 2Q$. Find expressions for the marginal and average revenue functions. Hence

(c) Calculate MR and AR for $Q = 0, 5, 10, 12.5, 15, 20, 25$. Comment on the relationship between MR and AR in (a) and (b).

Solution

(a) To find expressions for average and marginal revenue, first write down the equation for total revenue:

Step 1: $TR = P \cdot Q = 20Q$

Step 2: Marginal revenue is — Average revenue is

$$MR = \frac{d(TR)}{dQ} = \frac{d(20Q)}{dQ} = 20 \qquad AR = \frac{TR}{Q} = \frac{20Q}{Q} = 20$$

Comment: The AR function is always equal to the demand function; however, for a perfectly competitive firm, the MR function is also equal to the demand function. That is, $P = AR = MR = 20$. This condition is graphically demonstrated in Figure 6.8.

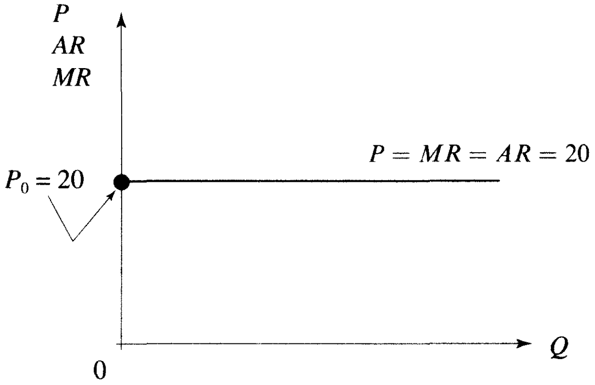


Figure 6.8 A perfectly competitive firm's AR and MR functions

Table 6.4 Marginal revenue and average revenue for a monopolist

(1) Q	**(2)** $P = 50 - 2Q$	**(3)** $TR = 50Q - 2Q^2$	**(4)** $MR = 50 - 4Q$	**(5)** $AR = 50 - 2Q$
0	50	0	50 (at $Q = 0$)	50
5	40	200	30 (at $Q = 5$)	40 (for each unit $Q = 1$ to $Q = 5$)
10	30	300	10 (at $Q = 10$)	30 (for each unit $Q = 1$ to $Q = 10$)
12.5	25	312.5	0 (at $Q = 12.5$)	25 (for each unit $Q = 1$ to $Q = 12.5$)
15	20	300	−10 (at $Q = 15$)	20 (for each unit $Q = 1$ to $Q = 15$)
20	10	200	−30 (at $Q = 20$)	10 (for each unit $Q = 1$ to $Q = 20$)
25	0	0	−50 (at $Q = 25$)	0 (for each unit $Q = 1$ to $Q = 25$)

(b) For the monopolist whose demand function is $P = 50 - 2Q$

Step 1: $TR = P \cdot Q = (50 - 2Q)Q = 50Q - 2Q^2$

Step 2: Marginal revenue is Average revenue is

$$MR = \frac{d(TR)}{dQ} = \frac{d(50Q - 2Q^2)}{dQ} = 50 - 4Q \qquad AR = \frac{TR}{Q} = \frac{50Q - 2Q^2}{Q} = 50 - 2Q$$

(c) As expected, the AR function is the same as the demand function; however, for the monopolist, the slope of the MR function (4) is double that of the demand function (2). Table 6.4 illustrates numerical values for the monopolist's average and marginal revenue functions. The functions are graphically demonstrated in Figure 6.9.

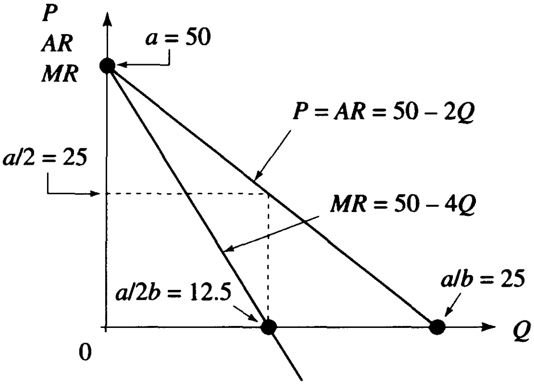


Figure 6.9 A monopolist's AR and MR functions

☐ Marginal and average revenue functions for a perfectly competitive firm and a monopolist: a summary

The relationship between the marginal and average revenue functions for a perfectly competitive firm and a monopolist are summarised in Table 6.5. Columns (2) and (4) illustrate the results as calculated in Worked Example 6.9 while columns (3) and (5) give general expressions for each type of market.

The following points are of interest:

- For monopoly, $MR < AR$ by the amount bQ. To prove this statement, write down the definition of MR in column (5) of Table 6.5:

  $$MR = a - 2bQ = (a - bQ) - bQ = AR - bQ, \text{ since } AR = (a - bQ)$$

  Hence, $MR = AR - bQ$ or $MR + bQ = AR$.

- The monopolist would not be interested in producing output where MR is negative. Thus the monopolist should only produce output up to the point where $MR = 0$. To find this point proceed as follows:

Table 6.5 Average revenue and marginal revenue functions: a summary

Functions	Perfect competition		Monopoly	
(1)	(2)	(3)	(4)	(5)
Demand function	$P = 20$	$P = P_0$	$P = 50 - 2Q$	$P = a - bQ$
Total revenue	$TR = 20Q$	$TR = P_0Q$	$TR = 50Q - 2Q^2$	$TR = aQ - bQ^2$
Average revenue	$AR = 20$	$AR = P_0$	$AR = 50 - 2Q$	$AR = a - bQ$
Marginal revenue	$MR = 20$	$MR = P_0$	$MR = 50 - 4Q$	$MR = a - 2bQ$

Example		**General expression**	
$MR = 50 - 4Q$		$MR = a - 2bQ$	
$0 = 50 - 4Q$	$MR = 0$	$0 = a - 2bQ$	$MR = 0$
$4Q = 50$		$2bQ = a$	
$Q = 12.5$		$Q = \frac{a}{2b}$	

Therefore, the monopolist should not produce any more than 12.5 units of output. (See Figure 6.9.)

❐ Total cost from average cost

Average cost is total cost divided by the level of output. Therefore, it can be shown that total cost is average cost multiplied by output, Q:

$$AC = \frac{TC}{Q} \quad \rightarrow \quad TC = AC \times Q \qquad (6.16)$$

WORKED EXAMPLE 6.10

DERIVE MARGINAL COST FROM AVERAGE COST

Find an expression for the MC function given the following average cost functions:

(a) $AC = 2Q + 5 + \frac{30}{Q}$

(b) $AC = 3Q^2 - 4Q + 6 + \frac{100}{Q}$

In each case (i) state the value of fixed cost and variable cost, and (ii) calculate the value of marginal cost when $Q = 50$.

Solution

(a) Since the AC function is given, derive the equation for TC from the definition of AC:

$$AC = \frac{TC}{Q} \quad \rightarrow \quad AC(Q) = TC$$

$$TC = AC \times Q = \left(2Q + 5 + \frac{30}{Q}\right)Q = 2Q^2 + 5Q + 30 \tag{6.17}$$

The MC function is then deduced by differentiating the total cost function,

$$MC = \frac{\mathrm{d}(TC)}{\mathrm{d}Q} = \frac{\mathrm{d}(2Q^2 + 5Q + 30)}{\mathrm{d}Q} = 4Q + 5 \tag{6.18}$$

(i) From the total cost function (equation (6.17)), $FC = 30$ and $VC = 2Q^2 + 5Q$.

(ii) Substitute $Q = 50$ into equation (6.18). Therefore, $MC = 4(50) + 5 = 205$.

(b) $TC = AC(Q)$

$$TC = \left(3Q^2 - 4Q + 6 + \frac{100}{Q}\right)Q = 3Q^3 - 4Q^2 + 6Q + 100 \tag{6.19}$$

$$MC = \frac{\mathrm{d}(TC)}{\mathrm{d}Q} = \frac{\mathrm{d}(3Q^3 - 4Q^2 + 6Q + 100)}{\mathrm{d}Q} = 9Q^2 - 8Q + 6 \tag{6.20}$$

(i) From the total cost function (equation (6.19)) $FC = 100$ and $VC = (3Q^3 - 4Q^2 + 6Q)$.

(ii) Substitute $Q = 50$ into equation (6.20), therefore, $MC = 9(50)^2 - 8(50) + 6 = 22106$.

Progress Exercises 6.3 Applications of Differentiation: MR, AR, MC, AC

1. The demand function for a monopolist is $Q = 120 - 3P$.

   (a) Find expressions for TR, MR and AR. Evaluate TR, MR and AR at $Q = 15$. Hence explain in words the meaning of each function at $Q = 15$.

   (b) Calculate the value of Q for which $AR = 0$. Calculate the value of MR when $AR = 0$. Is it sensible for a monopolist to sell this value of Q?

   (c) Graph TR, MR and AR on the same diagram.

2. The demand function for a good is $P = 125 - Q^{1.5}$.

   (a) Find expressions for TR, MR and AR. Is the slope of the MR curve twice the slope of the AR curve?

   (b) Evaluate TR, MR and AR at $Q = 10$ and at $Q = 25$. Hence, explain in words, the meaning of each function at $Q = 10$, $Q = 25$.

   (c) Calculate the value of Q for which $MR = 0$.
   Calculate the value of Q for which $AR = 0$.

At what point (value of Q) does the sale of further units first start to reduce total revenue? Explain.
(d) Graph TR, MR and AR on the same diagram.

3. A firm's fixed costs are 1000 and variable costs are given by $3Q$.
(a) Write down the equation for TC. Calculate the value of TC when $Q = 20$.
(b) Write down the equation for MC. Calculate the value of MC when $Q = 20$. Describe, in words, the meaning of MC for this function.

4. A firm has the following average cost function:

$$AC = 50 + \frac{10}{Q}$$

(a) Show by differentiation that AC decreases indefinitely as Q increases. Give an economic interpretation of this phenomenon.
(b) Write down the equation for total cost. Hence, write down the equation for total variable cost and average variable costs. State the value of fixed costs.
(c) Write down the equation for marginal costs. Comment on the relationship between TC and MC in this example.

5. A firm has an average cost function,

$$AC = Q^2 - 9Q + \frac{150}{Q} + 75$$

(a) Find an expression for the TC function and calculate TC when $Q = 15$.
(b) Write down the equations for FC and TVC.
(c) Find an expression for the MC function.
Show that $MC > 0$ for all Q. Plot the MC and AC curves on the same diagram.

6. Given the AR functions,
(i) $AR = 180 - 12Q$ (ii) $AR = 12$
(a) Write down the equations for the total revenue functions.
(b) Write down the equations for the marginal revenue functions.
(c) Comment on the relationship between the MR and AR.
(d) Graph TR, MR and AR on the same diagram.

6.2.3 Production functions and the marginal and average product of labour

Firms through the production process transform inputs (or factors of production) into units of output. There are a variety of inputs: labour, L; physical capital (buildings, machinery), K; raw materials, R; technology (including information technology), T_e; land, S; and enterprise, E.

A production function illustrates the relationship between input and output. Therefore, a production function may take the general form

$$Q = f(L, K, R, T_e, S, E \ldots)$$

This expression states that the level of output is dependent on the amounts of inputs used in the production process.

In the short run, the inputs K, R, T_e, S and E can be assumed to be fixed so the level of output, Q, is then only a function of labour, L. That is, $Q = f(L)$, such as $Q = -0.5L^3 + 20L^2$. The derivation of the marginal and average products of labour follow the methods outlined for the goods market in Section 6.2.1.

The **marginal product of labour** (MP_L) is the rate of change in total output, Q, with respect to labour input, L:

$$\text{Marginal product of labour: } MP_L = \frac{\mathrm{d}(Q)}{\mathrm{d}L} \tag{6.21}$$

The average product of labour (APL) is total output divided by the number of units of labour employed:

$$\text{Average product of labour: } APL = \frac{Q}{L} \tag{6.22}$$

Note: The APL is a measure of the average output per unit of labour, that is, it measures labour productivity.

WORKED EXAMPLE 6.11

DEDUCE THE EQUATION FOR THE MARGINAL AND AVERAGE PRODUCT OF LABOUR FROM A GIVEN PRODUCTION FUNCTION

Given the short-run production function $Q = 15L^2 - 0.5L^3$:

(a) Deduce the equation for the marginal product of labour. Calculate and comment on the marginal product of labour when 10 units of labour are employed.

(b) Derive the equation for the average product of labour. Calculate and comment on the average product of labour for the first 10 units of labour employed.

Solution

(a) Since the total output, Q, is given, differentiate Q with respect to L to obtain the equation of the marginal product of labour:

$$Q = 15L^2 - 0.5L^3 \qquad \text{total production}$$

$$MP_L = \frac{\mathrm{d}Q}{\mathrm{d}L} = 15(2L) - 0.5(3L^2) = 30L - 1.5L^2$$

When $L = 10$, $MP_L = 30(10) - 1.5(10)^2 = 150$.

Comment: At the point where 10 units of labour are employed, production is increasing at the rate of 150 units of output per additional unit of labour employed.

(b) The average product of labour is derived as

$$APL = \frac{Q}{L} = \frac{15L^2 - 0.5L^3}{L} = 15L - 0.5L^2$$

When $L = 10$, $APL = 15(10) - 0.5(10)^2 = 100$.

Comment: The average productivity per unit of labour is 100 units of output for each of the first 10 units of labour employed. Therefore, the total output of the first 10 units of labour is $APL \times L$, since

$$\frac{Q}{L} = APL$$

$$Q = APL \cdot L = (15L - 0.5L^2) \cdot L = [15(10) - 0.5(10)^2] \cdot 10 = 1000$$

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 6.

6.2.4 Perfectly competitive and monopsony labour markets

Worked Example 6.12 Find *TLC*, *MLC*, *ALC* given the labour supply functions

Worked Example 6.13 Find *TLC*, *MLC*, *ALC* for a perfectly competitive firm and a monopsonist

Figure 6.10 A perfectly competitive firm's *ALC* and *MLC* functions

Figure 6.11 A monopsonist's *ALC* and *MLC* functions

❐ Marginal and average labour costs for a perfectly competitive firm and a monopsonist: a summary

Table 6.6 Marginal and average labour cost functions: a summary

6.2.5 Marginal and average propensity to consume and save

❐ Marginal propensity to consume and save

The marginal propensity to consume, *MPC*, is defined as the change in consumption per unit change in income, and likewise, the marginal propensity to save, *MPS*, is defined as the change in savings per unit change in income. These were expressed in Chapter 3 as

$$MPC = \frac{\Delta C}{\Delta Y} \quad \text{and} \quad MPS = \frac{\Delta S}{\Delta Y}$$

These formulae are only exact when the consumption and savings functions are linear. When the consumption and savings functions are non-linear, the derivative should be used to measure the *MPC* and *MPS*.

Summary of definitions for MPC, APC, MPS, APS for the consumption function $C = C_0 + bY$

Function	**Consumption: $C = C_0 + bY$**	**Savings: $S = Y - C = Y - C_0 - bY$**
Marginal function	$MPC = \frac{\mathrm{d}(C)}{\mathrm{d}Y}$	$MPS = \frac{\mathrm{d}(S)}{\mathrm{d}Y}$
$0 < MPC < 1$	$MPC = \frac{\mathrm{d}C}{\mathrm{d}Y} = \frac{\mathrm{d}(C_0 + bY)}{\mathrm{d}Y} = b$	$MPS = \frac{\mathrm{d}S}{\mathrm{d}Y} = \frac{\mathrm{d}(Y - C_0 - \mathrm{b}Y)}{\mathrm{d}Y}$ $= 1 - b$
$0 < MPS < 1$		
Average function	$APC = \frac{C}{Y}$	$APS = \frac{S}{Y}$
	$APC = \frac{C}{Y} = \frac{C_0 + bY}{Y} = \frac{C_0}{Y} + b$	$APS = \frac{S}{Y} = \frac{Y - C_0 - bY}{Y}$ $= 1 - \frac{C_0}{Y} - b$

Note

- Since $Y = C + S$, then

$$\frac{\mathrm{d}Y}{\mathrm{d}Y} = \frac{\mathrm{d}C}{\mathrm{d}Y} + \frac{\mathrm{d}S}{\mathrm{d}Y} \quad \rightarrow \quad 1 = MPC + MPS$$

  Similarly, $APC + APS = 1$.
- Any reference to consumption and savings assumes planned consumption and planned savings respectively.
- $APC > MPC$ since

$$\frac{C_0}{Y} + b > b$$

- $MPS > APS$ since

$$1 - b > 1 - b - \frac{C_0}{Y}$$

WORKED EXAMPLE 6.14

MPC, MPS, APC, APS

Given the consumption function $C = 20 + 3Y^{0.4}$,

(a) Write down the equations for MPC and MPS.

(b) Write down the equations for APC and APS.

(c) Verify the inequalities $APC > MPC$ and $MPS > APS$ by comparing the values of MPC, APC, MPS and APS at $Y = 10$

Solution

(a) $$C = 20 + 3Y^{0.4}$$ (b)

$$MPC = \frac{dC}{dY} = 0 + 3(0.4Y^{0.4-1})$$

$$MPC = 1.2Y^{-0.6} = \frac{1.2}{Y^{0.6}}$$

$$MPS = 1 - MPC = 1 - \frac{1.2}{Y^{0.6}}$$

(c) Now evaluate MPC, APC, MPS, APS at $Y = 10$.

Marginal function at $Y = 10$	Average function at $Y = 10$
$MPC = \frac{1.2}{(10)^{0.6}} = 0.30$	$APC = \frac{20}{10} + \frac{3}{(10)^{0.6}} = 2.75$
$MPS = 1 - MPC = 1 - 0.30 = 0.70$	$APS = 1 - APC = 1 - 2.75 = -1.75$

These results verify the given inequalities $APC > MPC$ and $MPS > APS$.

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 6.

Section 6.2.6 Marginal utility

Worked Example 6.15 Calculating the marginal utility

Progress Exercises 6.4 Production, Labour Supply, Consumption, Savings and Utility

(Graphs may be plotted in Excel)

1. Given the labour supply functions
   (i) $w = 180$, perfectly competitive
   (ii) $w = 200 + 5L$, a monopsony
   (a) Find expressions for TLC, ALC and MLC.
   (b) Draw the ALC and MLC functions. Comment on the relationship between them.
2. Given the production functions, (i) $Q = 20L - L^2$, (ii) $Q = 225L - \frac{1}{3}L^3$
   (a) Find an expression for MP_L. Show that MP_L decreases indefinitely as labour input, L, increases (i.e. the law of diminishing returns applies).
   (b) Find an expression for the APL.
   (c) Plot the MP_L and APL. State the relationship between them.
3. Given the APL functions, (i) $APL = 150$, (ii) $APL = 30 - 2L$
   (a) Deduce an expression for the production function, Q.
   (b) Derive the equations for MP_L.
   (c) What is the relationship between MP_L and APL?

4. Given the consumption functions (i) $C = 40 + 0.8Y$ (ii) $C = 50 + 0.5Y^{0.8}$
   (a) Write down expressions for the marginal propensity to consume and the average propensity to consume. Show, in general, that the $APC > MPC$. Confirm this statement by evaluating APC and MPC at $Y = 10, 20$.
   (b) Deduce expressions for the marginal propensity to save and the average propensity to save. Show that the $MPS > APS$. Confirm this statement by evaluating APS and MPS at $Y = 10, 20$.
   (c) Plot the graphs of the APC, APS, MPC and MPS on the same diagram. Comment.
5. The benefit derived from study (at one sitting) is given by the utility function
   $$U = 5x^{0.8}$$
   where x is the number of hours spent studying.
   (a) Find an expression for the marginal utility, MU.
   (b) Show, by differentiation, that the level of utility decreases as the units of x increases (the law of diminishing marginal utility applies here).
   (c) Does MU ever become negative? Explain your answer.

6.3 Optimisation for Functions of One Variable

At the end of this section you should be able to:

- Use the slope of a curve to find turning points
- Use the change in slope and second derivatives to determine whether a turning point is a maximum or minimum
- Find the intervals along which a function is increasing or decreasing
- Outline the key features of curve sketching.

6.3.1 Slope of a curve and turning points

Optimisation is mainly concerned with finding maximum and minimum points, also known as stationary points, on a curve. Applications include finding maximum and minimum values for functions such as profit, cost, utility and production. In Section 6.1 the slope of a curve was shown to change at every point on the curve. Graphically, to 'see' the slope of a curve at a point, draw the tangent line to the curve at that point.

☺ **Remember,** the slope of the curve at a point is the same as the slope of the tangent at that point. Figure 6.12 shows four turning points, two minima and two maxima, with the tangents drawn at these points. Note these are called 'local' minimum and maximum points as opposed to 'global' or absolute minimum and maximum points.

In Figure 6.12 the tangents drawn at the turning points are horizontal, so they have no slope; that is, slope = 0. One can prove, with calculus that **at turning points the slope of a curve is always zero**. This fact provides the key to finding the exact coordinates of turning points. The argument is as follows:

- We know that dy/dx gives the equation for slope at any point (x, y).
- We know slope = 0 at turning points.
- So, $dy/dx = 0$ at turning points.

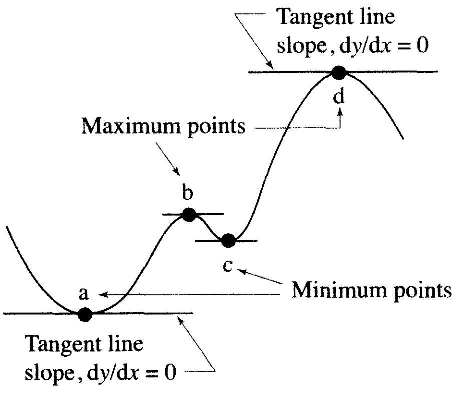


Figure 6.12 Turning points

Therefore, to find the x-coordinates of the turning points for a curve $y = f(x)$, the following method is used:

Step 1: Find dy/dx for the given curve $y = f(x)$.

Step 2: Solve the equation $dy/dx = 0$.

The solution of this equation gives the x-coordinates of the turning points.

Notes

(i) If the equation $dy/dx = 0$ has no solution, then there is no turning point.

(ii) While it is true that at turning points we can say $dy/dx = 0$, the converse is not always true. If $dy/dx = 0$ at a point, it may indeed be a turning point but it could be a point of inflection. Points of inflection are discussed in Section 6.5.2, but until then all examples and questions will deal with turning points only.

Before proceeding to worked examples, Table 6.7 briefly outlines some other terminology that is commonly used in optimisation.

Table 6.7 Some terminology used in optimisation

Terminology	Reference point, Figure 6.12	Explanation
Local minimum	*c*	Lowest point in a given interval
Local maximum	*b*	Highest point in a given interval
(Absolute) minimum	*a*	Lowest point along the entire curve
(Absolute) maximum	*d*	Highest point along the entire curve
Stationary points	*a*, *b*, *c*, *d*	Point at which slope of curve is zero

WORKED EXAMPLE 6.16

FINDING TURNING POINTS

Find the turning points for the following functions:

(a) (i) $y = 3x^2 - 18x + 34$ (b) $y = 1/x$

(ii) $AC = 3Q^2 - 18x + 34$

Solution

(i) and (ii) are identical equations in different variables.

Step 1: $y = 3x^2 - 18x + 34$

$$\text{slope} = \frac{dy}{dx} = 6x - 18$$

Step 2: Solve $dy/dx = 0$

$0 = 6x - 18$ slope = 0 at turning points

$-6x = -18$

$x = \dfrac{-18}{-6} = 3$ at turning point

when $x = 3$, $y = 3(3^2) - 18(3) + 34 = 7$

The turning point exists at $x = 3$, $y = 7$ This is identical to the AC curve (Figure 6.13) with $y \equiv AC$ and $x \equiv Q$

Step 1: $AC = 3Q^2 - 18Q + 34$

$$\text{slope} = \frac{d(AC)}{dQ} = 6Q - 18$$

Step 2: Solve $d(AC)/dQ = 0$

$0 = 6Q - 18$ slope = 0 at turning points

$-6Q = -18$

$Q = \dfrac{-18}{-6} = 3$ at turning point

when $Q = 3$, $AC = 3(3)^2 - 18(3) + 34 = 7$.

The turning point exists at $Q = 3$, $AC = 7$. The AC function is shown in Figure 6.13.

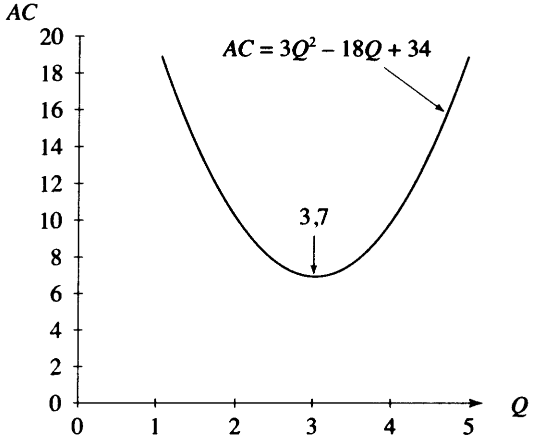


Figure 6.13 Locating turning points

(b) **Step 1:** $y = \dfrac{1}{x} = x^{-1}$ hence $\dfrac{dy}{dx} = -1\dfrac{x^{-1-1}}{1} = -\dfrac{x^{-2}}{1} = -\dfrac{1}{x^2}$

Step 2: Solve $dy/dx = 0 \rightarrow 0 = -\dfrac{1}{x^2} \rightarrow 0(x^2) = -1$

There is no value of x^2 (or x) which when multiplied by zero will have a value of -1.

☺ **Remember** $0 \times$ (any real number) $= 0$. Therefore, no value of x exists for which slope dy/dx is zero, so there is no turning point. The graph of $y = 1/x$ is illustrated in Figure 6.14.

☺ **Remember** the rectangular hyperbola of Chapter 4.

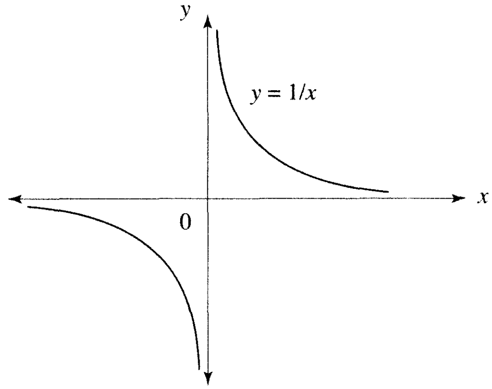


Figure 6.14 Graph of $y = 1/x$

Progress Exercises 6.5 Locate Turning Points

Locate the turning point(s) for the following functions:

1. $y = x^2 - 6x + 6$
2. $y = 2x^3 - 3x^2$
3. $P = -2Q^2 + 8Q$
4. $C = Y + 100$
5. $Q = 120 - 2.5P$
6. $y = x^3 - 3x^2 - 9x$
7. $TC = 144Q + \frac{1}{Q}$
8. $y = 8 - x^2 + x$
9. $U = 8x^{0.6}$
10. $y = x^4 - 2x^2$

6.3.2 Determining maximum and minimum turning points

The next step is to determine whether each turning point is a maximum or a minimum. We could of course plot the graphs, since we know the precise turning points as in Figure 6.13, but this is time-consuming and not practical in the long term. Before proceeding to a method that will enable us to determine whether a point is a maximum or a minimum, here are some points to note.

When y is increasing, the derivative or rate of change of y, i.e. dy/dx is positive.
When dy/dx is increasing, the derivative or rate of change of dy/dx is positive, i.e.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \text{ is positive}$$

When y is decreasing, the derivative or rate of change of y, i.e. dy/dx is negative.
When dy/dx is decreasing, the derivative or rate of change of dy/dx is negative, i.e.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \text{ is negative}$$

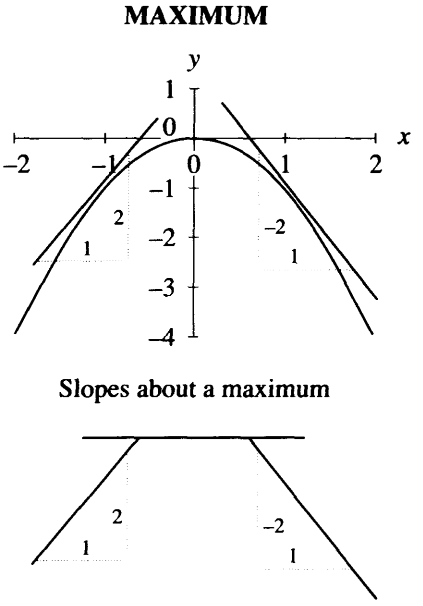
To summarise

When y is increasing then y' is positive When y is decreasing then y' is negative
When y' is increasing then y'' is positive When y' is decreasing then y'' is negative

For further detail on increasing and decreasing slopes, see Section 6.1.7 on the web site.

❑ Testing for minimum and maximum points

Figure 6.15 illustrates how slope changes as we move through maximum and minimum points.



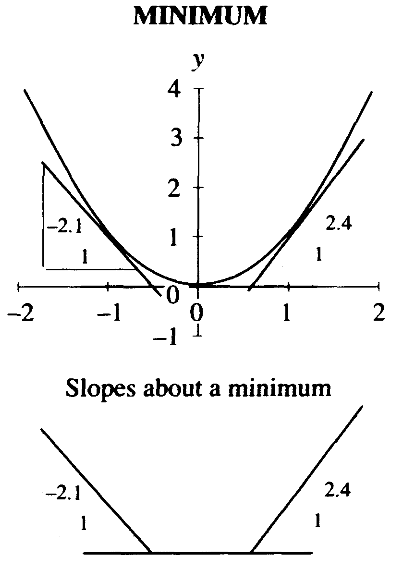
MAXIMUM

Slopes about a maximum

Slope → 2 → 0 → -2.2

Evaluated at selected points, the slope changes from positive to zero to negative therefore slope (y') is decreasing, hence y'' is negative in the region of a maximum

Figure 6.15(a) Maximum point



MINIMUM

Slopes about a minimum

Slope → -2.1 → 0 → 2.4

Evaluated at selected points, the slope changes from negative to zero to positive therefore slope (y') is increasing, hence y'' is positive in the region of a minimum

Figure 6.15(b) Minimum point

- As we move through a maximum point, illustrated in Figure 6.15(a), the slope y' changes from being positive before the maximum (as y increases towards the maximum), zero at the maximum and negative after the maximum (as y decreases down from the maximum).
- As we move through a minimum point, illustrated in Figure 6.15(b) the slope y' changes from being negative before the minimum (as y decreases down to the minimum), zero at the minimum and positive after (as y increases away from the minimum).

This behaviour provides two methods of determining whether a turning point is a maximum or a minimum. Start by calculating the value of x at which $\mathrm{d}y/\mathrm{d}x = 0$. Call this value x_0. Also determine $\mathrm{d}^2x/\mathrm{d}y^2$ and evaluate it at x_0. This is written symbolically as

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right\rvert_{x_0}$$

Method A: change of sign in slope	**Method B: second derivative sign test**
TEST FOR A MAXIMUM	
Evaluate slope $\mathrm{d}y/\mathrm{d}x$ at values of x immediately before and after x_0. If the slope changes from positive (before) to negative (after), the point is a maximum. See Figure 6.15(a)	Since the value of slope changes from positive through zero to negative as we move through a maximum, the value of slope (y') is decreasing, hence derivative of slope (y'') is negative If $\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right\rvert_{x_0}$ is negative, the point is a maximum
TEST FOR A MINIMUM	
Evaluate slope $\mathrm{d}y/\mathrm{d}x$ at values of x immediately before and after x_0. If the slope changes from negative (before) to positive (after), the point is a minimum. See Figure 6.15(b)	Since the value of slope changes from negative through zero to positive as we move through a minimum, the value of slope (y') is increasing, hence derivative of slope (y'') is positive If $\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right\rvert_{x_0}$ is positive, the point is a minimum

The second derivative sign test is the simplest to use and works for most common functions. However, if

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right\rvert_{x_0} = 0$$

the second derivative test provides no information. It is then necessary to use the change of sign in slope to decide whether the point is a turning point. If the slope does not change direction (or sign) immediately before and after the point, it may be a point of inflection (Section 6.5.2). The **max/min** method may be summarised as follows:

Step 1: Find $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}^2y/\mathrm{d}x^2$; both derivatives will be required.

Step 2: Solve the equation $\mathrm{d}y/\mathrm{d}x = 0$; the solution of this equation gives the x-coordinates of the turning points.

Step 2a: Calculate the y-coordinate of each turning point by substituting the x-coordinate of the turning point (from step 2) into the equation of the curve.

Step 3: Determine whether the turning point is a maximum or minimum by substituting the x-coordinate of the turning point (from step 2) into the equation of the second derivative (method B):

- The point is a maximum if the value of $\mathrm{d}^2y/\mathrm{d}x^2$ is negative at the point.
- The point is a minimum if the value of $\mathrm{d}^2y/\mathrm{d}x^2$ is positive at the point.

WORKED EXAMPLE 6.17

MAXIMUM AND MINIMUM TURNING POINTS

(a) Find the turning points for the curve, $y = -x^3 + 9x^2 - 24x + 26$.
(b) Determine which point is a maximum and which is a minimum by using the second derivatives.

Solution

The graph of $y = -x^3 + 9x^2 - 24x + 26$ is drawn in Figure 6.16 to confirm and show the answers found using the following method.

Step 1: Find first and second derivatives,

$$y = -x^3 + 9x^2 - 24x + 26$$

$$\text{Slope: } \frac{dy}{dx} = -3x^2 + 18x - 24$$

$$\text{Second derivative: } \frac{d^2y}{dx^2} = -6x + 18$$

Step 2: At turning points, slope is zero, therefore, solve the equation $dy/dx = 0$,

$$\frac{dy}{dx} = 0 \quad \text{at turning points}$$

$$-3x^2 + 18x - 24 = 0$$

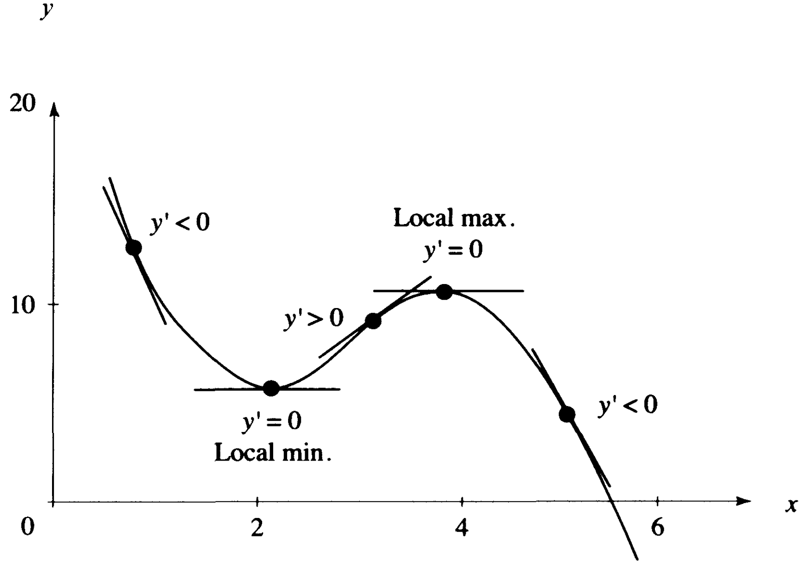


Figure 6.16 Graph of $y = -x^3 + 9x^2 - 24x + 26$

$$x^2 - 6x + 8 = 0 \quad \text{dividing both sides by } -3$$
$$(x-4)(x-2) = 0$$
$$x = 4 \text{ or } x = 2$$

(If it is difficult to find the factors for a quadratic, use the 'minus b' method to find the roots.) Therefore, there are two turning points, one at the point whose x-coordinate is $x = 4$ and the other at the point whose x-coordinate is $x = 2$.

Step 2a: Find y.
Since we know the x-coordinate, find y from the equation of the curve:

$$y = -x^3 + 9x^2 - 24x + 26$$
$$y = -(2)^3 + 9(2)^2 - 24(2) + 26 \quad \text{substituting } x = 2$$
$$= -8 + 36 - 48 + 26 = 6 \quad \therefore\ y = 6 \text{ when } x = 2$$

$$y = -(4)^3 + 9(4)^2 - 24(4) + 26 \quad \text{substituting } x = 4$$
$$= -64 + 144 - 96 + 26 = 10 \quad \therefore\ y = 10 \text{ when } x = 4$$

Step 3: Determine whether each turning point is a maximum or a minimum by substituting the value of x at the turning point into the expression for the second derivative.

Turning point at $x = 2$	**Turning point at $x = 4$**
$\frac{d^2y}{dx^2} = -6x + 18$	$\frac{d^2y}{dx^2} = -6x + 18$
$= -6(2) + 18 = 6$	$= -6(4) + 18 = -6$
POSITIVE	**NEGATIVE**
Therefore, a minimum at $x = 2$.	Therefore, a maximum at $x = 4$.
(2, 6) is a minimum point	**(4, 10) is a maximum point**

Progress Exercises 6.6 Determine Whether a Turning Point Is Maximum or Minimum

Find the maximum and/or minimum values (if any) for each of the following functions (these are the same functions as given in Progress Exercises 6.5)

1. $y = x^2 - 6x + 6$

2. $y = 2x^3 - 3x^2$

3. $P = -2Q^2 + 8Q$

4. $C = Y + 100$

5. $Q = 120 - 2.5P$

6. $y = x^3 - 3x^2 - 9x$

7. $TC = 144Q + \frac{1}{Q}$

8. $y = 8 - x^2 + x$

9. $U = 8x^{0.6}$

10. $y = x^4 - 2x^2$

6.3.3 Intervals along which a function is increasing or decreasing

Now that maximum and minimum points can be located exactly, stating the intervals along which a curve is increasing or decreasing is a very simple matter.

- If a point is a maximum point, then y must increase in the interval immediately before the maximum and decrease in the interval immediately after the maximum.
- If a point is a minimum point, then y must decrease in the interval immediately before the minimum and increase in the interval immediately after the minimum.

So, when the turning points are found, the intervals along which y is increasing or decreasing may be stated precisely. In Figure 6.16, y is decreasing towards a minimum at $x = 2$. Between the minimum and the maximum point at $x = 4$, y is increasing.

WORKED EXAMPLE 6.18

INTERVALS ALONG WHICH A CURVE IS INCREASING OR DECREASING

State the range of values of Q for which the average cost function

$$AC = -9Q + 0.5Q^2 + 43$$

is (a) increasing, (b) decreasing.

Solution

Before any statements are made about whether the function is increasing or decreasing, we must find the turning point(s), and also decide if they are a maximum or minimum. Work through the max./min. method as follows:

Step 1: Find first and second derivatives:

$$AC = -9Q + 0.5Q^2 + 43$$

$$\frac{\mathrm{d}(AC)}{\mathrm{d}Q} = -9 + Q$$

$$\frac{\mathrm{d}^2(AC)}{\mathrm{d}Q^2} = 1$$

Note: When the second derivative is a constant, there is only one turning point. The turning point is a minimum if the constant is positive, a maximum if the constant is negative.

Step 2: Solve slope = 0 as slope is zero at turning point(s):

$$\frac{\mathrm{d}(AC)}{\mathrm{d}Q} = 0$$

$$-9 + Q = 0 \quad \rightarrow \quad Q = 9$$

The turning point is at $Q = 9$.

Step 3: Determine whether the turning point is a maximum or minimum. Since the second derivative is a positive constant, then the only turning point is a minimum.

☺ **Conclusion:** AC is decreasing until $Q = 9$ (the minimum), increasing after $Q = 9$ as shown in Figure 6.17.

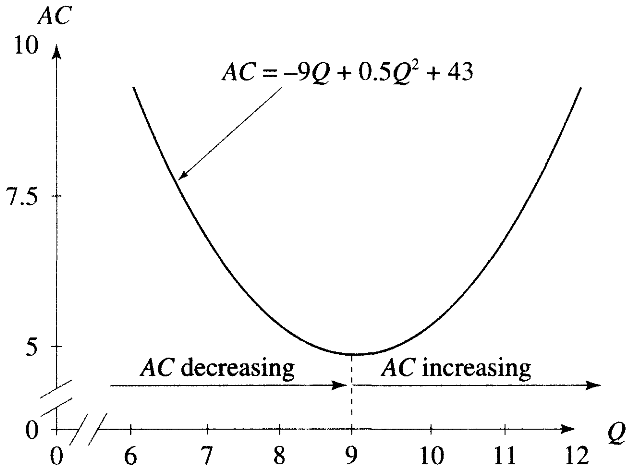


Figure 6.17 Interval along which AC is decreasing or increasing

6.3.4 Graphs of y, y', y'': derived curves

Earlier in this section it was stated that

- In the region of a maximum, the first derivative (slope) changes from positive to negative (y' decreases), so the second derivative is negative.
- In the region of a minimum the first derivative (slope) changes from negative to positive (y' increases), so the second derivative is positive.

These statements are illustrated graphically in Figure 6.18, Worked Example 6.19.

WORKED EXAMPLE 6.19

DERIVED CURVES

(a) Find the turning points for the curve, $y = 3x^2 - 0.1x^3$. Hence, determine the intervals along which y is increasing and decreasing.
(b) Plot the graphs of y against x, dy/dx against x and d^2y/dx^2 against x showing all turning points. Do the graphs of the first and second derivative confirm the rules stated above for finding optimum points?

Solution

(a) **Step 1:** Derive expressions for dy/dx and d^2y/dx^2 as follows:

One further comment about the point $x = 10$: From Figure 6.18 (b) slope is positive and decreasing up to the point at $x = 10$, and negative, but decreasing afterwards. So, at $x = 10$, the slope changes from being a positive slope to being a negative slope. This point is called a point of inflection. Points of inflection are studied later in the chapter.

Progress Exercises 6.7 Determine the Intervals along which a Curve Increases/Decreases

Determine the intervals along which each of the following curves is increasing or decreasing (consider the positive half of the plane, $x \geq 0$).

1. $y = x^2$

2. $AC = Q^2 - 20Q + 120$

3. $TR = 50Q - Q^2$

4. $AR = 50 - 2Q$

5. $MC = Q^2 - 18Q + 11$

6. $P = 10 - 2Q^{0.4}$

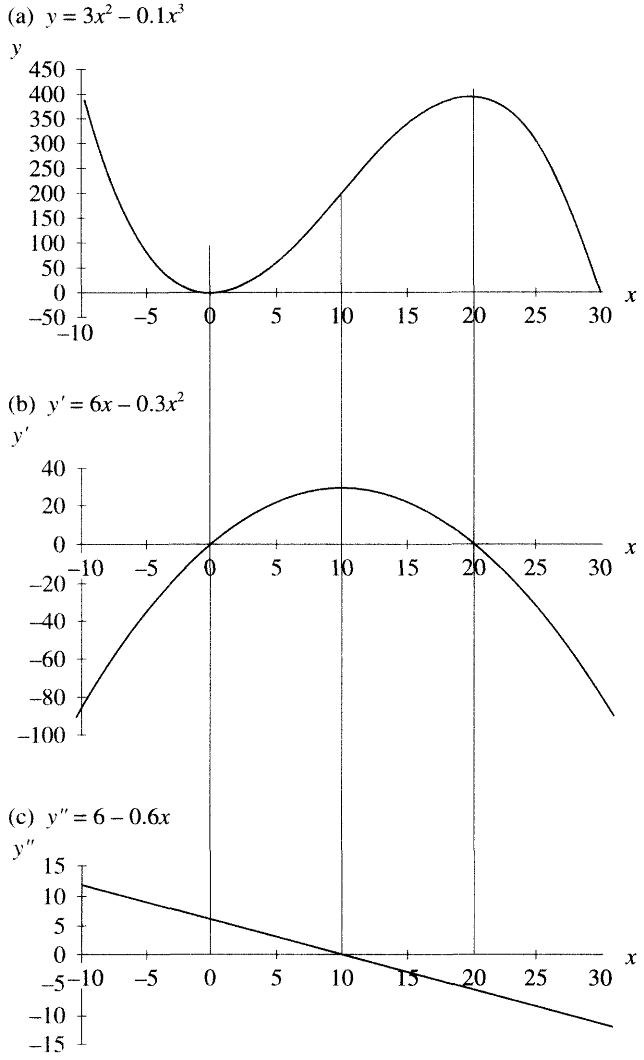
7. $TC = \frac{Q^3}{3} + 20$

8. $C = 90 + \sqrt{Y}$

6.3.5 Curve sketching and applications

Sketching curves outlines the general shape of the curve, but it also shows specific points such as turning points, points of intersection with the axes, etc. Thus, an economist and business person should be able to take most equations, such as a total cost function or a function outlining projected company sales, and determine when costs and sales increase or decrease; when the rate is increasing or decreasing; whether cost and sales ever reach a maximum or minimum, etc.

However, if a curve $y = f(x)$ is plotted by joining several points, key features of the curve may be missed out completely. For example, if the graph of $y = 1/(x - 0.23)$ is plotted as in Figure 6.19 from the points calculated in Table 6.9, a totally incorrect representation of the function in the interval $x = 0$ to $x = 1$ is obtained. The correct representation is shown in Figure 6.20.


(a) $y = 3x^2 - 0.1x^3$

(b) $y' = 6x - 0.3x^2$

(c) $y'' = 6 - 0.6x$

Figure 6.18 Graphs of y, y', y''

Table 6.8

x	**−10**	**−5**	**0**	**5**	**10**	**15**	**20**	**25**	**30**
(a) $y = 3x^2 - 0.1x^3$	400	87.5	0	62.5	200	337.5	400	312.5	0
(b) $\frac{dy}{dx} = 6x - 0.3x^2$	−90	−37.5	0	22.5	30	22.5	0	−37.5	−90
(c) $\frac{d^2y}{dx^2} = 6 - 0.6x$	12	9	6	3	0	−3	−6	−9	−12

One further comment about the point $x = 10$: From Figure 6.18 (b) slope is positive and decreasing up to the point at $x = 10$, and negative, but decreasing afterwards. So, at $x = 10$, the slope changes from being a positive slope to being a negative slope. This point is called a point of inflection. Points of inflection are studied later in the chapter.

Progress Exercises 6.7 Determine the Intervals along which a Curve Increases/Decreases

Determine the intervals along which each of the following curves is increasing or decreasing (consider the positive half of the plane, $x \geq 0$).

1. $y = x^2$
2. $AC = Q^2 - 20Q + 120$
3. $TR = 50Q - Q^2$
4. $AR = 50 - 2Q$
5. $MC = Q^2 - 18Q + 11$
6. $P = 10 - 2Q^{0.4}$
7. $TC = \frac{Q^3}{3} + 20$
8. $C = 90 + \sqrt{Y}$

6.3.5 Curve sketching and applications

Sketching curves outlines the general shape of the curve, but it also shows specific points such as turning points, points of intersection with the axes, etc. Thus, an economist and business person should be able to take most equations, such as a total cost function or a function outlining projected company sales, and determine when costs and sales increase or decrease; when the rate is increasing or decreasing; whether cost and sales ever reach a maximum or minimum, etc.

However, if a curve $y = f(x)$ is plotted by joining several points, key features of the curve may be missed out completely. For example, if the graph of $y = 1/(x - 0.23)$ is plotted as in Figure 6.19 from the points calculated in Table 6.9, a totally incorrect representation of the function in the interval $x = 0$ to $x = 1$ is obtained. The correct representation is shown in Figure 6.20.

Table 6.9 Points for Figure 6.19

x	−2	−1	0	1	2	3
$y = \frac{1}{(x - 0.23)}$	−0.448	−0.81301	−4.34783	1.298701	0.564972	0.36101

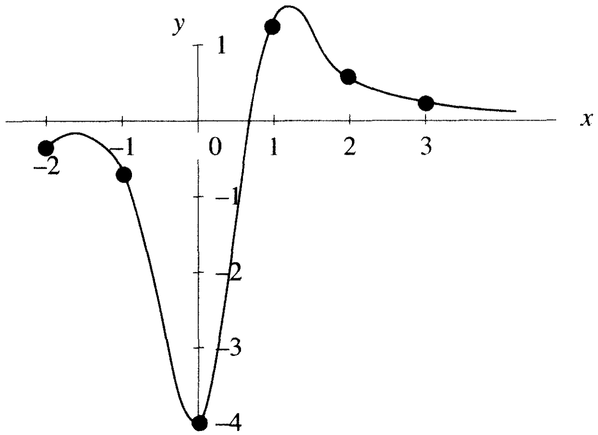


Figure 6.19 Incorrect curve of $y = 1/(x - 0.23)$

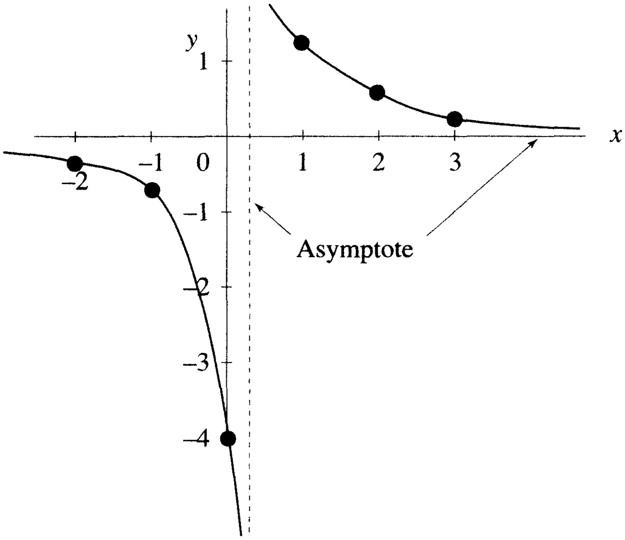


Figure 6.20 Correct curve of $y = 1/(x - 0.23)$

☹ Figure 6.19 is incorrect as:

- It loses the key feature of the correct graph, namely the asymptote at $x = 0.23$.
- It shows turning points when there are no turning points in the correct graph. You can prove this by differentiating the function.

It is therefore important to get an overview or sketch of a function at first, then the curve can be plotted accurately, point by point, over any interval if required.

☐ Some key features to look for when sketching a curve

The following points act as a useful guide to graph sketching:

1. Is the curve similar to any standard curve, for example a quadratic or polynomial, exponential, log, etc?
2. Is there any value that causes division by zero? If there is, then there will be a vertical asymptote at this point (see Chapter 4).
3. Check that the curve exists in every region of the plane, for example $y = \sqrt{x}$ is not a real number when x is negative, so there is no graph in the negative half of the plane.
4. Find the points of intersection with the axes.
5. Find turning points and points of inflection (Section 6.5.2).
6. Define the curvature over any interval if required. This is particularly important for graphs that do not have turning points, such as exponential graphs. See Worked Example 6.17(b) and Section 6.5.

The examples which follow should demonstrate how points 1 to 6 are used in the general approach to graph sketching. Points 1, 2 and 3 are normally checked first, that is, state whether the function in question is similar to any of the well-known curves, whether there are any vertical asymptotes as a result of division by zero, and whether there are regions of the plane where there are no graphs. Then points 4 and 5 are checked: the usual points of intersection with axes and turning points are calculated, and finally, check for points of inflection and curvature, particularly if there are no turning points (see Section 6.5). A number of points may be plotted, particularly if there is no turning point or point of intersection with the axes or if you need convincing.

WORKED EXAMPLE 6.20

SKETCHING FUNCTIONS

Sketch the following functions over the stated interval.

(a) $Q = 100 - P^2 \qquad -1 < P < 10$

(b) $AC = \dfrac{5}{Q} \qquad 0 < Q < 4$

Solution

(a) Plot Q on the vertical, P on the horizontal as in Figure 6.21. This is a polynomial, therefore there are no problems with discontinuities and we proceed as follows:

Where does the graph cross the axes?

The graph crosses the Q-axis at $P = 0$, that is, $Q = 100$.
The graph crosses the P-axis at $Q = 0$, that is, $P^2 = 100$, that is, at $P = 10$ and $P = -10$.

Find the turning point(s)

$$Q = 100 - P^2$$

$$\frac{dQ}{dP} = -2P$$

$$= 0 \quad \text{when } P = 0, Q = 100$$

$$\frac{d^2Q}{dP^2} = -2, \text{ negative. } \therefore \text{ a maximum}$$

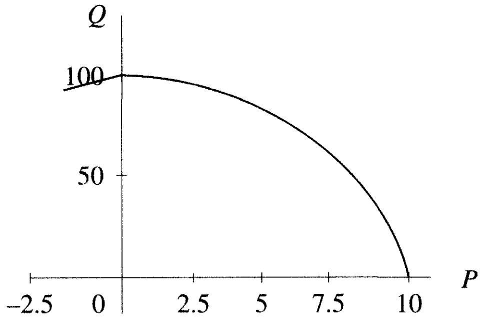


Figure 6.21 Graph of $Q = 100 - P^2$

(b) Plot AC on the vertical, Q on the horizontal as in Figure 6.22. First, this is a familiar graph (for example, see Worked Example 6.16(b), also Chapter 4, Hyperbolic functions). There is division by zero when $Q = 0$, so the AC-axis is a vertical asymptote. Also there are no turning points, as shown below. Proceed as follows:

Where does the graph cross the axes?

Graph crosses the Q-axis at $AC = 0$, i.e.

$$AC = \frac{5}{Q}$$

$$0 = \frac{5}{Q}$$

$$0(Q) = 5$$

There is no value, Q, that can be multiplied by zero to give 5; therefore, there is no solution, so the graph does not cut the Q-axis.

Graph crosses the AC-axis at $Q = 0$, i.e.

$$AC = \frac{5}{Q}$$

$$AC = \frac{5}{0}$$

$$AC(0) = 5$$

Again, there is no solution, so the graph does not cross the AC-axis. Division by zero gives the vertical asymptote.

Show that there is no turning point(s):

Step 1: Find derivatives

$$AC = \frac{5}{Q}$$

$$\frac{\mathrm{d}(AC)}{\mathrm{d}Q} = -\frac{5}{Q^2}$$

$$\frac{\mathrm{d}^2(AC)}{\mathrm{d}Q^2} = \frac{10}{Q^3}$$

Step 2: Find turning points: Solve slope = 0

$$\frac{\mathrm{d}(AC)}{\mathrm{d}Q} = 0$$

$$\frac{-5}{Q^2} = 0$$

$$-5 = 0(Q^2) = 0$$

No solution, therefore, no turning point and, hence no need to do step 3 to determine whether point is a maximum or minimum.

Table 6.10 gives some points for $AC = 5/Q$, which are then plotted as Figure 6.22.

Table 6.10 Points for Figure 6.22

Q	0	0.5	1	2	3	4
$AC = \frac{5}{Q}$	?	10	5	2.5	1.67	1.25

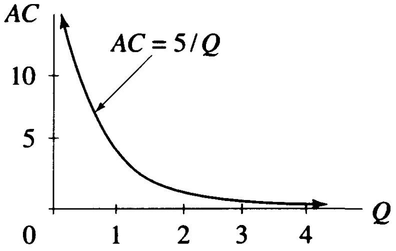


Figure 6.22 Graph of $AC = 5/Q$

Progress Exercises 6.8 Sketch Curves, Indicating Turning Points and Intersection with Axes

The following curves were given in Progress Exercises 6.5 and 6.6. Sketch each by finding the points of intersection with the axes and using the maximum and minimum points already calculated in Progress Exercises 6.6.

1. $y = x^2 - 6x + 6$
2. $y = 2x^3 - 3x^2$
3. $P = -2Q^2 + 8Q$
4. $C = Y + 100$
5. $Q = 120 - 2.5P$
6. $y = x^3 - 3x^2 - 9x$
7. $TC = 144Q + \frac{1}{Q}$
8. $y = 8 - x^2 + x$
9. $U = 8x^{0.6}$
10. $y = x^4 - 2x^2$

Confirm each of the above sketches by plotting the graphs in Excel, if available.

6.4 Economic Applications of Maximum and Minimum Points

At the end of this section you should be able to:

- Sketch curves
- Find the value of output at which TR is at a maximum
- Find the value of output at which profit (π) is at a maximum

- Understand the use of the rule: $MC = MR$ in profit optimisation
- Show the profit-maximising point of a price-discriminating firm
- Show the profit-maximising points for a perfectly competitive firm and a monopoly in the goods market.

In Section 6.2 the first derivatives of economic functions were called marginal functions. Therefore, the optimum value of functions, such as revenue, profit, cost, etc., will all occur when the corresponding marginal function (first derivative) is zero as will be shown in the following worked examples.

WORKED EXAMPLE 6.21

MAXIMUM *TR* AND A SKETCH OF THE *TR* FUNCTION

The demand function for a good is given as $P = 50 - 2Q$.
(a) Write down expressions for the *TR* and *MR* functions.
(b) Calculate the output at which *TR* is a maximum and confirm that marginal revenue is zero at this point.
(c) Sketch the *TR* and *MR* functions.

Solution

(a)

$$TR = P \times Q = (50 - 2Q)Q$$
$$= 50Q - 2Q^2$$
$$MR = \frac{\mathrm{d}(TR)}{\mathrm{d}Q} = 50 - 4Q$$

(b) To find the output level at which *TR* is at a maximum, work through the max./min. method for finding turning points as follows:

Step 1: Get derivatives	**Step 2:** Find turning point(s)	**Step 3:** Max. or min.?
$TR = 50Q - 2Q^2$ $\frac{\mathrm{d}(TR)}{\mathrm{d}Q} = 50 - 4Q$ $\frac{\mathrm{d}^2(TR)}{\mathrm{d}Q^2} = -4$	$\frac{\mathrm{d}(TR)}{\mathrm{d}Q} = 0$ at turning point $50 - 4Q = 0$ $Q = \frac{50}{4} = 12.5$ **Step 2a:** At $Q = 12.5$, $TR = [50 - 2(12.5)]12.5 = 312.5$	The second derivative is a negative constant, therefore *TR* has a maximum value. $TR = 312.5$ at $Q = 12.5$

In general, *TR* is maximised when

$$\frac{\mathrm{d}(TR)}{\mathrm{d}Q} = MR = 0 \qquad \textbf{(6.28)}$$

and

$$\frac{\mathrm{d}^2(TR)}{\mathrm{d}Q^2} = \frac{\mathrm{d}(MR)}{\mathrm{d}Q} = (MR)' < 0$$

Note: When $MR = 0$, and $(MR)' > 0$, then TR is a minimum. This situation does not arise in this example.

To confirm these rules in this example, calculate MR when $Q = 12.5$, the value at which TR is maximised:

$$MR = 50 - 4Q = 50 - 4(12.5) = 0$$

(c) The TR and MR functions are sketched in Figure 6.23.

Sketching the total revenue function: TR is a quadratic and was sketched in Worked Example 4.7, Figure 4.8. Therefore, find

(1) Points of intersection with the axes. TR crosses the horizontal axis when $TR = 0$, that is, at $Q = 0$ and $Q = 25$. See Worked Example 4.7. Therefore, the points of intersection with the horizontal axis are (0, 0) and (25, 0).

(2) The turning point for the total revenue function was calculated in part (b), at $TR = 312.5$ and $Q = 12.5$.

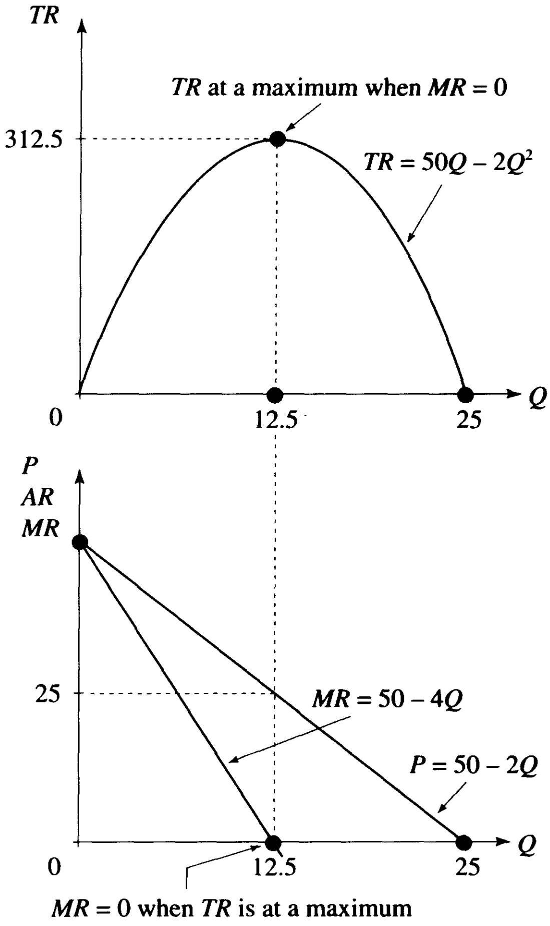


Figure 6.23 TR is at a maximum when $MR = 0$

☺ (**Remember:** In Chapter 4 it was stated that the optimum point for a quadratic occurred halfway between the points of intersection with the horizontal axis.)

Sketching the marginal revenue function: $MR = 50 - 4Q$ is a linear function, therefore two points are required to sketch its graph. The MR function intersects the vertical axis at $Q = 0$, therefore, $MR = 50 - 4(0) = 50$. The MR function intersects the horizontal axis when $MR = 0$, therefore $0 = 50 - 4Q \rightarrow Q = 12.5$. The demand function is also sketched in Figure 6.23.

WORKED EXAMPLE 6.22

BREAK-EVEN, PROFIT, LOSS AND GRAPHS

The demand function for a good is given by the equation $P = 50 - 2Q$, while total cost is given by $TC = 160 + 2Q$.

(a) Write down the equation for (i) total revenue and (ii) profit.
(b) (i) Sketch the total cost and total revenue functions on the same diagram.
    (ii) From the graph estimate, in terms of Q, when the firm breaks even, makes a profit and makes a loss.
Confirm these answers algebraically.
(c) Determine the maximum profit and the value of Q at which profit is a maximum. Sketch the profit function.
(d) Compare the levels of output at which profit and total revenue are maximised.

Solution

(a) (i) $TR = PQ = (50 - 2Q)Q = 50Q - 2Q^2$

(ii)
$$\begin{aligned}\pi = TR - TC &= 50Q - 2Q^2 - (160 + 2Q)\\ &= -2Q^2 + 48Q - 160\\ &= -2(Q^2 - 24Q + 80)\end{aligned}$$

(b) (i) **Sketching the total cost function:** $TC = 160 + 2Q$ is a linear function, therefore two points are required to sketch its graph. For example, at $Q = 0$, $TC = 160 + 2(0) = 160$ and taking any other Q value, say $Q = 20$, then $TC = 160 + 2(20) = 200$. The total cost function is drawn in Figure 6.24.

Sketching the total revenue function: This total revenue function is the same as that sketched in Worked Example 6.21, Figure 6.23. TR and TC are graphed in Figure 6.24.

(ii) From the graph, break-even occurs when $TC = TR$, at $Q = 4$ and $Q = 20$.

The break-even points are calculated algebraically as follows:

$$TC = TR$$
$$160 + 2Q = 50Q - 2Q^2$$
$$2Q^2 - 48Q + 160 = 0$$
$$Q^2 - 24Q + 80 = 0 \quad \text{dividing both sides by 2}$$
$$(Q - 4)(Q - 20) = 0$$

Therefore the break-even quantities are $Q = 4$ and $Q = 20$.

As shown in Figure 6.24, losses are incurred below break-even point $Q = 4$ (where $TC > TR$) and above break-even point $Q = 20$ (where $TC > TR$). Profits are incurred between the two break-even points since $TC < TR$ in this interval.

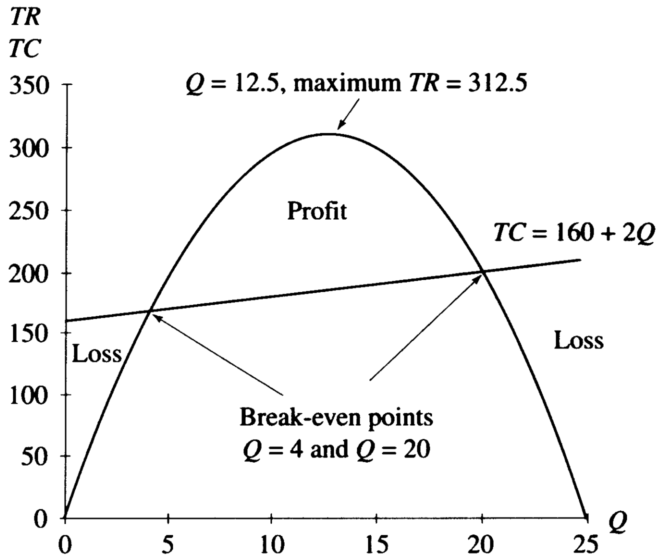


Figure 6.24 Total revenue and total cost functions

(c) Maximum profits are calculated by finding the turning point(s) for the profit function. This is done using the three-step max./min. method:

Step 1: Get derivatives

$$\pi = -2Q^2 + 48Q - 160$$
$$\frac{d\pi}{dQ} = -4Q + 48$$
$$\frac{d^2\pi}{dQ^2} = -4$$

Step 2: Find turning point(s)

$$\frac{d\pi}{dQ} = 0$$

at turning point

$$-4Q + 48 = 0$$
$$Q = \frac{48}{4} = 12$$

Step 2a: At $Q = 12$,

$$\pi = -2(12)^2 + 48(12) - 160 = 128$$

Step 3: Max. or min.?

The second derivative is a negative constant therefore profits are a maximum

$\pi = 128$ at $Q = 12$

The profit function has only one turning point and this is a maximum.

Sketching the profit function: The profit function is sketched in Figure 6.25. To sketch the profit function, find

(1) the points of intersection with the axes and **(2)** the turning point.

(1) The graph cuts the Q-axis when $\pi = 0$.

$$\begin{aligned} \pi &= 0 \\ -2(Q^2 - 24Q + 80) &= 0 \\ Q^2 - 24Q + 80 &= 0 \quad \text{dividing both sides by } -2 \\ (Q-4)(Q-20) &= 0 \\ \text{therefore} \quad Q &= 4 \quad \text{and} \quad Q = 20 \end{aligned}$$

These values of Q where $\pi = 0$ are also the break-even points.

(2) We have just shown that profit has a maximum value of 128 when $Q = 12$. This provides another point for sketching the profit function.

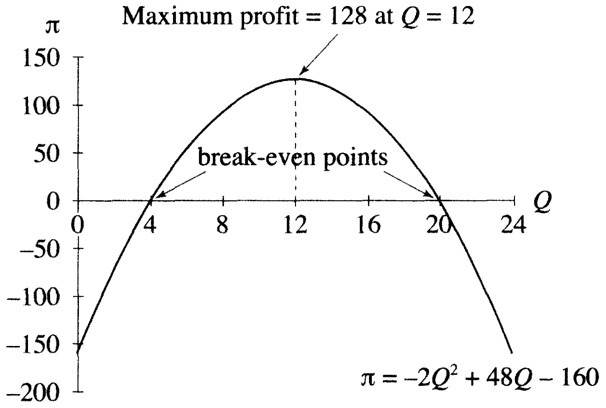


Figure 6.25 Profit function

(d) Maximum profit of 128 occurs when $Q = 12$.
Maximum total revenue of 312.5 occurs when $Q = 12.5$.

WORKED EXAMPLE 6.23

MAXIMUM AND MINIMUM OUTPUT FOR A FIRM OVER TIME

The output for a firm over time is given by the function,

$$Q = \frac{t^3}{30} - \frac{t^2}{5} + \frac{3t}{10} + 120$$

Determine the years (t) in which output is at a maximum and a minimum.

Solution

In order to find the years in which output is at a maximum and minimum the turning points of the function must be calculated. The three steps for finding the turning points are outlined briefly as follows:

Step 1: Find derivatives

$$Q = \frac{t^3}{30} - \frac{t^2}{5} + \frac{3t}{10} + 120$$

$$\frac{dQ}{dt} = \frac{3t^2}{30} - \frac{2t}{5} + \frac{3}{10}$$

$$\frac{d^2Q}{dt^2} = \frac{6t}{30} - \frac{2}{5}$$

$$= 0.2t - 0.4$$

Step 2: Find turning point(s)

$dQ/dt = 0$ at turning point

$$\frac{3t^2}{30} - \frac{2t}{5} + \frac{3}{10} = 0$$

$$\frac{t^2}{10} - \frac{4t}{10} + \frac{3}{10} = 0$$

$$\frac{1}{10}(t^2 - 4t + 3) = 0$$

$$t^2 - 4t + 3 = 0$$

$$(t - 3)(t - 1) = 0$$

Turning points at $t = 3$ and at $t = 1$

Step 3: Max. or min.?

$$\frac{d^2Q}{dt^2} = 0.2t - 0.4$$

At $t = 3$,

$$\frac{d^2Q}{dt^2} = 0.2(3) - 0.4$$

$$= 0.2$$

Positive, so a minimum

At $t = 1$,

$$\frac{d^2Q}{dt^2} = 0.2(1) - 0.4$$

$$= -0.2$$

Negative, so a maximum

Turning points: minimum at $t = 3$, $Q = 120.000$, maximum at $t = 1$, $Q = 120.13$

The firm produces maximum output in year 1 and minimum output in year 3. It may have been first on the market with a new good. Initially, its sales are high; however, other firms are quick to enter the market supplying alternative goods. The first firm incurs a drop in sales as there is a slump in the demand for its good or maybe some of the rival firms have produced a better alternative good.

Sketching the Q function: Next, find where the graph crosses the axes, if possible. The graph crosses the vertical axis when $t = 0$, that is, at $Q = 120$. The graph crosses the horizontal axis at $Q = 0$, that is, at

$$\frac{t^3}{30} - \frac{t^2}{5} + \frac{3t}{10} + 120 = 0$$

This is a cubic equation, which is not easily solved. So we resort to calculating several points on the curve between $t = 0$ and $t = 5$ as in Table 6.11. The graph is then sketched in Figure 6.26.

Table 6.11 Points for sketching the Q function in Figure 6.26

t	0	1	2	3	4	5
$Q = \frac{t^3}{30} - \frac{t^2}{5} + \frac{3t}{10} + 120$	120	120.13	120.067	120	120.13	120.67

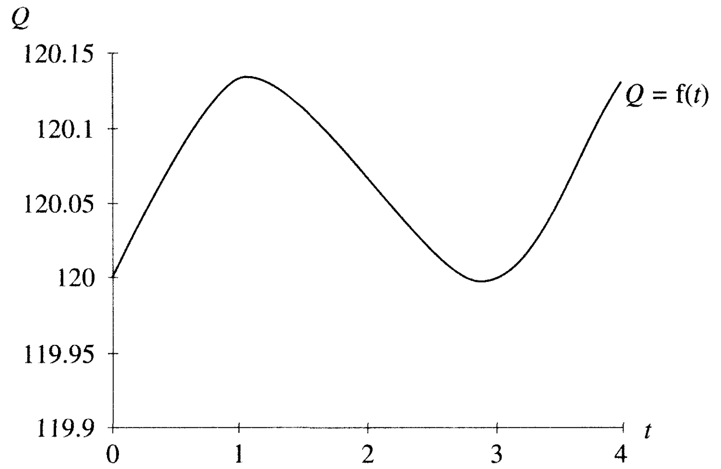


Figure 6.26 A firm's output function over time

❒ To show that $MR = MC$ and $(MR)' < (MC)'$ when profit is maximised

In the previous worked examples, profit, π, was maximised when $(\pi)' = 0$ and $(\pi)'' < 0$. However, an alternative and sometimes simpler method based on the marginal revenue and marginal cost function is derived as follows. To derive the rule, work through the usual max./min. method, but write the first derivatives of TR and TC as MR and MC as follows:

$$\pi = TR - TC$$

$$\frac{d(\pi)}{dQ} = \frac{d(TR)}{dQ} - \frac{d(TC)}{dQ} \quad \text{or} \quad (\pi)' = MR - MC$$

$$\frac{d^2(\pi)}{dQ^2} = \frac{d(MR)}{dQ} - \frac{d(MC)}{dQ} \quad \text{or} \quad (\pi)'' = (MR)' - (MC)'$$

Apply the usual conditions for a maximum: $(\pi)' = 0$ and $(\pi)'' < 0$. That is,

$$(\pi)' = MR - MC = 0 \quad \rightarrow \quad MR = MC \qquad \textbf{(6.29)}$$

and, apply the second-order condition

$$(\pi)'' = (MR)' - (MC)' < 0 \quad \rightarrow \quad (MR)' < (MC)' \qquad \textbf{(6.30)}$$

Equation (6.30) gives the second-order condition for profit maximisation: the derivative of MC is greater than the derivative of MR: $(MC)' > (MR)'$. In Worked Example 6.25 equations (6.29) and (6.30) provide a simpler alternative method for finding maximum profit.

❒ Price discrimination

In the previous worked examples it was assumed, rather unrealistically, that a firm charges all consumers the same price for its good or services. Worked Example 6.25 shows that a firm may make a greater profit when different prices are charged in markets with different demand functions.

WORKED EXAMPLE 6.24

PROFIT MAXIMISATION AND PRICE DISCRIMINATION

A hotel charges different prices for the same meal in two different markets. The demand function in each market is given by the respective equations

$$P_1 = 50 - 4Q_1 \quad \text{and} \quad P_2 = 80 - 3Q_2$$

The company's cost function is given by the equation

$$TC = 120 + 8Q$$

where $Q = Q_1 + Q_2$.

(a) For each market, write down the marginal revenue and marginal cost.

(b) Determine the price and quantity at which profit is maximised. Calculate the overall profit from the two markets.

(c) If price discrimination is declared illegal, what price should the company charge to maximise profits? Calculate the overall profit. How does it compare with the profit made under price discrimination?

Market 1	**Market 2**
$TR = (50 - 4Q_1)Q_1$	$TR = (80 - 3Q_2)Q_2$
$MR = 50 - 8Q_1$	$MR = 80 - 6Q_2$
$MC = 8$	$MC = 8$

Solution (a)

(b) Profit is maximised when $MR = MC$, subject to second-order conditions, namely $(MR)' < (MC)'$. Alternatively, profit is maximised when $(\pi)' = 0$ and $(\pi)'' < 0$.

In this example, the first method shall be used to find maximum profit.

Market 1	**Market 2**
$MR = MC$	$MR = MC$
$50 - 8Q_1 = 8$	$80 - 6Q_2 = 8$
$42 = 8Q$	$72 = 6Q$
$Q_1 = 5.25 \quad (P_1 = 29)$	$Q_2 = 12 \quad (P_2 = 44)$
Second-order conditions confirm a maximum:	Second-order conditions confirm a maximum:
$(MR)' = -8\colon\ (MC)' = 0$	$(MR)' = -6\colon\ (MC)' = 0$
therefore $(MR)' < (MC)'$, as required.	therefore $(MR)' < (MC)'$, as required.

Maximum profit from both markets:

$$\begin{aligned}\pi &= TR - TC = P_1Q_1 + P_2Q_2 - (120 + 8Q) \text{ since } Q = Q_1 + Q_2\\ &= 29(5.25) + 44(12) - 120 - 8(12 + 5.25) = 422.25\end{aligned}$$

(c) If price discrimination is not permitted, then $P_1 = P_2 = P$. The overall demand is the sum of the two separate demands: $Q = Q_1 + Q_2$.

So, let $P_1 = P_2 = P$ in the two separate demand functions. From these demand functions find an expression for $Q_1 + Q_2$, then replace it with Q. This is the equation of the overall demand function:

$$\begin{aligned} 3P &= 3(50) - 12Q_1 && \text{demand function (1)} \times 3 \\ 4P &= 4(80) - 12Q_2 && \text{demand function (2)} \times 4 \\ 7P &= 470 - 12(Q_1 + Q_2) && \text{adding} \\ 7P &= 470 - 12Q \\ P &= 67.14 - 1.714Q \end{aligned}$$

This overall demand function may now be used to find maximum profit by the same method as in part (b):

$$MR = MC \quad \rightarrow \quad 67.14 - 3.428Q = 8$$

$$Q = 17.25 \quad (P = 37.57)$$

Second-order conditions are satisfied: $(MR)' - (MC)' = -3.428 - 0 < 0$.
Overall profit $= TR - TC = (37.57)(17.25) - (120 + 8(17.25)) = 390.08$.
Overall profit with price discrimination = 422.25 from part (b).

The profits from no price discrimination, £390.08, are less than those from price discrimination, £422.25.

Price discrimination pays.

❐ Profit maximisation in perfect competition and monopoly (goods market)

In the following worked examples some calculations are left to the reader as similar problems have already been covered. However, the answers are given.

WORKED EXAMPLE 6.25

PROFIT MAXIMISATION FOR A PERFECTLY COMPETITIVE FIRM

A perfectly competitive firm has a demand function $P = 121$ and costs given as

$$TVC = \frac{1}{2}Q^3 - 15Q^2 + 175Q, \quad TFC = 500$$

(a) Write down the equations for TC, TR and π.

(b) Find the output, Q, at which profit is minimised and maximised. Comment on the relationship between MC and MR at these points.

(c) Show that $MC = MR$ is a necessary but not sufficient condition for maximum profit.

(d) Sketch TC, TR, π on one diagram, and MR, MC on a diagram vertically below it.

Solution

(a) $TC = TFC + TVC$

$$TC = \frac{1}{2}Q^3 - 15Q^2 + 175Q + 500$$

$$TR = 121Q$$

$$\pi = TR - TC = 121Q - \frac{1}{2}Q^3 + 15Q^2 - 175Q - 500$$

$$= -\frac{1}{2}Q^3 + 15Q^2 - 54Q - 500$$

(b) Work through the max./min. method:

Step 1: Get derivatives

$$\pi = -\frac{1}{2}Q^3 + 15Q^2 - 54Q - 500$$

$$\frac{d\pi}{dQ} = -\frac{3}{2}Q^2 + 30Q - 54$$

$$\frac{d^2\pi}{dQ^2} = -\frac{6}{2}Q + 30 = -3Q + 30$$

Step 2: Solve $d\pi/dQ = 0$

$$-\frac{3}{2}Q^2 + 30Q - 54 = 0$$

Multiplying both sides by -2 gives

$$3Q^2 - 60Q + 108 = 0$$

Solve this quadratic for Q.

$$Q = \frac{-(-60) \pm \sqrt{(-60)^2 - 4(3)(108)}}{2(3)}$$

$$= \frac{60 \pm 48}{6}$$

$$Q = 18 \quad \text{or} \quad Q = 2$$

Step 3: Max. or min.?

$$\frac{d^2\pi}{dQ^2} = -3Q + 30$$

At $Q = 18$.

$$(\pi)'' = -3(18) + 30 = -24$$

Negative, so a max. at $Q = 18$, $\pi = 472$

At $Q = 2$,

$$(\pi)'' = -3(2) + 30 = 24$$

Positive, so a min. at $Q = 2$, $\pi = -552$

- The profit function has a minimum value at $Q = 2$, $\pi = -552$. Note, at $Q = 2$, $MR = MC = 121$.
- The profit function has a maximum value at $Q = 18$, $\pi = 472$. Note, at $Q = 18$, $MR = MC = 121$.

(c) Therefore, following from part (b), $MR = MC$ for both maximum and minimum profit. This is a necessary but not a sufficient condition for profit maximisation. For a maximum point, the second derivative, π'', must be negative at that point or $(MR)' - (MC)' < 0$ from equation (6.30). Descriptively, in this example, the maximum and minimum profit points are distinguished by the slope of the MC curve at the point of intersection with the MR curve. Figure 6.27 clearly illustrates that profit is maximised at the point where the slope of the MC curve is greater than the slope of the MR curve.

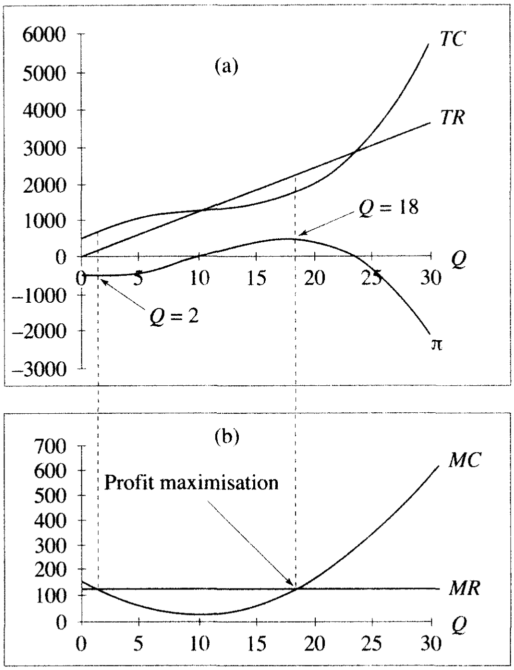


Figure 6.27 *TR*, *TC*, *MR*, *MC* and π functions for a perfectly competitive firm

(d) The *TC*, *TR*, *MR*, *MC* and π functions are plotted in Figure 6.27 from the data in Table 6.12. (Again, Excel may be used to calculate and plot these functions.)

Table 6.12 Points for profit maximisation of a *PC* firm

Q	*TC*	*TR*	π	*MC*	*MR*
0	500	0	−500	175	121
5	1062.5	605	−457.5	62.5	121
10	1250	1210	−40	25	121
15	1437.5	1815	377.5	62.5	121
20	2000	2420	420	175	121
25	3312.5	3025	−287.5	362.5	121
30	5750	3630	−2120	625	121

WORKED EXAMPLE 6.26

PROFIT MAXIMISATION FOR A MONOPOLIST

A monopolist faces a demand function $P = 152.5 - 3Q$ and has costs given as

$$TVC = \frac{1}{2}Q^3 - 15Q^2 + 175Q, \quad TFC = 300$$

(a) Write down the equations for TC, TR and π.
(b) Find the output, Q, at which profit is minimised and maximised.
(c) Confirm the condition $MC = MR$ for maximum profit.
(d) Sketch the TC, TR, π on one diagram, and MR, MC on a diagram vertically below it.

Solution

(a)
$$TC = \frac{1}{2}Q^3 - 15Q^2 + 175Q + 300$$
$$TR = (152.5 - 3Q)Q$$
$$\pi = TR - TC = (152.5Q - 3Q^2) - \left(\frac{1}{2}Q^3 - 15Q^2 + 175Q + 300\right)$$
$$= -\frac{1}{2}Q^3 + 12Q^2 - 22.5Q - 300$$

(b) The turning points of the profit function are calculated using the max./min. method.

Step 1: Get derivatives

$$\pi = -\frac{1}{2}Q^3 + 12Q^2 - 22.5Q - 300$$
$$\frac{d\pi}{dQ} = -\frac{3}{2}Q^2 + 24Q - 22.5$$
$$\frac{d^2\pi}{dQ^2} = -\frac{6}{2}Q + 24 = -3Q + 24$$

Step 2: Solve $d\pi/dQ = 0$

$$-\frac{3}{2}Q^2 + 24Q - 22.5 = 0$$

Multiplying both sides by -2 gives

$$3Q^2 - 48Q + 45 = 0$$

Solve this quadratic for Q

$$Q = \frac{-(-48) \pm \sqrt{(-48)^2 - 4(3)(45)}}{2(3)}$$
$$= \frac{48 \pm 42}{6}$$
$$Q = 15 \quad \text{or} \quad Q = 1$$

Step 3: Max. or min.?

$$\frac{d^2\pi}{dQ^2} = -3Q + 24$$

When $Q = 15$,

$$(\pi)'' = -3(15) + 24$$
$$= -21$$

Negative, so a max. at

$Q = 15$, $\pi = 375$

When $Q = 1$,

$$(\pi)'' = -3(1) + 24$$
$$= 21$$

Positive, so a min. at

$Q = 1$, $\pi = -311$

(c) In Worked Example 6.25 it was shown that profit is either maximised or minimised when $MR = MC$. We can solve for the value of Q at which profit is a maximum as follows:

$$MR = MC$$
$$152.5 - 6Q = \frac{3}{2}Q^2 - 30Q + 175$$
$$-\frac{3}{2}Q^2 + 24Q - 22.5 = 0$$
$$3Q^2 - 48Q + 45 = 0 \quad \text{multiplying both sides by } -2$$

This is the same quadratic as in part (b), step 2 where profit is maximised or minimised at $Q = 15$ and $Q = 1$. So $MR = MC$ is a necessary but not sufficient condition for profit maximisation. To determine which value of Q gives the maximum profit, use the second-order condition, or be practical! Calculate profit at each value of Q: at $Q = 15$, $\pi = 375$ and at $Q = 1$, $\pi = -311$. It is obvious that profit is maximised at $Q = 15$.

(d) The TC, TR, MR, MC and π functions are sketched in Figure 6.28 from the data in Table 6.13. (Again, Excel can be easily used to plot these functions.)

Table 6.13 Points for profit maximisation of a monopolist

Q	TC	TR	π	MC	MR
0	300	0	−300	175	152.5
5	862.5	687.5	−175	62.5	122.5
10	1050	1225	175	25	92.5
15	1237.5	1612.5	375	62.5	62.5
20	1800	1850	50	175	32.5
25	3112.5	1937.5	−1175	362.5	2.5

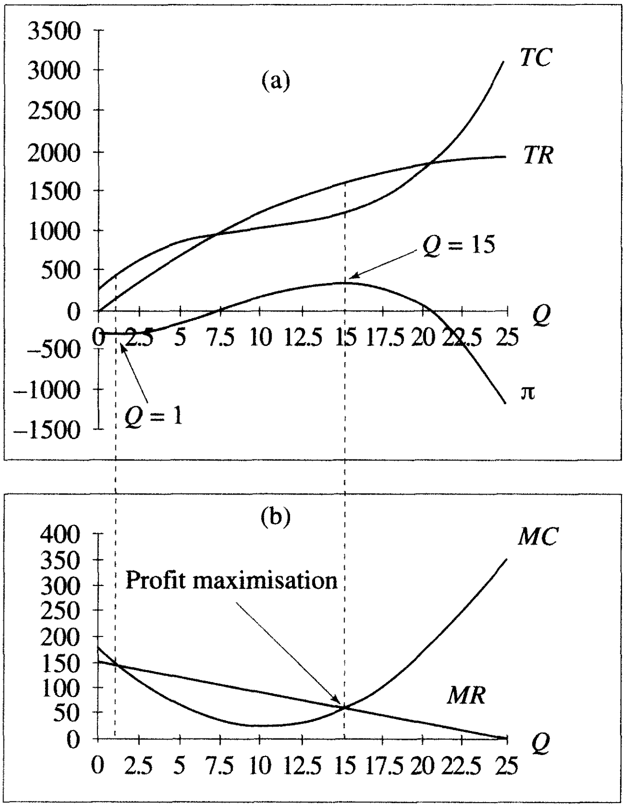


Figure 6.28 TR, TC, MR, MC and π functions for a monopolist

❒ Summary

The use of first and second derivatives to locate and determine the nature of turning points is summarised in Figure 6.29. y' and y'' mean the first and second derivatives respectively. The three steps are summarised as follows:

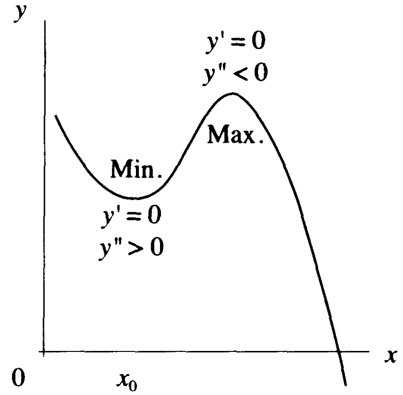


Figure 6.29 Summary of turning points

Step 1: Find y' and y''

Step 2: Solve $y' = 0$ to get the x-coordinate of the turning point(s), (x_0).

Step 2a: For each (x_0) find the y-coordinate.

Step 3: Evaluate y'' at each (x_0). If $y'' < 0$, then the point at (x_0) is a maximum. If $y'' > 0$, then the point at (x_0) is a minimum.

Progress Exercises 6.9 Maximum/Minimum and Graphs for TR, MR, AR, TC, MC, AC

1. A firm's total revenue function is given by the equation $TR = 90Q - 3Q^2$.
   (a) Find the value of Q for which TR is maximised, hence calculate the maximum TR.
   (b) Write down the equations of the average revenue and marginal revenue functions. Describe how AR and MR change before and after maximum TR.
   (c) Sketch TR, MR and AR on the same diagram. From the graph, write down the values of AR and MR when TR is zero and when TR is a maximum.
2. Given the demand functions,
   (i) $Q = 150 - 0.5P$ (ii) $P = 80 - 2Q$ (iii) $P = 45$
   (a) Write down the equations for TR.
   (b) Calculate the number of units which must be sold to maximise TR. Calculate the maximum TR.
   (c) State the values of Q at which MR and AR are zero.
3. A shop which sells T-shirts has a demand function and a total cost function given by the equations

$$P = 240 - 10Q \quad \text{and} \quad TC = 120 + 8Q$$

(a) Write down the equations for TR and profit.
(b) Calculate the number of T-shirts which must be sold to maximise
  (i) profit, (ii) total revenue
(c) Write down the equations for MR and MC.
Show that $MR = MC$ when profit is maximised.
(d) Plot the graphs of
  (i) TR and TC on the same diagram. From the graph estimate the break-even points (confirm your answer algebraically).
  (ii) MR and MC on the same diagram. What is the significance of the point of intersection of these two graphs?

4. The average cost and average revenue functions for a particular brand of mobile phone are given by the equations

$$AC = 15 + \frac{8000}{Q} \quad \text{and} \quad AR = 25$$

(a) Write down the equations for TR, TC, MR, MC.
(b) How many mobile phones must be made and sold to break-even?
(c) Write down the equation of the profit function. Show, by differentiation, that neither profit nor revenue ever reach a maximum. Use MR and MC to explain why there is no maximum.
(d) Plot the graphs of TR, TC and the profit function on one diagram. Plot MR and MC on a separate diagram. Comment on the relationship between the two diagrams.

5. A company supplying fitted kitchens is considering two different pricing strategies:
I: $P = 2374$ II: $P = 5504 - 0.8Q$
The total cost function is $TC = 608580 + 120Q$
(a) Write down the equations for MR, MC and the profit functions. Calculate the price and quantity for which profit is maximised/minimised by
  (i) using the condition $MR = MC$.
  (ii) Differentiating the profit function.
If a maximum profit is not possible give an explanation.
(b) Calculate the profit for both pricing strategies at the value of Q for which profit in market II is maximised.
(c) Graph the TC function and both TR functions on the same diagram. Estimate the break-even points. Estimate the price and quantity at which the company makes the same total revenue by both pricing schemes.
(d) Graph both profit functions on the same diagram. Estimate the point at which the company would make the same profit by both pricing schemes. Compare your answer with part (c). Comment. Briefly compare the pricing strategies.

6.5 Curvature and Other Applications

At the end of this section you should be familiar with the following:

- The second derivative and curvature
- Finding the point of inflection
- The relationship between Q, MP_L and APL

- The relationship between *TC*, *TVC* and *MC*
- The relationship between *MC*, *AVC*, *AC*.

6.5.1 Second derivative and curvature

The first and second derivatives (y' and y'') have already been used to find maximum and minimum points of various functions. In this section, both y' and y'' will be used to describe curvature and find points of inflection.

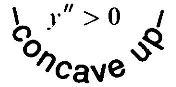
In Section 6.3, Worked Example 6.19, the first derivative and second derivative of the graph $y = 3x^2 - 0.1x^3$ were plotted in Figure 6.18 to demonstrate that first derivatives are zero at turning points, while the second derivative is positive in the region of a minimum, negative in the region of a maximum and zero at a point of inflection.

In the region of a minimum, where $y'' > 0$, the graph curves upwards and is described as concave up, while in the region of a maximum, where $y'' < 0$, the graph curves downwards and is described as concave down. Hence, the second derivative may also be used to define the 'curvature' of a curve as follows:

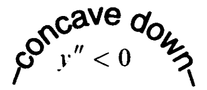
Therefore, to determine the curvature of a curve along any interval, determine whether second derivatives are negative or positive in the required interval.

The rules which use the first and second derivatives to determine maximum points, minimum points, points of inflection and define curvature are extremely important. The results from Worked Example 6.19, Figure 6.18 (a), (b) and (c) demonstrate graphically,

The curvature in the interval about a minimum (when $y'' > 0$) is described as concave up.



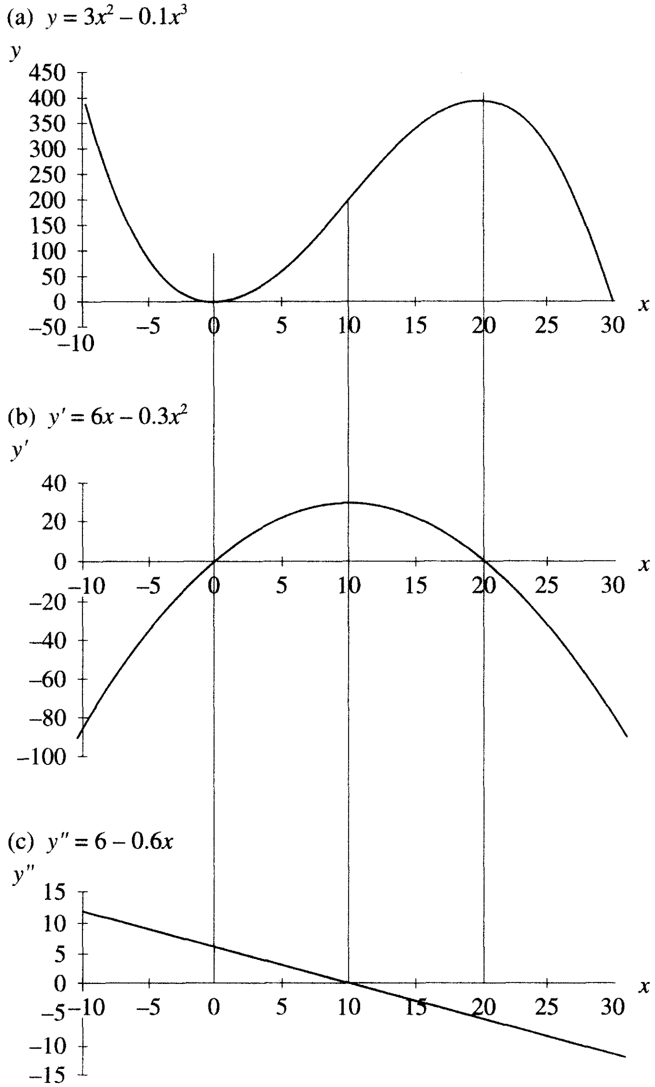
The curvature in the interval about a maximum (when $y'' < 0$) is described as concave down.



- How y, y' and y'' work together to determine maximum and minimum points
- How to use y'' to define the curvature of a curve within an interval
- How to use y'' to find the point of inflection.

Figure 6.18 is reproduced here for convenience.

- In Figure 6.18(a), y has a minimum at $x = 0$ and a maximum at $x = 20$.
- In Figure 6.18(b), the slope, y', is zero at $x = 0$ and at $x = 20$.
- In Figure 6.18(c), $y'' > 0$ around the minimum point at $x = 0$ and $y'' < 0$ around the maximum point at $x = 20$.
- In Figure 6.18(c), $y'' > 0$ in the region surrounding the minimum point where the graph curves up: curvature is described as concave up. Curvature is described as concave down in the region surrounding the maximum point, where $y'' < 0$.
- The second derivative, y'', is zero at $x = 10$. This is called a **point of inflection**. It is the point at which the curvature changes from concave up to concave down (or vice versa). Points of inflection shall be discussed in more detail later in this chapter.


(a) $y = 3x^2 - 0.1x^3$

(b) $y' = 6x - 0.3x^2$

(c) $y'' = 6 - 0.6x$

Figure 6.18 Graphs of y, y', y''

Note: It is beyond the scope of this book to deal with cases where the second derivative is zero or non-existent.

❒ Curvature in economics

In Figure 6.18 curvature in the region of a minimum, where $y'' > 0$, is described as concave up, while that in the region of a maximum, where $y'' < 0$, is described as concave down. These rules for determining curvature based on the sign of the second derivative apply whether the interval contains an actual turning point or not. The only requirement is that slope is

Table 6.14 Summary of relationship between y, y' and y''

	At $x = 0$	**At $x = 20$**
Graph (a) $y = 3x^2 - 0.1x^3$	y has a minimum point	y has a maximum point
Graph (b) $\frac{dy}{dx} = 6x - 0.3x^2$	y' is zero	y' is zero
Graph (c) $\frac{d^2y}{dx^2} = 6 - 0.6x$	y'' is positive	y'' is negative

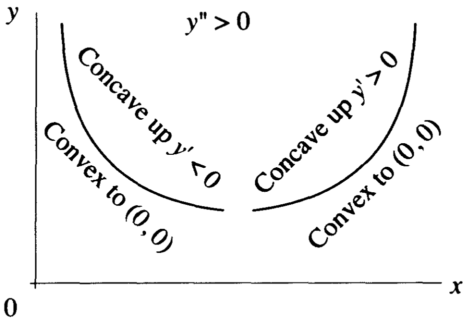

y
y" > 0
Concave up y' < 0
Convex to (0, 0)
Concave up y' > 0
Convex to (0, 0)
0
x

Figure 6.30 Concave up

increasing, hence $y'' > 0$ for the graph to curve upwards, and that slope is decreasing, hence $y'' < 0$, for the graph to curve downwards as shown in Figures 6.30 and 6.31. See also Figures 6.4 and 6.5

In economic applications, curves are usually economically meaningful in the first quadrant only. In such applications, curves are frequently described as convex towards the origin instead of concave up and concave towards the origin instead of concave down.

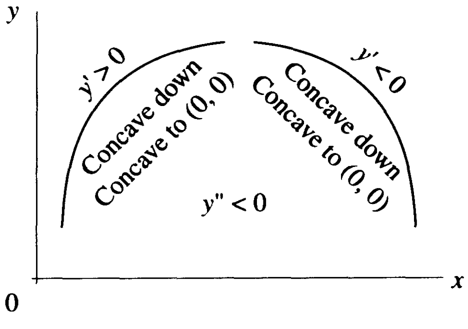

y
y' > 0
Concave down
Concave to (0, 0)
y' < 0
Concave down
Concave to (0, 0)
y" < 0
0
x

Figure 6.31 Concave down

WORKED EXAMPLE 6.27

CURVATURE OF CURVES: CONVEX OR CONCAVE TOWARDS THE ORIGIN

Use the first and second derivatives to determine the curvature of each of the following curves:

$$\text{(a) } y = 3x^4 + 20 \qquad \text{(b) } Q = \frac{25}{L}$$

in terms of whether
(i) The curve is increasing or decreasing
(ii) Curvature is concave up or concave down
(iii) Curvature is convex or concave towards the origin (in first quadrant only)

Make a rough sketch of each curve.

Solution

In each case, find the first and second derivatives. Then check whether the derivatives are positive or negative.

(a) $y = 3x^4 + 20$

$$\text{(i) } y' = 12x^3 \begin{cases} \text{when } x > 0,\ y' > 0 \rightarrow y \text{ is increasing as } x \text{ increases} \\ \text{when } x < 0,\ y' < 0 \rightarrow y \text{ is decreasing as } x \text{ increases} \end{cases}$$

(ii) $y'' = 36x^2$ positive → concave up for any value of x.

(iii) The curve is graphed in the first quadrant of Figure 6.32, and it may be described as convex towards the origin.

(iv) Plot about four or five points as shown in Table 6.15 to find the general shape of the curve.

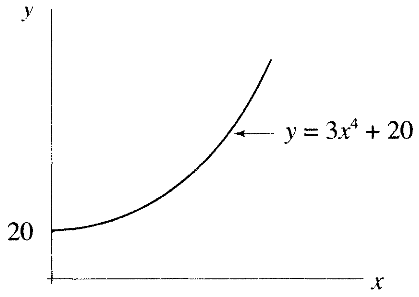


Figure 6.32 Graph of $y = 3x^4 + 20$

Table 6.15 Points for Figure 6.32

x	**0**	**1**	**2**	**3**	**4**	**5**
$y = 3x^4 + 20$	20	23	68	263	788	1895

(b) $Q = \frac{25}{L} = 25L^{-1}$

(i) $Q' = 25(-L^{-2}) = -25L^{-2} = -\frac{25}{L^2}$

$\rightarrow$ negative for all $L \rightarrow$ curve is falling

(ii) $Q'' = -25(-2L^{-3}) = 50L^{-3} = \frac{50}{L^3}$

$\rightarrow$ positive for all $L > 0 \rightarrow$ concave up

(iii) In the first quadrant this curve is described as convex towards the origin.

(iv) To plot the curve, calculate some points as shown in Table 6.16 then plotted in Figure 6.33.

Again note **division by 0** when $x = 0$, hence a vertical asymptote.

Table 6.16 Points for Figure 6.33

L	0	1	2	3	4	5
$Q = \frac{25}{L}$	?	25	12.5	8.33	6.25	5

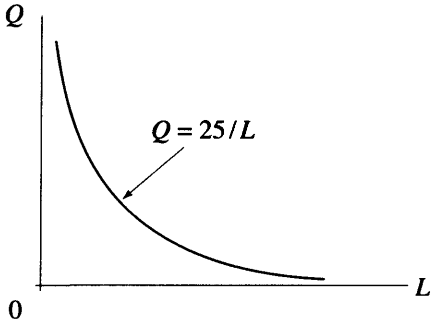
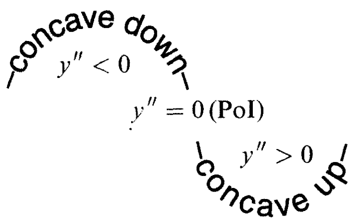


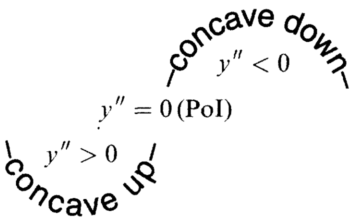
Figure 6.33 Graph of $Q = 25/L$

6.5.2 Points of inflection

In Figure 6.18(c) the point at which $y'' = 0$ was called **a point of inflection (PoI)**. This occurs at the point where curvature changes from concave up ($y'' > 0$) to concave down ($y'' < 0$), or vice versa.

Logically, if curvature changes from concave down to concave up, y'' changes from being negative to positive. This change can only occur at $y'' = 0$. Two points of inflection are shown on different curves in Figure 6.34, one at point A1 and the other at point A2.


concave down
$y'' < 0$
$y'' = 0$ (PoI)
$y'' > 0$
concave up


concave down
$y'' < 0$
$y'' = 0$ (PoI)
$y'' > 0$
concave up

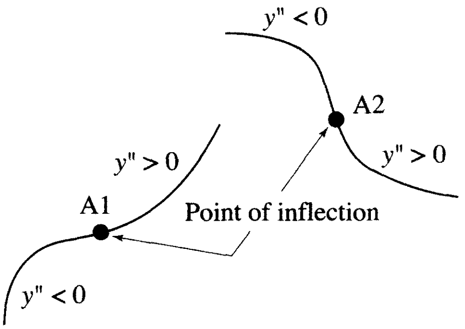

$y'' < 0$
A2
$y'' > 0$
$y'' > 0$
A1
Point of inflection
$y'' < 0$

Figure 6.34 Point of inflection

At points of inflection, $y'' = 0$ *and* y'' changes sign when evaluated before and after the point of inflection.

❐ Stationary points of inflection

If both y' and y'' are zero, then the point of inflection is said to be stationary. See Figure 6.35, points B1 and B2.

❐ Points of inflection in economics

A point of inflection may also be described as the point at which slope (rate of change) changes from being an increasing slope, as indicated when $y'' > 0$, to being a decreasing slope

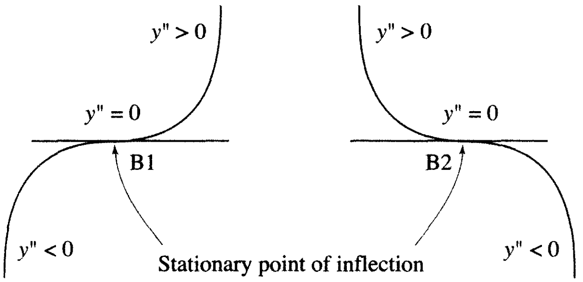

$y'' > 0$
$y'' = 0$
B1
$y'' < 0$
$y'' > 0$
$y'' = 0$
B2
$y'' < 0$
Stationary point of inflection

Figure 6.35 Stationary points of inflection

when $y'' < 0$, or vice versa. This description is useful in economics, where slope is the marginal function. Therefore, the PoI is the point at which the marginal function (slope) changes from being an increasing marginal function (slope) with $y'' > 0$ to being a decreasing marginal function (slope) with $y'' < 0$, or vice versa.

Before proceeding with further worked examples, the general method for finding points of inflection is outlined:

Step 1: Find the derivatives y', y'' and y'''.
Step 2: Find points of inflection. Solve $y'' = 0$ for the x-coordinate of the point of inflection.
Step 3: Confirm that the point in step 2 is actually a point of inflection by checking for change in the sign of y'' immediately before and after the point in question.

A simple alternative test for confirming a point of inflection for polynomials of degree 3 is to show that y''' is not zero at the point. Since y'' must change sign before and after a PoI, then y'' is either increasing or decreasing, so the rate of change of $y'' = y'''$ is either positive or negative, but not zero.

WORKED EXAMPLE 6.28

LOCATE THE POINT OF INFLECTION, PoI = POINT AT WHICH MARGINAL RATE CHANGES

Show that the curve $y = 3x^2 - 0.1x^3$ given in Worked Example 6.19 has a point of inflection at $x = 10$.

Solution

The first, second and third derivatives are

$$y = 3x^2 - 0.1x^3 \quad \rightarrow \quad \frac{dy}{dx} = 6x - 0.3x^2 \quad \rightarrow \quad \frac{d^2y}{dx^2}$$

$$= 6 - 0.6x \quad \rightarrow \quad \frac{d^3y}{dx^3} = -0.6$$

To find the point of inflection solve the equation, second derivative equal to 0:

$$6 - 0.6x = 0 \quad \rightarrow \quad x = 10$$

Figure 6.18(c) confirms that the second derivative is zero at $x = 10$ indicating a possible point of inflection. To confirm that this is a point of inflection, from Figure 6.18(c) we see that the second derivative is positive for values of x before $x = 10$ and negative for values of x after $x = 10$. Alternatively, y''' is not zero at $x = 10$, confirming that this is a point of inflection.

Progress Exercises 6.10 Curvature, Points of Inflection

For each of the following

(a) Determine the curvature along which the interval $x > 0$.
(b) Find the points of inflection (if any).
(c) Sketch the function (use Excel, if available).

1. $y = x^2 + 2x$

2. $y = -x^3 + 5x + 6$

3. $y = 3x^4 + x^2$

4. $L = 0.5K^{0.25}$

5. $TC = Q^3 - 6Q^2 + 3Q$

6. $TC = -Q^4 + 200$

7. $AC = Q^2 + \dfrac{2}{Q} + 8$

8. $TR = Q(120 - 0.8Q)$

9. $y = \dfrac{(x+2)}{x^2} - \dfrac{2x}{x^3}$

10. $AFC = \dfrac{100}{Q}$

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 6.

❒ Points of inflection and curvature for production functions

Worked Example 6.29 Relationship between the *APL* and MP_L functions

Table 6.17 Points for plotting short-run production function and MP_L, *APL* functions

Figure 6.36 Short-run production function MP_L and *APL* functions

Worked Example 6.30 Point of inflection on the production function: law of diminishing returns

❒ Points of inflection and curvature for total cost functions

Many total cost functions do not contain maximum or minimum points. The point of inflection defines the level of output at which marginal cost changes from decreasing to increasing.

WORKED EXAMPLE 6.31

RELATIONSHIP BETWEEN *TC* AND *MC*

Given the total cost function

$$TC = \frac{1}{2}Q^3 - 15Q^2 + 175Q + 1000$$

find the point of inflection and give an economic interpretation in terms of total costs and marginal costs.

Solution

The point of inflection for the total cost function occurs when the second derivative is zero, so find the first, second and third derivatives:

$$TC = \frac{1}{2}Q^3 - 15Q^2 + 175Q + 1000$$

$$MC = \frac{d(TC)}{dQ} = \frac{1}{2}(3Q^2) - 15(2Q) + 175(1) = 1.5Q^2 - 30Q + 175$$

$$\frac{d^2(TC)}{dQ^2} = 1.5(2Q) - 30 = 3Q - 30$$

$$\frac{d^3(TC)}{dQ^3} = 3$$

Equate the second derivative to zero, hence solve for Q:

$$3Q - 30 = 0, \text{ therefore, } Q = 10$$

The third derivative is not zero, so, there is a point of inflection at $Q = 10$, $TC = 1750$.

To give an economic interpretation, it is necessary to determine whether marginal cost (slope) is increasing, $TC'' > 0$, or decreasing, $TC'' < 0$, for values of $Q < 10$ and $Q > 10$. $TC'' = 3Q - 30$, therefore:

- For $Q < 10$, TC'' is negative, indicating that MC, or slope, is decreasing. The cost of producing an extra unit of output is decreasing, as in Figure 6.37(b).
- For $Q > 10$, TC'' is positive, indicating that MC, or slope, is increasing. The cost of producing an extra unit of output is increasing, as in Figure 6.37(b).

At $Q = 10$, the point of inflection, marginal cost changes from decreasing to increasing marginal costs. At $Q = 10$, marginal costs are a minimum.

The point of inflection is shown in Figure 6.37(a). As the TC and TVC functions are simply vertical translations of each other, then the point of inflection of the TVC function is also at $Q = 10$.

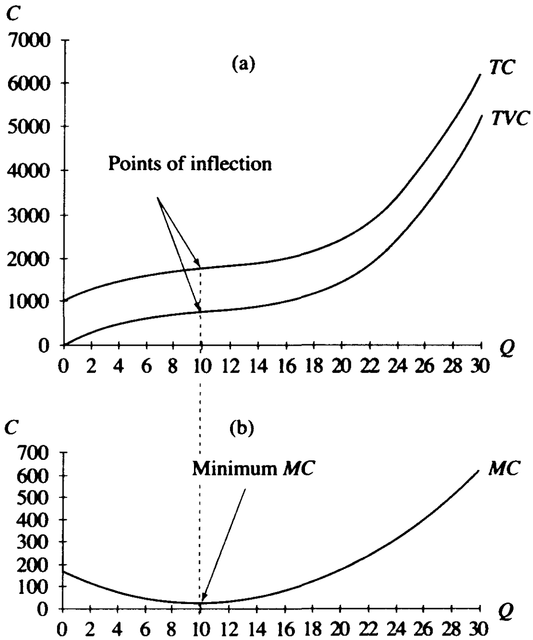


Figure 6.37 *TC*, *TVC* and *MC* functions

WORKED EXAMPLE 6.32

RELATIONSHIP BETWEEN *AC, AVC, AFC* AND *MC* FUNCTIONS

The total variable cost, *TVC*, and total fixed cost, *TFC*, functions for a good are given as

$$TVC = \frac{Q^3}{2} - 15Q^2 + 175Q, \qquad TFC = 1000$$

($TC = TVC + TFC$. This is the same *TC* function as in Worked Example 6.32.)

(a) Write down the equations for
  (i) Total costs, *TC*
  (ii) Average costs, *AC*
  (iii) Average variable cost, *AVC*
  (iv) Average fixed cost, *AFC*
  (v) Marginal cost, *MC*.

(b) Find the values of Q for which MC, AVC, AFC and *AC* are minimised and sketch the graphs of these functions.

(c) Show that the *MC* curve passes through the minimum points of the *AC* and *AVC* curves. (That is $MC = AVC$ when *AVC* is at a minimum and $MC = AC$ when *AC* is at a minimum.)

Solution

(a)

$$TC = TVC + TFC = \frac{Q^3}{2} - 15Q^2 + 175Q + 1000$$

$$AC = \frac{TC}{Q} = \frac{\frac{Q^3}{2} - 15Q^2 + 175Q + 1000}{Q} = \frac{Q^2}{2} - 15Q + 175 + \frac{1000}{Q}$$

$$AVC = \frac{TVC}{Q} = \frac{\frac{Q^3}{2} - 15Q^2 + 175Q}{Q} = \frac{Q^2}{2} - 15Q + 175$$

$$AFC = \frac{TFC}{Q} = \frac{1000}{Q}$$

$$MC = \frac{\mathrm{d}(TC)}{\mathrm{d}Q} = \frac{3Q^2}{2} - 30Q + 175 \text{ or } MC = \frac{\mathrm{d}(TVC)}{\mathrm{d}Q} = \frac{3Q^2}{2} - 30Q + 175$$

Notice that $AC = AVC + AFC$.

(b) The turning points for MC, AVC, AFC and *AC* are now calculated.

- To find the minimum values for the *MC* and *AVC* functions equate the first derivatives to zero. The second derivative should be positive at each minimum point.

The graphs of the *AVC* and *MC* functions are plotted in Figure 6.38(b) from the points in columns (1), (6) and (8) of Table 6.18.

$$MC = \frac{3Q^2}{2} - 30Q + 175$$

Step 1: Find first and second derivatives

$$\frac{\mathrm{d}(MC)}{\mathrm{d}Q} = \frac{3(2Q)}{2} - 30(1) = 3Q - 30$$

$$\frac{\mathrm{d}^2(MC)}{\mathrm{d}Q^2} = 3$$

Step 2: Find turning points by solving first derivative = 0.

$$3Q - 30 = 0 \quad \rightarrow \quad Q = 10$$

Step 2a: When $Q = 10$, $MC = 25$
Step 3: Confirm the minimum.
The second derivative is a positive constant, so the only turning point is a minimum.

$$AVC = \frac{Q^2}{2} - 15Q + 175$$

Step 1: Find first and second derivatives

$$\frac{\mathrm{d}(AVC)}{\mathrm{d}Q} = \frac{(2Q)}{2} - 15(1) = Q - 15$$

$$\frac{\mathrm{d}^2(AVC)}{\mathrm{d}Q^2} = 1$$

Step 2: Find turning points by solving first derivative = 0.

$$Q - 15 = 0 \quad \rightarrow \quad Q = 15$$

Step 2a: When $Q = 15$, $AVC = 62.5$
Step 3: Confirm the minimum.
The second derivative is a positive constant, so the only turning point is a minimum.

- The graph of $AFC = 1000/Q$ is a hyperbolic curve with the vertical axis as asymptote

Step 1: Get derivatives:

$$AFC = 1000Q^{-1}$$

$$\frac{\mathrm{d}(AFC)}{\mathrm{d}Q} = 1000(-Q^{-2}) = -1000Q^{-2} = -\frac{1000}{Q^2}$$

$$\frac{\mathrm{d}^2(AFC)}{\mathrm{d}^2Q} = -1000(-2Q^{-3}) = 2000Q^{-3} = \frac{2000}{Q^3}$$

Step 2: At a minimum, the first derivative is zero. Solve this equation for Q.

$$-\frac{1000}{Q^2} = 0 \rightarrow -1000 = 0(Q^2) = 0.$$

This statement, $-1000 = 0$, is a contradiction, therefore, no solution. To put it another way, there is no value of Q^2 which can be multiplied by 0 to give -1000. Therefore, since we cannot find a value of Q for which slope is zero, there is no turning point.

Curvature: Slope

$$\frac{\mathrm{d}(AFC)}{\mathrm{d}Q} = -\frac{1000}{Q^2}$$

is negative for all values of Q, therefore, this is a decreasing curve. The second derivative,

$$\frac{\mathrm{d}^2(AFC)}{\mathrm{d}Q^2} = \frac{2000}{Q^3}$$

is positive for all values of $Q > 0$, therefore, the curvature is concave up or convex towards the origin.

- The point where AC is at a minimum is not so easily calculated. Equating the derivative of AC to zero and solving for Q produces a cubic equation, as shown below:

$$AC = 0.5Q^2 - 15Q + 175 + 1000Q^{-1}$$

$$\frac{d(AC)}{dQ} = 0.5(2Q) - 15(1) + 1000(-Q^{-2}) = Q - 15 - 1000Q^{-2}$$

At the minimum AC, the first derivative is zero,

$$Q - 15 + \frac{1000}{Q^2} = 0$$

$$\frac{Q}{1} - \frac{15}{1} + \frac{1000}{Q^2} = 0$$

$$\frac{Q^3 - 15Q^2 + 1000}{Q^2} = 0$$

$$Q^3 - 15Q^2 + 1000 = 0 \qquad \text{multiplying both sides by } Q^2$$

The solution to cubic equations is beyond the scope of this text, unless it is possible to factor the cubic function, thereby reducing the problem to the product of linear and quadratic functions. However, you are expected to be resourceful, so when all else fails, sketch the graph by plotting the points from columns (1) and (7) in Table 6.18, and estimate from the graph that minimum AC is approximately at $Q = 18.1$ and $AC = 123$. In this case, it is quite safe to plot points, since polynomials, like this one, have no infinite jumps etc. The AC function is plotted in Figure 6.38(b).

Since the minima for the MC and AVC are at $Q = 10$ and $Q = 15$ respectively, sketch the graphs from $Q = 0$ to $Q = 30$ to get a reasonable picture of the main

Table 6.18 Points for plotting the total, average and marginal cost functions

(1) Q	(2) TFC	(3) TVC	(4) TC	(5) AFC	(6) AVC	(7) AC	(8) MC
0	1000	0	1000	–	–	–	–
3	1000	403.5	1403.5	333.33	134.5	467.83	98.5
6	1000	618	1618	166.66	103	269.66	49
9	1000	724.5	1724.5	111.11	80.5	191.61	26.5
10	1000	750	1750	100	75	175	25
12	1000	804	1804	83.33	67	150.33	31
15	1000	937.5	1937.5	66.66	62.5	129.16	62.5
18	1000	1206	2206	55.55	67	122.55	121
18.1	1000	1218.22	2218.22	55.24	67.30	122.54	123.41
21	1000	1690.5	2690.5	47.61	80.5	128.11	206.5
24	1000	2472	3472	41.66	103	144.66	319
27	1000	3631.5	4631.5	37.03	134.5	171.53	458.5
30	1000	5250	6250	33.33	175	208.33	625

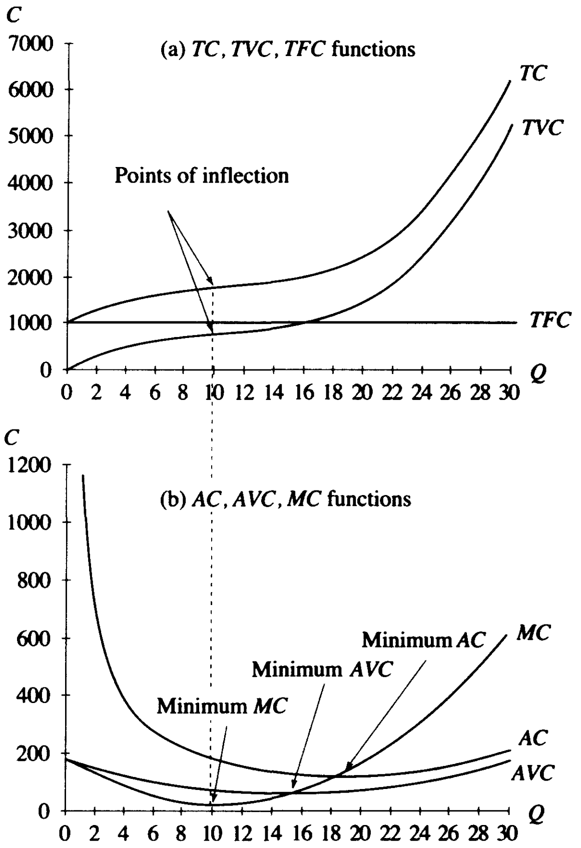


Figure 6.38 Total, average and marginal cost functions

points of all curves. Use Excel, if available, to calculate the tables of points and plot the graphs.

(c) From Figure 6.38(b), we can deduce the following:

- **The MC curve passes through the minimum point of the AVC curve** ($MC = AVC$ when AVC is at a minimum.)

The minimum value of AVC is at $Q = 15$, $AVC = 62.5$. To show that MC and AVC curves intersect at this point, evaluate MC at $Q = 15$.

$$\begin{aligned} MC &= 1.5Q^2 - 30Q + 175 \\ &= 1.5(15)^2 - 30(15) + 175 \\ &= 337.5 - 450 + 175 = 62.5 \end{aligned}$$

So both curves pass through the point $Q = 15$, $MC = AVC = 62.5$. This confirms that $MC = AVC$ at minimum AVC.

- **The MC curve passes through the minimum point of the AC curve** ($MC = AC$ when AC is a minimum.)

AC is at a minimum at $Q = 18.1$ and $AC = 123$. To show that MC and AC curves intersect at the point $Q = 18.1$, evaluate MC at $Q = 18.1$.

$$\begin{aligned} MC &= 1.5Q^2 - 30Q + 175 \\ &= 1.5(18.1)^2 - 30(18.1) + 175 \\ &= 491 - 543 + 175 = 123 \end{aligned}$$

This confirms that $MC = AC$ when AC is a minimum.

To summarise:

When AC is at a minimum, $MC = AC$.
When AC is falling, $MC < AC$; when AC is rising, $MC > AC$.
When AVC is at a minimum, $MC = AVC$.
Minimum $MC <$ Minimum $AVC <$ Minimum AC.

Progress Exercises 6.11 Applications of Differentiation, Curvature and Points of Inflection MP_L, APL, TC, MC etc. Relationships between $MC/AVC/AFC$

1. The quantity of sandwiches, Q, made in a small coffee shop is given by the equation

$$Q = -2L^3 + 12L^2$$

where L is the number of labour-hours hired.

(a) Write down the equations for the marginal product of labour (MP_L) and average product of labour (APL). Give a verbal description of MP_L and APL at $L = 1.5$.

(b) Calculate the units of labour at which the MP_L and APL are maximised, and plot both graphs on the same diagram. Confirm algebraically that MP_L and APL are equal when APL is at a maximum.

(c) Find the turning points and point of inflection for the production function. Plot the production function.

(d) Use the graphs and any other results calculated in (a) and (b) to describe how productivity (Q = number of sandwiches) and the rate of change in productivity as the number of labour units employed (L) increases.

Is there any relationship between the turning points and points of inflection in the graphs plotted in (b) and (c)?

2. Consider a firm whose total cost function is

$$TC = \frac{1}{3}Q^3 - 30Q^2 + 2800Q + 900$$

(a) Write down the equations for MC, AC, AVC and AFC.

(b) Find the minimum MC.

(c) Find the minimum AVC. Show that the MC and AVC curves intersect at the minimum point on the AVC curve.
(d) Plot the MC, AFC and AVC curves on the same diagram. Comment on the points at which these curves intersect.

3. The total cost and total revenue functions for a company are

$$TC = \frac{1}{3}Q^3 - 15Q^2 + 480Q + 750 \qquad TR = Q(536 - 2Q)$$

(a) Plot the graph of TC and TR for $Q = 0$ to 50. From the graph write down the range of values of Q for which the firm makes a profit.
(b) Write down the equation for the profit function. Hence
(i) Determine the level of output, Q, for which profit is maximised and minimised
(ii) State an alternative method for locating the optimum profit
(c) Write down the equation of the marginal profit curve. Hence find the maximum value of marginal profit.

4. Given the total cost function $TC = \frac{1}{3}Q^3 - 36Q^2 + 1500Q + 9900$ and a total revenue function, $TR = 1360Q$
(a) Plot the graph of TC and TR for $Q = 0$ to 110. From the graph, write down the range of values of Q for which the firm makes a profit.
(b) Write down the equation for the profit function. Hence determine the level of output, Q, for which profit is maximised and minimised.

6.6 Further Differentiation and Applications

At the end of this section you should be able to:

- Differentiate other standard functions
- Use the chain rule to differentiate functions of functions
- Use the product and quotient rules
- Use differentiation to describe and analyse other economic models.

6.6.1 Derivatives of other standard functions

So far we have only used the general power rule for differentiation which is restated here as

$$y = x^n \quad \text{then} \quad \frac{dy}{dx} = nx^{n-1}$$

However, when more versatile models are developed, differentiation must be extended and used in the analysis of these models. We begin by writing down the rules for finding the derivatives of other standard functions which are frequently used in economics, management and finance. The differentiation of these functions is very straightforward, it simply involves following the rules which are summarized in Table 6.19.

Table 6.19 Rules for finding derivatives

	$f(x)$	$f'(x)$
Power rule	x^n	nx^{n-1}
Exponential, base e	e^x	e^x
Log to base e	$\ln(x)$	$\frac{1}{x}$

WORKED EXAMPLE 6.33

DERIVATIVES OF OTHER STANDARD FUNCTIONS

Find the first, second and third derivatives for each of the following functions:

(a) $C = 100(1 - e^Y)$

(b) $TC = 10 + \ln Q$

Solution

(a)

$$C = 100 - 100e^Y$$

$$\frac{dC}{dY} = -100(e^Y)$$

$$\frac{d^2C}{dY^2} = -100(e^Y)$$

$$\frac{d^3C}{dY^3} = -100(e^Y)$$

(b)

$$TC = 10 + \ln Q$$

$$\frac{d(TC)}{dQ} = \frac{1}{Q} = Q^{-1}$$

$$\frac{d^2(TC)}{dQ^2} = -1Q^{-2} = -\frac{1}{Q^2}$$

$$\frac{d^3(TC)}{dQ^3} = -1(-2Q^{-3})$$

$$= 2Q^{-3} = \frac{2}{Q^3}$$

Progress Exercises 6.12 Differentiation of Logs and Exponentials

Determine the first derivative in questions 1 to 12.

1. $y = 10e^x$

2. $y = \frac{e^x}{10}$

3. $TVC = 243\frac{1}{x} + \ln(x)$

4. $P = 20 + 20 \ln Q$

5. $Q = \sqrt[3]{L} + 4$

6. $P = 150(1 - e^t)$

7. $P = 87(1.2)\frac{e^{2t}}{e^t}$

8. $AC = \ln Q + \frac{2}{Q}$

9. $Q = \frac{\ln(P)}{12} - \frac{12}{\sqrt{4P}}$

10. $x = 10.85(e^t)$

11. $C = \frac{150}{Y} + 0.8e^Y$

12. $C = x^2 - 5e^x + \ln(x)$

Determine the indicated derivative in questions 13 to 16.

13. $T - \dfrac{12}{Y} = e^Y$ find $\dfrac{dT}{dY}, \dfrac{d^2T}{dY^2}$

14. $e^{y+12} = x$ find $\dfrac{dy}{dx}$

15. $\ln(Y) = 15 + t$ find $\dfrac{dY}{dt}, \dfrac{d^2Y}{dt^2}$

16. $\ln(Y + 10) = x + 5$ find $\dfrac{dY}{dx}, \dfrac{d^2Y}{dx^2}$

6.6.2 Chain rule for differentiating a function of a function

❑ What is a function of a function?

In Chapter 2 a function was loosely defined as a rule for operating on x (resulting in one y-value). For example,

$5x + 7$: is a function of x → multiply x by 5, then add 7
e^x: is a function of x → raise x to the base e
$\ln(x)$: is a function of x → get the natural log of x

If a function of x is operated on again, we then have a function of a function of x. For example, if the first function is **$(5x + 7)$**

(a) $(5x + 7)^{-4}$ → **(first function)** raised to the power of (-4)
(b) $e^{(5x+7)}$ → raise the **(first function)** to base e
(c) $\ln(5x + 7)$ → get the natural log of the **(first function)**

To differentiate a function of a function, do so in stages as summarised below in Worked Example 6.34.

❑ Stages of the chain rule

Stage 1

Call the first function (or inside function) u,

(**u = first function in terms of x**).

Then write out y, the main function, in terms of u (**y in terms of u**).
Differentiate each function separately to get

$$\frac{du}{dx} \quad \text{and} \quad \frac{dy}{du}$$

Stage 2

Multiply (or chain) the results together so that the intermediate step, u, cancels out.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}} \qquad \textbf{(6.31)}$$

WORKED EXAMPLE 6.34

USING THE CHAIN RULE FOR DIFFERENTIATION

Find the first derivative of each of the following:

(a) $(5x+7)^{-4}$
(b) $e^{(5x+7)}$
(c) $\ln(5x+7)$

Solution

(a) $y=(5x+7)^{-4}$

This is essentially a base to a power, so treat the base as the single variable, u. The stages are outlined as follows:

Stage 1

Treat the base as the variable, u, therefore,

$$u = 5x+7$$

hence

$$y = u^{-4}$$

We have two equations, so differentiate:

$$\frac{du}{dx}=5$$

and

$$\frac{dy}{du}=-4u^{-4-1}=-4u^{-5}$$

Result

If $y=(5x+7)^{-4}$ then

$$\frac{dy}{dx}=-\frac{20}{(5x+7)^5}$$

Stage 2

Chain the two results together so that u cancels using equation (6.31):

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

$$=(-4u^{-5})(5)=-20(u)^{-5}$$

$$=-20(5x+7)^{-5}$$ substitute $5x+7=u$

$$=-\frac{20}{(5x+7)^5}$$

(b)

Stage 1

$$y=e^{5x+7}$$

$$y=e^u \quad \text{where} \quad u=5x+7$$

$$\frac{dy}{du}=e^u \quad \text{and} \quad \frac{du}{dx}=5$$

Result

If $y=e^{5x+7}$ then

$$\frac{dy}{dx}=5e^{5x+7}$$

Stage 2

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$$

$$=(e^u)(5)$$

$$=5e^{5x+7}$$

(c)

Stage 1

$y = \ln(5x+7)$

$y = \ln(u)$ where $u = 5x + 7$

$\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 5$

Stage 2

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= \left(\frac{1}{u}\right)\left(\frac{5}{1}\right)$$
$$= \left(\frac{5}{u}\right)$$
$$= \left(\frac{5}{5x+7}\right)$$

Result

If $y = \ln(5x+7)$ then

$$\frac{dy}{dx} = \frac{5}{5x+7}$$

Progress Exercises 6.13 Differentiation with the Chain Rule

Differentiate each of the following:

1. $y = (2x-5)^7$
2. $y = (4-5x)^3$
3. $y = (1-0.8x)^{-5}$
4. $y = \frac{1}{\sqrt{5x+12}}$
5. $P = \sqrt{Q^2+12}$
6. $TC = \sqrt{Q^5+3Q^2}$
7. $Q = 0.85\sqrt[3]{L^2+3L}$
8. $Q = 120\left(\frac{10}{P+5}\right)$
9. $y = 12e^{0.81x}$
10. $P = 1223e^{1.09t}$
11. $P = 800e^{-1.4Q}$
12. $S = 185(1-e^{-1.2t})$
13. $y = (1-e^{2.5t})$
14. $P = 27 + \frac{80}{e^{4t}}$
15. $y = \frac{15}{1+e^t}$
16. $T = 200 + 30e^{0.88Y}$
17. $C = 1240 + \ln(Y^2+4)$
18. $Q = 500\ln(K^3+8K)$
19. $y = 12e^{-1.57} + \frac{2}{x+2} - \ln(2x+2)$
20. $P = \ln\left(\frac{Q^2-2Q}{Q}\right)$
21. $P = 8023(1.25e^{-0.5t})$
22. $C = 425 + 1.2\ln\left(\frac{Y}{Y+2}\right)$
23. $S = 248(1-e^{-0.65Y})$
24. $P = \frac{4}{\sqrt{Q}} + 5\ln(Q)$

6.6.3 Product rule for differentiation

In differentiation, a 'product' refers to two distinct, independent functions of x multiplied together; for example,

(a) $y = x^2e^x$ → x^2 is multiplied by e^x *or* the product of x^2 and e^x

(b) $P = 2\sqrt{Q}(Q+5)$ → $2\sqrt{Q}$ is multiplied by $(Q+5)$ *or* the product of $2Q^{0.5}$ and $(Q+5)$

(c) $C = (Y+4)\ln(Y)$ → $(Y+4)$ is multiplied by $\ln(Y)$ *or* product of $(Y+4)$ and $\ln(Y)$

❒ Product rule

The above products are differentiated according to the product rule (which can be derived from first principles, calculus again) given as equation (6.32).

If $y = u(x)v(x)$ usually written as $y = uv$, then

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} \tag{6.32}$$

where u is the first function of x and v is the second function of x.

❒ Stages of the product rule

Stage 1	Stage 2
State the equation for u. Find du/dx. State the equation for v. Find dv/dx.	Fill in the right-hand side of the product rule given in equation (6.32) and simplify your answer: $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

WORKED EXAMPLE 6.35

USING THE PRODUCT RULE FOR DIFFERENTIATION

Differentiate each of the following:

(a) $y = x^2e^x$

(b) $P = 2\sqrt{Q}(Q+5)$

(c) $C = (Y+4)\ln(Y)$

Solution

(a)

Stage 1

State u and v and find $\mathrm{d}u/\mathrm{d}x$ and $\mathrm{d}v/\mathrm{d}x$.

$$y = x^2e^x$$

$$u = x^2 \quad \text{and} \quad v = e^x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^x$$

Stage 2

Fill in the product rule, equation (6.32).

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}\\ &= (e^x)(2x) + (x^2)(e^x)\\ &= e^x(2x + x^2)\\ &= xe^x(2 + x)\end{aligned}$$

Result

$y = x^2e^x$ then $\mathrm{d}y/\mathrm{d}x = xe^x(2 + x)$

(b)

Stage 1

State u and v and find $\mathrm{d}u/\mathrm{d}Q$ and $\mathrm{d}v/\mathrm{d}Q$.

$$\begin{aligned}P &= 2\sqrt{Q}(Q + 5)\\ &= 2(Q)^{1/2}(Q + 5)\end{aligned}$$

$$u = 2Q^{(0.5)} \quad \text{and} \quad v = Q + 5$$

$$\frac{\mathrm{d}u}{\mathrm{d}Q} = 2(0.5Q^{0.5-1}) \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}Q} = 1$$

$$\begin{aligned}&= 2(0.5Q^{-0.5})\\ &= Q^{-0.5}\end{aligned}$$

Stage 2

Fill in the product rule, equation (6.32)

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}Q} &= v\frac{\mathrm{d}u}{\mathrm{d}Q} + u\frac{\mathrm{d}v}{\mathrm{d}Q}\\ &= (Q + 5)(Q^{-0.5}) + (2Q^{0.5})1\\ &= Q^{0.5} + 5Q^{-0.5} + 2Q^{0.5}\\ &= 3Q^{0.5} + 5Q^{-0.5}\end{aligned}$$

Result

$P = 2\sqrt{Q}(Q + 5)$ then

$$\frac{\mathrm{d}P}{\mathrm{d}Q} = 3Q^{0.5} + \frac{5}{Q^{0.5}}$$

(c)

Stage 1

State u and v and find $\mathrm{d}u/\mathrm{d}Y$ and $\mathrm{d}v/\mathrm{d}Y$.

$$C = (Y + 4)\ln(Y)$$

$$u = (Y + 4) \quad \text{and} \quad v = \ln(Y)$$

$$\frac{\mathrm{d}u}{\mathrm{d}Y} = 1 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}Y} = \frac{1}{Y}$$

Stage 2

Fill in the product rule, equation (6.32).

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}Y} &= v\frac{\mathrm{d}u}{\mathrm{d}Y} + u\frac{\mathrm{d}v}{\mathrm{d}Y}\\ &= \ln(Y)(1) + (Y + 4)\frac{1}{Y}\\ &= \ln(Y) + \frac{Y + 4}{Y}\\ &= \ln(Y) + 1 + \frac{4}{Y}\end{aligned}$$

Result

$C = (Y+4)\ln(Y)$ then $\dfrac{\mathrm{d}C}{\mathrm{d}Y} = \ln(Y) + 1 + \dfrac{4}{Y}$

Progress Exercises 6.14 Differentiation with the Product Rule

Find the first derivatives for each of the following:

1. $y = x\ln(x)$
2. $TR = (50 - 2\sqrt{Q})Q$
3. $y = (t^2)e^t + 10$
4. $AC = \dfrac{1}{Q} - Q\ln(Q)$
5. $Q = L\sqrt{L+5}$
6. $y = x^{1.5}\sqrt{9x}$
7. $P = 15 - Qe^Q$
8. $C = Q\sqrt{1+2Q}$
9. $S = 100Ye^{-0.5Y}$
10. $I = \dfrac{1}{Y}\ln(Y+8)$
11. $C = 200 + \sqrt{Y}\ln(8Y)$
12. $P = \dfrac{3}{20}xe^{x-5}$
13. $TC = \dfrac{Q^3}{15} + Q\sqrt{3Q-1} - Q + 250$
14. $P = 20 - \sqrt{5Q}\ln Q$
15. $S = t(3t-5)^{-4}$
16. $x = y^3\ln(y^3)$

6.6.4 Quotient rule for differentiation

In differentiation, a 'quotient' refers to a fraction in which a function of x is divided by another function in x. For example,

(a) $\dfrac{x^2}{1+e^x} \rightarrow x^2$ is divided by $(1+e^x)$

(b) $\dfrac{Y+4}{\ln Y} \rightarrow (Y+4)$ is divided by $\ln(Y)$

(c) $\dfrac{Q}{3Q+5} \rightarrow Q$ is divided by $(3Q+5)$

❐ Quotient rule

These quotients are differentiated according to the quotient rule given in equation (6.33):

If $y = \dfrac{u(x)}{v(x)}$, usually written as $y = \dfrac{u}{v}$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \tag{6.33}$$

where u is the function above the line and v is the function below it.

☐ Stages of the quotient rule

Stage 1

State the equation for u. Find $\mathrm{d}u/\mathrm{d}x$.

State the equation for v. Find $\mathrm{d}v/\mathrm{d}x$.

Stage 2

Fill in the right-hand side of the quotient rule given in equation (6.33) and simplify your answer.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x}-u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

WORKED EXAMPLE 6.36

USING THE QUOTIENT RULE FOR DIFFERENTIATION

Differentiate each of the following:

(a) $y=\dfrac{x^2}{1+e^x}$

(b) $C=\dfrac{Y+4}{\ln(Y)}$

(c) $P=\dfrac{Q}{3Q+5}$

Solution

(a)

Stage 1

State u and v and find $\mathrm{d}u/\mathrm{d}x$ and $\mathrm{d}v/\mathrm{d}x$.

$$y=\frac{x^2}{1+e^x}$$

$$u=x^2 \qquad v=1+e^x$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=2x \qquad \frac{\mathrm{d}v}{\mathrm{d}x}=e^x$$

Stage 2

Fill in the quotient rule, equation (6.33).

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x}-u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

$$=\frac{(1+e^x)(2x)-x^2(e^x)}{(1+e^x)^2}$$

$$=\frac{x(2+2e^x-xe^x)}{1+2e^x+e^{2x}}$$

Result

If $y=\dfrac{x^2}{(1+e^x)}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x(2+2e^x-xe^x)}{1+2e^x+e^{2x}}$

(b)

Stage 1

State u and v and find $\mathrm{d}u/\mathrm{d}Y$ and $\mathrm{d}v/\mathrm{d}Y$.

$$C = \frac{Y+4}{\ln(Y)}$$

$$u = (Y+4) \qquad v = \ln(Y)$$

$$\frac{\mathrm{d}u}{\mathrm{d}Y} = 1 \qquad \frac{\mathrm{d}v}{\mathrm{d}Y} = \frac{1}{Y}$$

Stage 2

Fill in the quotient rule, equation (6.33).

$$\frac{\mathrm{d}C}{\mathrm{d}Y} = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}Y} - u\dfrac{\mathrm{d}v}{\mathrm{d}Y}}{v^2}$$

$$= \frac{(\ln Y)(1) - (Y+4)\dfrac{1}{Y}}{(\ln Y)^2}$$

$$= \frac{\ln(Y) - 1 - \dfrac{4}{Y}}{(\ln Y)^2}$$

Result

If $C = \dfrac{Y+4}{\ln(Y)}$ then

$$\frac{\mathrm{d}C}{\mathrm{d}Y} = \frac{\ln(Y) - 1 - \dfrac{4}{Y}}{(\ln Y)^2}$$

(c)

Stage 1

State u and v and find $\mathrm{d}u/\mathrm{d}Q$ and $\mathrm{d}v/\mathrm{d}Q$.

$$P = \frac{Q}{3Q+5}$$

$$u = Q \qquad v = 3Q+5$$

$$\frac{\mathrm{d}u}{\mathrm{d}Q} = 1 \qquad \frac{\mathrm{d}v}{\mathrm{d}Q} = 3$$

Stage 2

Fill in the quotient rule, equation (6.33)

$$\frac{\mathrm{d}P}{\mathrm{d}Q} = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}Q} - u\dfrac{\mathrm{d}v}{\mathrm{d}Q}}{v^2}$$

$$= \frac{(3Q+5)(1) - (Q)(3)}{(3Q+5)^2}$$

$$= \frac{3Q+5-3Q}{(3Q+5)^2}$$

$$= \frac{5}{(3Q+5)^2}$$

Result

If $P = \dfrac{Q}{3Q+5}$ then $\dfrac{\mathrm{d}P}{\mathrm{d}Q} = \dfrac{5}{(3Q+5)^2}$

Progress Exercises 6.15 Differentiation with the Quotient Rule

Find the first derivative for each of the following and simplify your answer.

1. $y = \dfrac{\ln(x)}{x}$ **2.** $P = \dfrac{50-Q}{50+Q}$ **3.** $y = \dfrac{t^2+1}{t^2-1}$

4. $AC = \dfrac{\sqrt{Q+1}}{Q}$ **5.** $P = 100 - \dfrac{Q}{Q+1}$ **6.** $C = 919\left(1 - \dfrac{e^{-0.8Y}}{Y}\right)$

7. $y = \dfrac{2x\ln(x) + 1}{x}$ **8.** $T = 205\,\dfrac{\ln(Y)}{Y}$

WORKED EXAMPLE 6.37

FIND *MC* GIVEN A LOGARITHMIC *TC* FUNCTION

A total cost function is given by the equation $TC = 120\ln(Q + 10)$.

(a) Write down the value of *TFC*. Sketch the *TC* function.

(b) Write down the equation for marginal cost.

(c) Show that *TC* and *MC* do not have turning points.

Solution

(a) $TC = 120\ln(Q + 10)$. Therefore, when $Q = 0$, $TFC = 120\ln 10 = 276.3$. The *TC* function is plotted in Figure 6.39 from the points in Table 6.20.

(b) $MC = \dfrac{d(TC)}{dQ} = 120\,\dfrac{1}{Q+10} = \dfrac{120}{Q+10}$

(c) *TC* has turning points if $d(TC)/dQ = 0$, that is, $120/Q + 10 = 0 \rightarrow 120 = 0(Q + 10) \rightarrow 120 = 0$. This is a contradiction, so there is no turning point. *MC* has turning points if $d(MC)/dQ = 0$, that is,

$$\frac{d}{dQ}\left(\frac{120}{Q+10}\right) = 0 \quad \rightarrow \quad -\frac{120}{(Q+10)^2} = 0$$

$$\rightarrow \quad -120 = 0(Q+10)^2 \rightarrow -120 = 0$$

Again a contradiction, therefore, no turning point.

Table 6.20 Points for plotting $TC = 120\ln(Q + 10)$

Q	**0**	**5**	**10**	**15**	**20**	**25**	**30**
TC	276.3	324.9	359.5	386.3	408.1	426.6	442.7

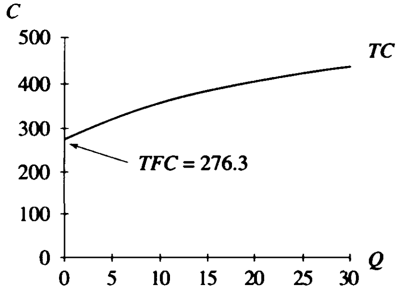


Figure 6.39 *TC* function

WORKED EXAMPLE 6.38

DEMAND, *TR*, *MR* EXPRESSED IN TERMS OF EXPONENTIALS

A firm's demand function is given by the equation $P = 150/e^{0.02Q}$.

(a) Sketch the demand function.

(b) Write down the equations for TR and MR.

(c) Determine the output Q at which TR is a maximum.

Solution

(a) The demand function is sketched in Figure 6.40 from the points in Table 6.21.

(b) $TR = PQ = 150Qe^{-0.02Q}$.

To find MR, differentiate TR. As TR is a product, use the product rule to differentiate w.r.t. Q.

Let $u = 150Q$ and $v = e^{-0.02Q}$ hence,

$$\frac{du}{dQ} = 150 \quad \text{and} \quad \frac{dv}{dQ} = -0.02e^{-0.02Q}$$

Now, substitute, u, du/dQ, v and dv/dQ into the equation for the product rule, equation 6.32.

$$\frac{d(TR)}{dQ} = v\frac{du}{dQ} + u\frac{dv}{dQ}$$

$$= e^{-0.02Q}(150) + 150Q(-0.02e^{-0.02Q})$$

$$= 150e^{-0.02Q}(1 - 0.02Q) \quad \text{factoring out the common terms, } 150e^{-0.02Q}$$

Therefore, $MR = 150e^{-0.02Q}(1 - 0.02Q)$.

Table 6.21 Points for plotting $P = 150e^{-0.02Q}$

Q	0	5	10	15	20	25	30
P	150	135.7	122.8	111.1	100.5	91.0	82.3

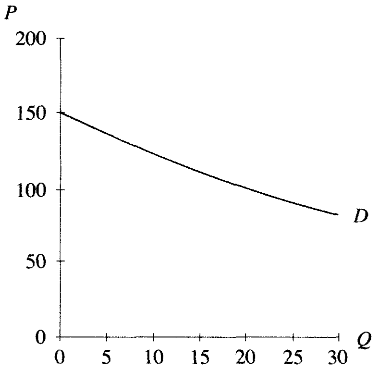


Figure 6.40 Demand function $P = 150e^{-0.02Q}$

(c) Work through the usual max./min. method to find the turning points on the *TR* curve.

Step 1: Get derivatives,

$$\frac{d(TR)}{dQ} = 150e^{-0.02Q}(1 - 0.02Q)$$

$d^2(TR)/dQ^2$ is found by differentiating $d(TR)/dQ$ which is a product. The reader is asked to show that

$$\frac{d^2(TR)}{dQ^2} = 150e^{-0.02Q}(0.0004Q - 0.04)$$

Step 2: Solve $\frac{d(TR)}{dQ} = 0$:

$$150e^{-0.02Q}(1 - 0.02Q)$$

$$\underbrace{150e^{-0.02Q}}_{\neq 0}\ \underbrace{(1 - 0.02Q)}_{1 - 0.02Q = 0 \ \rightarrow\ Q = 50} = 0$$

Step 3: Max. or min.?
Evaluate the second derivative at $Q = 50$:

$$TR'' = 150e^{-0.02(50)}[0.0004(50) - 0.04] = 55.182[-0.02] = -1.1036 < 0$$

Since $TR'' < 0$, *TR* is a maximum at $Q = 50$, $TR = 2759.1$

Progress Exercises 6.16 Further Applications, Using Chain, Product and Quotient Rule

1. A demand function is given by the equation $P = 24 - 6\ln(Q)$.
   (a) Write down the equation for *TR*. Determine the value of *Q* at which *TR* is maximised.
   (b) Write down the equation for *MR*. Show, by differentiation, that *MR* decreases but never reaches zero: that *MR* is concave up.
2. A consumption function is given by the equation $C = 800(1 - e^{-0.2Y})$.
   (a) Write down the equation for the *MPC*. Hence describe how consumption changes as income increases.
   (b) Use differentiation to show that consumption has no maximum value.
   (c) Plot the consumption function. How would you describe the curvature of the consumption function? Use the second derivative to confirm your answer.
3. A utility function is given by the equation $U = 20xe^{-0.1x}$, where x is the number of glasses of wine consumed.
   (a) Show that this utility function has a maximum value and calculate the maximum utility.
   (b) Describe how marginal utility changes for glasses of wine consumed after the maximum utility is reached. Do you consider this reasonable? Give an explanation.

4. A firm's average revenue function is $AR = 50/e^{0.05Q}$.

(a) Determine the equations for TR and MR.

(b) Find the value of Q for which TR is a maximum. Calculate the price and TR when TR is maximised.

5. The enjoyment derived from watching a particular film is given by the equation $U = 95 - \ln(t + 5)$, where t is the viewing time in minutes.

(a) Graph the utility function for the first two hours of viewing. Hence describe how enjoyment changes as the film progresses.

(b) Determine the equation for marginal utility and plot its graph.

6. Given

$$AC = \frac{100e^{0.05(Q-10)}}{Q+1}$$

(a) Find the minimum AC and graph the AC function for $Q = 0$ to $Q = 25$.

(b) Write down the equation of TC.

6.7 Elasticity and the Derivative

At the end of this section you should be familiar with:

- The use of the derivative to calculate the coefficient of point elasticity of demand
- The constant elasticity demand function
- The relationship between price elasticity of demand, marginal revenue, total revenue and price changes.

6.7.1 Point elasticity of demand and the derivative

In Section 2.6 the concept of elasticity was introduced. Price elasticity of demand was defined as

$$\varepsilon_{\mathrm{d}} = \frac{\%\text{ change in quantity}}{\%\text{ change in price}} \quad \rightarrow \quad \varepsilon_{\mathrm{d}} \times (\%\text{ change in } P) = (\%\text{ change in } Q) \qquad \textbf{(6.34)}$$

The formula for price elasticity of demand measured at a point on the demand function was given as

$$\varepsilon_{\mathrm{d}} = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q} = -\frac{1}{b} \cdot \frac{P}{Q}$$

This formula is exact when the demand function is linear. When the demand function is non-linear, the derivative $\mathrm{d}Q/\mathrm{d}P$ must be used instead of $\Delta Q/\Delta P$, that is,

$$\boxed{\varepsilon_{\mathrm{d}} = \frac{\mathrm{d}Q}{\mathrm{d}P} \cdot \frac{P}{Q}} \qquad \textbf{(6.35)}$$

Note: If the demand function is linear, then both

$$\varepsilon_d = \frac{\Delta Q}{\Delta P}\cdot\frac{P}{Q} = -\frac{1}{b}\cdot\frac{P}{Q} \quad \text{and} \quad \varepsilon_d = \frac{dQ}{dP}\cdot\frac{P}{Q}$$

can be used interchangeably as either formula provides the same answer (slope $= \Delta Q/\Delta P$ is a constant value).

In the worked examples which follow the price elasticity of demand shall be determined by the following method:

Step 1: From the equation of the demand function determine an expression for dP/dQ or dQ/dP, whichever is most convenient. Since the definition of elasticity requires dQ/dP, this is easily obtained by inverting dP/dQ, since

$$\frac{dQ}{dP} = \frac{1}{\frac{dP}{dQ}}.$$

Step 2: The expression for ε_d is obtained by substituting the expression for dQ/dP into equation (6.35). Frequently, the expression for ε_d may be simplified to an expression in Q only or P only. See Worked Example 6.40.

Step 3: To evaluate ε_d at a given price (P) or a given quantity (Q), find the simplest expression for ε_d. Then substitute the given value(s) of Q or P or both into the simplified form of equation (6.35).

WORKED EXAMPLE 6.39

EXPRESSIONS FOR POINT ELASTICITY OF DEMAND IN TERMS OF *P*, *Q* OR BOTH FOR LINEAR AND NON-LINEAR DEMAND FUNCTIONS

Given the demand functions

$$P = 100 - 2Q \qquad \text{and} \qquad Q = 80 - 10\ln(P)$$

(a) For each function, derive an expression for the price elasticity of demand, ε_d, in terms of

- (i) Q only
- (ii) P only.

Hence evaluate ε_d at

- (i) $P = 60$
- (ii) $Q = 80$

(b) Calculate the percentage change in demand in response to a 5% price increase by

- (i) Using the definition of ε_d in equation (6.34)
- (ii) Calculating the exact percentage change in Q.

Give reasons for the different answers obtained in (i) and (ii).

Solution

(a) (i)

$$P = 100 - 2Q$$

Step 1: $\dfrac{dP}{dQ} = -2 \rightarrow \dfrac{dQ}{dP} = \dfrac{1}{-2} = -\dfrac{1}{2}$

Steps 2 and 3:

$$\varepsilon_d = \frac{dQ}{dP}\frac{P}{Q} = -\frac{1}{2}\frac{P}{Q} = \frac{-0.5}{1}\frac{P}{Q}$$

$$\varepsilon_d = \frac{-0.5}{1} \times \frac{100 - 2Q}{Q} = \frac{-50 + Q}{Q}$$

$$\varepsilon_d = 1 - \frac{50}{Q}$$

This is ε_d expressed in terms of Q.
At $Q = 80$,

$$\varepsilon_d = 1 - \frac{50}{Q} = 1 - \frac{50}{80} = 0.375$$

ε_d is positive since $P < 0$ when $Q = 80$

$$Q = 80 - 10\ln(P)$$

Step 1: $\dfrac{dQ}{dP} = -\dfrac{10}{P}$

Steps 2 and 3:

$$\varepsilon_d = \frac{dQ}{dP}\frac{P}{Q} = -\frac{10}{P}\frac{P}{Q} = -\frac{10}{Q}$$

$$\varepsilon_d = -\frac{10}{Q}$$

This is ε_d expressed in terms of Q.
At $Q = 80$,

$$\varepsilon_d = -\frac{10}{Q} = -\frac{10}{80} = -0.125$$

(a) (ii)

$$\varepsilon_d = \frac{dQ}{dP}\frac{P}{Q} = -\frac{1}{2}\frac{P}{Q}$$

$$= -\frac{1}{2} \times \frac{P}{50 - 0.5P} = \frac{P}{-100 + P}$$

$$= \frac{P}{P - 100}$$

$$\varepsilon_d = \frac{P}{P - 100}$$

This is ε_d expressed in terms of P.
At $P = 60$,

$$\varepsilon_d = \frac{P}{P - 100} = \frac{60}{60 - 100} = -1.5$$

$$\varepsilon_d = \frac{dQ}{dP}\frac{P}{Q} = \frac{-10}{P}\frac{P}{Q} = \frac{-10}{Q}$$

$$= \frac{-10}{80 - 10\ln(P)} \times \frac{-0.1}{-0.1}$$

$$= \frac{1}{\ln(P) - 8}$$

$$\varepsilon_d = \frac{1}{\ln(P) - 8}$$

This is ε_d expressed in terms of P.
At $P = 60$,

$$\varepsilon_d = \frac{1}{\ln(P) - 8} = \frac{1}{\ln(60) - 8} = -0.256$$

(b) (i) To calculate the percentage change in Q when P increases by 5%, use the definition of elasticity given in equation (6.34):

$$\varepsilon_d = \frac{\text{\% change in quantity}}{\text{\% change in price}} \quad \rightarrow \quad \varepsilon_d \times (\text{\% change in } P) = (\text{\% change in } Q)$$

$$\varepsilon_d \times (5\%) = (\text{\% change in } Q)$$

$$(-1.5)(5) = -7.5\%$$

So, demand drops by 7.5% when price increases by 5%.

$$\varepsilon_d \times (5\%) = (\text{\% change in } Q)$$

$$(-0.256)(5) = -1.28\%$$

So, demand drops by 1.28% when price increases by 5%.

(ii) When price increases by 5%, the new price is $P = \frac{105}{100} \times 60 = 63$.

The corresponding value of Q for each function is

$$P = 100 - 2Q \rightarrow Q = 50 - 0.5P$$

At $P = 60$, $Q = 50 - 0.5(60) = 20.0$

At $P = 63$, $Q = 50 - 0.5(63) = 18.5$

Therefore, the percentage change in Q

$$= \frac{\text{change in } Q}{\text{initial value of } Q} \times 100$$

$$= \frac{18.5 - 20}{20} \times 100 = -7.5\%$$

This is exactly the same result as that calculated using elasticity.

$$Q = 80 - 10\ln(P)$$

At $P = 60$,

$$Q = 80 - 10\ln(60) = 39.057$$

At $P = 63$,

$$Q = 80 - 10\ln(63) = 38.569$$

Therefore, the percentage change in Q

$$= \frac{\text{change in } Q}{\text{initial value of } Q} \times 100$$

$$= \frac{38.569 - 39.057}{39.057} \times 100$$

$$= -1.249\%$$

This is *not* exactly the same result as that calculated using elasticity. The reason: for non-linear demand functions, the change over a small interval, $P = 60$ to 63, is approximately equal to the change at a point.

Therefore, for non-linear demand functions, the equation

$$\varepsilon_d = \frac{\%\text{ change in quantity}}{\%\text{ change in price}} \rightarrow \varepsilon_d \times (\%\text{ change in } P) = (\%\text{ change in } Q)$$

may be used to calculate the approximate percentage change in demand, Q, as a result of small percentage changes in P.

WORKED EXAMPLE 6.40

POINT ELASTICITY OF DEMAND FOR NON-LINEAR DEMAND FUNCTIONS

(a) Given the demand function $P = 60 - Q^2$, calculate the coefficient for point elasticity of demand at $P = 44$.

(b) Given the demand function $Q = 45e^{-0.04P}$, calculate the coefficient for point elasticity of demand at $P = 10$.

Solution

(a) $P = 60 - Q^2$

Step 1: Find $\mathrm{d}Q/\mathrm{d}P$. Since the demand function is written as $P = 60 - Q^2$, find $\mathrm{d}P/\mathrm{d}Q$.

$$\frac{\mathrm{d}P}{\mathrm{d}Q} = -2Q, \text{ then } \frac{\mathrm{d}Q}{\mathrm{d}P} = \frac{1}{\dfrac{\mathrm{d}P}{\mathrm{d}Q}} = \frac{1}{-2Q}$$

Step 2: Derive an expression for ε_d in terms of P.

$$\varepsilon_\mathrm{d} = \frac{\mathrm{d}Q}{\mathrm{d}P} \cdot \frac{P}{Q} = -\frac{1}{2Q}\frac{P}{Q} = -\frac{P}{2Q^2}$$

$$= -\frac{P}{2(60-P)}$$

Step 3: Evaluate ε_d at $P = 44$:

$$\varepsilon_\mathrm{d} = -\frac{P}{2(60-P)} = -\frac{44}{2(60-44)}$$

$$= -1.375$$

Since $\varepsilon_\mathrm{d} < -1$, demand is elastic.

(b) $Q = 45e^{-0.04P}$

Step 1: Find $\mathrm{d}Q/\mathrm{d}P$. Since the demand function is written as $Q = 45e^{-0.04P}$,

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = 45(-0.04)e^{-0.04P} = -1.8e^{-0.04P}$$

Step 2: Derive an expression for ε_d in terms of P.

$$\varepsilon_\mathrm{d} = \frac{\mathrm{d}Q}{\mathrm{d}P} \cdot \frac{P}{Q} = \frac{-1.8e^{-0.04P}}{1} \cdot \frac{P}{45e^{-0.04P}}$$

$$\varepsilon_\mathrm{d} = -0.04P$$

Step 3: Evaluate ε_d at $P = 10$:

This one is easy!

$$\varepsilon_\mathrm{d} = -0.04P = -0.04(10) = -0.4$$

Since $\varepsilon_\mathrm{d} > -1$, demand is inelastic.

6.7.2 Constant elasticity demand function

In Section 2.6 it was illustrated that the coefficient of price elasticity of demand varies along a linear demand function. The same is true for a non-linear demand function. For example, choose several values for P and prove that the coefficient of point elasticity varies along the demand function, $P = 60 - Q^2$.

There is, however, a case when the price elasticity of demand is constant along a non-linear demand function. This is known as the **constant elasticity demand function**.

WORKED EXAMPLE 6.41

CONSTANT ELASTICITY DEMAND FUNCTION

Show that a demand function of the form $Q = a/P^c$, where a and c are constants, has a constant elasticity of demand $\varepsilon_\mathrm{d} = -c$, that is, for every value of (P, Q), $\varepsilon_\mathrm{d} = -c$. Hence, show that $Q = 200/P^2$ has a constant elasticity of demand, $\varepsilon_\mathrm{d} = -2$.

Solution

General expression	**Example**
Step 1:	**Step 1:**
$Q = \frac{a}{P^c} = aP^{-c}$	$Q = \frac{200}{P^2} = 200P^{-2}$
$\frac{dQ}{dP} = -caP^{-c-1}$	$\frac{dQ}{dP} = (-2)200P^{-2-1}$
Step 2: $\frac{dQ}{dP} = -caP^{-c-1}$	**Step 2:** $\frac{dQ}{dP} = (-2)200P^{-2-1}$
$\varepsilon_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$	$\varepsilon_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$
$= -\frac{caP^{-c-1}}{1} \frac{P}{Q}$	$= \frac{-2(200)P^{-3}}{1} \frac{P}{Q}$
$= \frac{-caP^{-c}}{Q}$ adding indices on P	$= \frac{-2(200)P^{-2}}{Q}$ adding indices on P
$= -\frac{cQ}{Q}$ since $Q = aP^{-c}$	$= \frac{(-2)Q}{Q}$ since $Q = 200P^{-2}$
$= -c$ Qs cancel	$= -2$ Qs cancel

6.7.3 Price elasticity of demand, marginal revenue, total revenue and price changes

- The relationship between price elasticity of demand and marginal revenue may be expressed by equation (6.36).
- The relationship between price elasticity of demand and the change in total revenue with respect to price may be expressed by equation (6.37).

$$MR = P\left(1 + \frac{1}{\varepsilon_d}\right) \tag{6.36}$$

$$\frac{d(TR)}{dP} = Q(1 + \varepsilon_d) \tag{6.37}$$

Equations (6.36) and (6.37) are derived as follows.

Regard TR as a product, $(P \times Q)$, and differentiate using the product rule. $TR = PQ$, let $u = P$ and $v = Q$.

Differentiate w.r.t. Q $\quad MR = \dfrac{\mathrm{d}(TR)}{\mathrm{d}Q}$

let $u = P$, $\quad v = Q$, hence

$$\frac{\mathrm{d}u}{\mathrm{d}Q} = \frac{\mathrm{d}P}{\mathrm{d}Q} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}Q} = \frac{\mathrm{d}Q}{\mathrm{d}Q} = 1$$

substitute for u, v, $\quad \dfrac{\mathrm{d}u}{\mathrm{d}Q}, \dfrac{\mathrm{d}v}{\mathrm{d}Q}$

$$MR = \frac{\mathrm{d}(TR)}{\mathrm{d}Q} = u\frac{\mathrm{d}v}{\mathrm{d}Q} + v\frac{\mathrm{d}u}{\mathrm{d}Q}$$

$$MR = P\frac{\cancel{\mathrm{d}Q}}{\cancel{\mathrm{d}Q}} + Q\frac{\mathrm{d}P}{\mathrm{d}Q}$$

$$MR = P + Q \cdot \frac{\mathrm{d}P}{\mathrm{d}Q} \qquad \textbf{(6.38)}$$

Now: $\dfrac{\varepsilon_\mathrm{d}}{1} = \dfrac{\mathrm{d}Q}{\mathrm{d}P} \cdot \dfrac{P}{Q}$

$\dfrac{1}{\varepsilon_\mathrm{d}} = \dfrac{\mathrm{d}P}{\mathrm{d}Q} \cdot \dfrac{Q}{P}$ $\quad$ inverting both sides

Multiply $Q\dfrac{\mathrm{d}P}{\mathrm{d}Q}$ in equation (6.38) by $\dfrac{P}{P}$.

$$MR = P + \frac{P}{1}\left(\frac{Q}{P} \cdot \frac{\mathrm{d}P}{\mathrm{d}Q}\right) = P + P\left(\frac{1}{\varepsilon_\mathrm{d}}\right)$$

$$MR = P\left(1 + \frac{1}{\varepsilon_\mathrm{d}}\right) \qquad \text{equation (6.36)}$$

Differentiate w.r.t. P $\quad \dfrac{\mathrm{d}(TR)}{\mathrm{d}P}$

let $u = P$, $\quad v = Q$, hence

$$\frac{\mathrm{d}u}{\mathrm{d}P} = \frac{\mathrm{d}P}{\mathrm{d}P} = 1 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}P} = \frac{\mathrm{d}Q}{\mathrm{d}P}$$

substitute for u, v, $\quad \dfrac{\mathrm{d}u}{\mathrm{d}Q}, \dfrac{\mathrm{d}v}{\mathrm{d}Q}$

$$\frac{\mathrm{d}(TR)}{\mathrm{d}P} = u\frac{\mathrm{d}v}{\mathrm{d}P} + v\frac{\mathrm{d}u}{\mathrm{d}P}$$

$$\frac{\mathrm{d}(TR)}{\mathrm{d}P} = P\frac{\mathrm{d}Q}{\mathrm{d}P} + Q\frac{\cancel{\mathrm{d}P}}{\cancel{\mathrm{d}P}}$$

$$\frac{\mathrm{d}(TR)}{\mathrm{d}P} = P\frac{\mathrm{d}Q}{\mathrm{d}P} + Q \qquad \textbf{(6.39)}$$

Now: $\dfrac{\varepsilon_\mathrm{d}}{1} = \dfrac{\mathrm{d}Q}{\mathrm{d}P} \cdot \dfrac{P}{Q}$

Multiply $P\dfrac{\mathrm{d}Q}{\mathrm{d}P}$ in equation (6.39) by $\dfrac{Q}{Q}$.

$$\frac{\mathrm{d}(TR)}{\mathrm{d}P} = Q + \frac{Q}{1}\left(\frac{P}{Q} \cdot \frac{\mathrm{d}Q}{\mathrm{d}P}\right) = Q + Q(\varepsilon_\mathrm{d})$$

$$\frac{\mathrm{d}(TR)}{\mathrm{d}P} = Q(1 + \varepsilon_\mathrm{d}) \qquad \text{equation (6.37)}$$

Equation (6.36) may also be written as

$$MR = P + \frac{P}{\varepsilon_\mathrm{d}} \quad \rightarrow \quad \frac{MR - P}{1} = \frac{P}{\varepsilon_\mathrm{d}} \quad \rightarrow \quad \frac{1}{MR - P} = \frac{\varepsilon_\mathrm{d}}{P} \quad \rightarrow \quad \varepsilon_\mathrm{d} = \frac{P}{MR - P} \qquad \textbf{(6.40)}$$

There are various approaches to demonstrating the relationship between ε_d, MR and TR as P varies. An explanation based on equation (6.40) and Figure 6.41 will now be presented:

☺ **Remember:** Price, $P = AR$.

Go to the web site www.wiley.co.uk/bradley2ed for additional material from Chapter 6.

☐ Price elasticity of demand, TR, MR and P (price)

Table 6.22 Price elasticity of demand, marginal and total revenue

Worked Example 6.42 Elasticity, marginal revenue and total revenue

Table 6.23 Elasticity, total revenue and marginal revenue

Figure 6.42 Elasticity, total revenue and marginal revenue

Progress Exercises 6.17 Differentiation and Point Elasticity

1. (a) Define the price elasticity of demand ε_d.
   (b) The price elasticity of demand for a good is -0.8. What is the percentage change in the quantity demanded when price increases by 5%? Write down the general expression of the function which has a constant elasticity of demand of -0.8.
2. Given the demand functions

   (i) $P = 80 - 2Q$ (ii) $Q = 120 - 4P$ (iii) $P = 432$ (iv) $P = a - bQ$

   (a) Determine expressions for the price elasticity of demand in terms of

   (i) P and (ii) Q.

   (b) Evaluate the price elasticity of demand at

   (i) $P = 50$ and (ii) $Q = 30$.
3. The demand for family membership of a sports club is given by the equation

   $P = 500e^{-0.01Q}$, where P is the monthly fee.

   (a) Derive an expression for the price elasticity of demand in terms of Q. Evaluate the ε_d when $Q = 100$. Describe how demand changes for price increases when $Q < 100$ and when $Q > 100$.
   (b) If the club has 150 members, calculate the fee per membership (price). If the fee increases by 10%, use elasticity to calculate the approximate percentage change in demand. Why is the answer approximate?
4. The demand for seats at a cup final is given by the equation $Q = 192 - P^2$.
   (a) Derive an expression for the price elasticity of demand in terms of P. Calculate the price which should be charged when $\varepsilon_d = -1$. How many seats are available at this price?
   (b) If 10 more seats become available when $\varepsilon_d = -1$, calculate the approximate percentage change in price. Calculate the new price.
5. (a) Show that the function $Q = a/P^c$, where a and c are constants, has a constant elasticity of demand: $\varepsilon_d = -c$.
   (b) (i) Determine the elasticity of demand for train journeys on a given route when the demand function is $Q = 1200/P^{1.2}$, where Q is the number of fares in thousands.
   (ii) If the fare increases by 5%, use elasticity to calculate the percentage change in demand.
   (iii) If the fare decreases by 5%, use elasticity to calculate the percentage change in demand.
6. (See question 5)
   (a) If the company charges £30 per fare, calculate the corresponding number of fares demanded to the nearest fare.
   (b) If the fare increases by 5%, calculate the percentage change in demand exactly. How does this compare with 5(b)(ii) above? Give reasons for the difference.

7. The demand for soft drinks is given by the equation: $Q = 100 - 2P$ where P pence is the price per can, Q is the number of cans demanded.
(a) Write down the equations for TR, MR, AR.
(b) Determine the price and quantity at which revenue is maximised.
(c) Derive an expression for the price elasticity of demand in terms of

(i) P and (ii) Q.

(d) Show that TR is a maximum, MR is zero when $\varepsilon_d = -1$.

8. (See problem 7)
(a) Plot the graphs of TR, MR, AR (suitable for Excel).
(b) Find the value of Q at which $\varepsilon_d = -1$. Hence indicate on the diagram the intervals along which $\varepsilon_d < -1$ and $\varepsilon_d > -1$.
Comment on the relationship between P, Q, TR, MR and elasticity.

9. The demand for a certain wine is given by the equation: $P = 1500e^{-0.025Q}$ where P (French Francs), is the price per bottle, Q is the number of bottles demanded.
(a) Write down the equations for TR, MR, AR.
(b) Determine the price and quantity at which revenue is maximised
(c) Derive an expression for the price elasticity of demand in terms of

(i) P (ii) Q

(d) Show that $\varepsilon_d = -1$ when TR is maximised.

10. Derive the equation

$$MR = P\left(1 + \frac{1}{\varepsilon_d}\right)$$

(a) The demand function is of the form $P = a - bQ$. Would you expect ε_d to have the same value at different prices? Justify your answer.
(b) If the price of a new car is £45 400 and the price elasticity of demand is –0.9:
  (i) Calculate the marginal revenue.
  (ii) Deduce the equation of the demand function $P = a - bQ$, if it is known that $P = 800$ when $Q = 90$. Hence,
  (iii) Use elasticity, or otherwise, to determine the price and quantity at which TR is maximised.

11. (a) Show that when profit is maximised

$$P = \frac{MC}{\left(1 + \frac{1}{\varepsilon_d}\right)}$$

(b) A PC centre has a demand and total cost function for printers given by the respective equations

$$P = 300 - Q: \quad TC = 800 + 2Q.$$

Find the price, quantity and ε_d at which profit is maximised. Verify that the equation given in (a) holds when profit is maximised but not at other values of P and Q on the demand function.

6.8 Summary

□ Mathematics

Slope of the straight line $y = mx + c$ is m

Slope of a curve $y = \mathrm{f}(x)$ over a small horizontal distance Δx is $\Delta y/\Delta x$

Slope of a curve $y = \mathrm{f}(x)$ at a point (x_0, y_0) is $\mathrm{d}y/\mathrm{d}x$ evaluated at $(x = x_0, y = y_0)$

Derivatives of standard functions:

Function: $y = \mathrm{f}(x)$	**Derivative:** $\dfrac{\mathrm{d}y}{\mathrm{d}x}$
x^n	nx^{n-1}
e^x	e^x
$\ln(x)$	$\dfrac{1}{x}$
K	0
Kx^n	$K(nx^{n-1})$
$K_1x^n + K_2x^m$	$K_1(nx^{n-1}) + K_2(mx^{m-1})$

Chain rule: $y = \mathrm{f}(\mathrm{g}(x))$. Let $u = \mathrm{g}(x)$, hence $y = \mathrm{f}(u)$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\cancel{\mathrm{d}u}}\frac{\cancel{\mathrm{d}u}}{\mathrm{d}x}$$

Useful derivatives to remember:

(1) $y = e^{\mathrm{f}(x)} \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x) \times e^{\mathrm{f}(x)}$

(2) $y = \ln(\mathrm{f}(x)) \quad \rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$

Product rule: $y = \mathrm{f}(x)\mathrm{g}(x) = uv$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

Quotient rule:

$$y = \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \frac{u}{v}, \text{ then } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Maximum/minimum and points of inflection

	First derivative $\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$ evaluated at (x_0, y_0)	
Maximum point at (x_0, y_0)	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} < 0$	or check for changes in sign of dy/dx
Minimum point at (x_0, y_0)	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} > 0$	
Point of inflection at (x_0, y_0)	$\frac{dy}{dx} = 0$ for a horizontal PoI $\frac{dy}{dx} \neq 0$ for a non-horizontal PoI	$\frac{d^2y}{dx^2} = 0$ and $\frac{d^2y}{dx^2}$ changes sign	

Curvature

The curvature along an interval is **concave up** if $d^2y/dx^2 > 0$ along that interval

The curvature along an interval is **concave down** if $d^2y/dx^2 < 0$ along that interval

❒ Applications

Marginal function is the derivative of the total function:

$$MC = \frac{d(TC)}{d(Q)}: \quad MR = \frac{d(TR)}{d(Q)}: \quad MP_L = \frac{d(Q)}{d(L)}$$

Revise the relationships between marginal and average functions.

Average function: $AC = \frac{TC}{Q}$: $AR = \frac{TR}{Q}$: $APL = \frac{Q}{L}$

Optimisation: particular rules to note

1. Maximum profit when $MR = MC$ and
$$\frac{d(MR)}{dQ} < \frac{d(MC)}{dQ}$$
2. MC and AVC intersect at the minimum point on the AVC curve
3. APL and MP_L intersect at the maximum point on the APL curve

Elasticity: point elasticity of demand

1. Definition: $P = f(Q)$, then
$$\varepsilon_d = \frac{\%\Delta Q}{\%\Delta P} = \frac{dQ/Q}{dP/P} = \frac{dQ}{dP}\frac{P}{Q}$$
2. The function $Q = a/P^c$ has a constant elasticity of demand $\varepsilon_d = -c$
3. Relationship between MR and ε_d: $MR = P\left(1 + \frac{1}{\varepsilon_d}\right)$

Production and labour

A short-run production function $Q = \mathrm{f}(L)$
Costs:

$$TLC = wL$$

$$ALC = \frac{TLC}{L}$$

$$MLC = \frac{\mathrm{d}(TLC)}{\mathrm{d}L}$$

Test Exercises 6

1. (a) Determine the first derivative of the following:

   (i) $y = 7x^4$ (ii) $y = \dfrac{1}{x^2}$ (iii) $y = \sqrt{x} + 4x - 8$

   (b) A firm has the following total cost and total revenue functions:

   $$TC = \frac{1}{3}Q^3 - 9Q^2 + 200Q + 5050.\ TR = Q(120 - 10Q)$$

   Deduce the equations for the following functions:

   (i) marginal cost (ii) marginal revenue (iii) average cost (iv) average revenue
2. The output from a pottery is related to the number of labour units employed according to the production function $Q = 9L^2 - 0.1L^3$.

   (a) Determine the number of labour units which should be employed to maximise output, Q.

   (b) Find the point of inflection.

3. (See question 2)

   (a) Derive the equation for the marginal product of labour and the average product of labour.

   (b) Hence determine the number of labour units which maximise

   (i) MP_L and (ii) APL.

   (c) Show that the MP_L and APL curves intersect at the maximum point on the APL curve.

4. (See questions 2 and 3)
   Graph the production function MP_L and APL, indicating the relationship between turning point(s) and point of inflection.

5. A firm has an average cost function $AC = 10 - 3Q + Q^2$.

   (a) Write down the equations for TC, MC.

   (b) Determine the values of Q at which (i) MC and (ii) AC are minimised.

   (c) Plot the AC and MC curves on the same diagram. Confirm algebraically that the curves intersect at the minimum point on the AC curve.

6. (a) Show that profit is maximised/minimised when $MR = MC$. How would you determine whether profit is a maximum and not a minimum when $MR = MC$?

(b) The demand and total cost functions for a good are given by the equations

$$P = 80 - 4Q: \quad TC = \frac{1}{3}Q^3 - 18Q^2 + 240Q + 1500$$

(i) Write down the equations for *MR*, *MC* and profit.
(ii) Determine the number of goods which must be produced and sold to maximise profit.

7. (a) Write down the definition for the price elasticity of demand, ε_d. If the demand function is of the form $P = a - bQ$, show that the value of ε_d is not influenced by the value of slope.
(b) A firm faces different demand functions for its product in two separate markets

Market I: $P = 120 - 3Q$ Market II: $P = 180e^{-0.6Q}$

For each market
(i) Determine the price elasticity of demand when $P = 10$
(ii) Calculate the response to a 5% increase in price when $P = 10$ using

(i) the definition of elasticity (ii) direct calculation. Comment.

8. A firm may sell its product in two different markets in which the demand functions are

Market I: $P = 120 - 2Q$ Market II: $P = 492 - 10Q$

The total cost function is given by the equation $TC = 15 + 12Q$.

(a) Calculate the price which should be charged in each market to maximise the firm's profit.
(b) Calculate the price elasticity of demand when profit is maximised in each market. What is the relationship between price elasticity of demand and the price charged in different markets?

9. (See question 8)
(a) Calculate the price charged for the product if profits are maximised with no price discrimination.
(b) Compare the profit made with and without price discrimination.

10. (a) Determine the first derivatives of the following:

(i) $y = x - x\ln(x)$ (ii) $y = 2xe^{-0.1x}$

(b) The demand function for a good is given by the equation $P = 200e^{-0.1Q}$.
(i) Write down the equation for total revenue. (ii) Hence determine the value of Q at which total revenue is a maximum.

11. Determine the first and second derivatives of the following:

(a) $y = \dfrac{x}{x + 10}$ (b) $y = 10xe^{5x}$ (c) $P = (20Q - 2)^{0.5}$

CHAPTER

7

Functions of Several Variables

At the end of this chapter you should understand the term partial differentiation and its application in economics and business. The chapter is divided into the following sections:

7.1 Partial differentiation
7.2 Applications of partial differentiation
7.3 Unconstrained optimisation
7.4 Constrained optimisation and Lagrange multipliers
7.5 Summary

In Chapter 6, differentiation was confined to functions of one independent variable only. However, more realistic economic models are usually functions of more than one variable. For example, in Chapter 6 the short-run production function $Q = \mathrm{f}(L)$ was described as a function of only one input, labour. In the long run, a more realistic model assumes that output, $Q = \mathrm{f}(L, K)$, is a function of two inputs, labour and capital.

In this chapter, those basic differentiation techniques introduced in Chapter 6 are extended to the differentiation of functions of two or more variables, with applications such as optimisation, rates of change, etc.

7.1 Partial Differentiation

At the end of this section you should be able to:

- Understand the term functions of two or more variables and graph such functions
- Find first-order partial derivatives of functions of two or more variables.
- Find second-order partial derivatives of functions of two or more variables.
- Determine differentials and small changes.

7.1.1 Functions of two or more variables

Functions of two variables are written in general form as

$$z = \mathrm{f}(x, y)$$

for example,

$$z = x + 2y + 4 \tag{7.1}$$

where x and y are the independent variables and z is the dependent variable, that is, its value depends on the values of x and y.

Graphically, functions of two variables are represented by a three-dimensional diagram. Equation (7.1) is graphed in Figure 7.1, where both x and y are represented on the two horizontal axes and z on the vertical axis. To plot the function, values are assigned to both x and y in order to calculate the corresponding value of the dependent variable z, such as those given in Table 7.1.

Graphically, the function, $z = x + 2y + 4$, is a plane in three dimensions. A plane in three dimensions is represented by the general linear function in two variables:

$$z = ax + by + c$$

Otherwise, if there is any power on either x or y, or products of x and y terms, the function is a surface in three dimensions which may contain humps (relative maxima), hollows (relative minima), etc.

❐ Graphical representation of functions of two variables in economics

Diagrams in three dimensions are difficult to draw and to visualise. In economics, functions of two variables are frequently plotted by fixing one of the variables, thereby reducing the

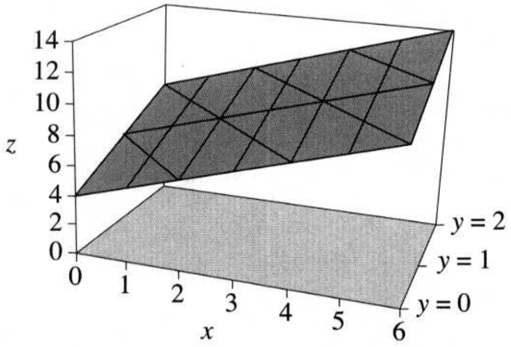


Figure 7.1 $z = x + 2y + 4$: a three-dimensional plane

problem of graphical representation and mathematical analysis to a series of two-dimensional curves.

To see how this two-dimensional representation comes about, consider a production function whose equation is

$$Q = 5LK \tag{7.2}$$

That is, output is a function of the amounts of inputs of labour, L and capital, K.

If output, Q, is fixed at some constant value, for example, at $Q = 5$, then equation (7.2) may be rearranged to express K in terms of L:

$$Q = 5LK \quad \rightarrow \quad 5 = 5LK \quad \rightarrow \quad 1 = LK \quad \rightarrow \quad K = \frac{1}{L}$$

This is a curve in a plane defined by an L and a K axis. In effect, it is a single slice of the three-dimensional diagram along the plane $Q = 5$. This downward-sloping curve is a hyperbolic function as described in Chapter 4. The graph is plotted in Figure 7.3. In economics, it is called an **isoquant** (isoquant means equal quantity). It shows the various combinations of inputs L and K for which quantity (production) is constant at $Q = 5$.

WORKED EXAMPLE 7.1

PLOT ISOQUANTS FOR A GIVEN PRODUCTION FUNCTION

Given the production function $Q = 5LK$, write down the equations for the isoquants defined by (i) $Q = 5$, (ii) $Q = 10$, (iii) $Q = 15$, (iv) $Q = 18$.

(a) Calculate a table of points for each isoquant for values $L = 0.02, 0.04, \ldots, 0.16$, hence show that any pair of points on a given isoquant give the same output, Q.

(b) Graph the isoquants on the same diagram.

(c) Deduce an expression for the slope of the isoquants. Use the second derivative to show that the isoquants are convex towards the origin.

Solution

Substitute the fixed value of Q into the equation of the production function, $Q = 5LK$, then write the resulting equation in the form $K = f(L)$. The equations of the corresponding isoquants are given in Table 7.2.

(a) The points for each isoquant are calculated in Table 7.3.

Note: when Q is calculated for any point (L, K) on a given isoquant, as expected, the result is the given fixed value of Q. For example, on the isoquant $Q = 10$, calculate $Q = 5LK$ for selected points,

$(L = 0.02, K = 100) \rightarrow Q = 5(0.02)(100) = 10$

$(L = 0.14, K = 14.29) \rightarrow Q = 5(0.14)(14.29) = 10.003$ (rounding error of 0.003)

$(L = 0.16, K = 12.50) \rightarrow Q = 5(0.16)(12.5) = 10.$

(b) The isoquants are graphed in three dimensions in Figure 7.2, using Excel. The graphs are all in parallel LK planes. In most economics textbooks, isoquants will

Table 7.2 Production functions and isoquants

Output, Q	Equation of production function, given Q	Equation of isoquant
$Q = 5$	$5 = 5LK$	$K = \frac{1}{L}$
$Q = 10$	$10 = 5LK$	$K = \frac{2}{L}$
$Q = 15$	$15 = 5LK$	$K = \frac{3}{L}$
$Q = 18$	$18 = 5LK$	$K = \frac{3.6}{L}$

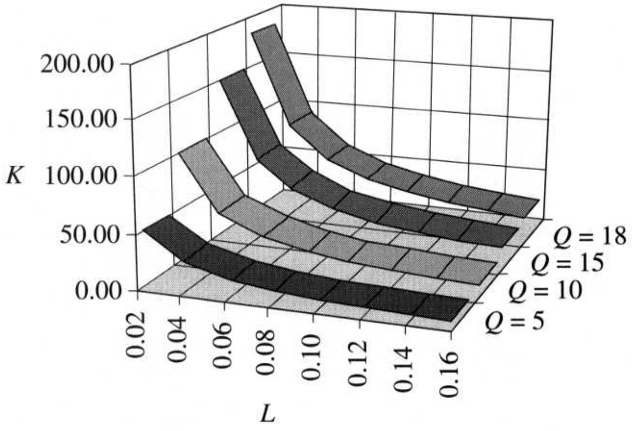


Figure 7.2 A three-dimensional view of isoquants

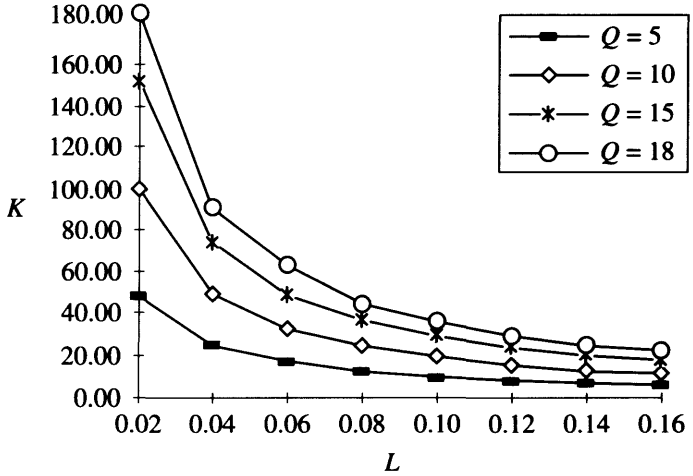


Figure 7.3 A two-dimensional view of isoquants

normally be drawn in the LK plane as two-dimensional diagrams such as shown in Figure 7.3. The standard convention is to plot L on the horizontal axis and K on the vertical one. Furthermore, note that isoquants that lie further in a north-easterly direction represent higher levels of quantity.

(c) The expression for the slope of any function $y = f(x)$, is found by differentiation. The equation of each isoquant is of the same general form:

$$Q_0 = 5LK \quad \to \quad K = \frac{Q_0}{5L} \quad \text{where } Q_0 \text{ is constant}$$

Therefore, the expression for slope is simply the equation of the first derivative:

$$K = \frac{Q_0}{5L} = \frac{1}{5}Q_0L^{-1}$$

$$\frac{dK}{dL} = \frac{1}{5}Q_0(-1L^{-2}) = -\frac{1}{5}\frac{Q_0}{L^2}$$

So the slope of any isoquant is negative, therefore K is decreasing as L increases.

Furthermore, the second derivative is positive, indicating that the curve is convex towards the origin, agreeing with the graphs in part (b):

$$\frac{dK}{dL} = -\frac{1}{5}\frac{Q_0}{L^2} \quad \to \quad \frac{d^2K}{dL^2} = \frac{2}{5}\frac{Q_0}{L^3} > 0 \qquad \text{for } L > 0$$

7.1.2 Partial differentiation: first-order partial derivatives

Partial differentiation is equivalent to differentiating along an isoquant: one of the three variables is fixed, then differentiation proceeds as normal with the two remaining variables.

For example, in Worked Example 7.1, $Q = Q_0$:

$$K = \frac{Q_0}{5L} = \frac{1}{5}\,Q_0 L^{-1}$$

$$\frac{\mathrm{d}K}{\mathrm{d}L} = \frac{1}{5}\,Q_0(-1L^{-2}) = -\frac{1}{5}\frac{Q_0}{L^2}$$

However, it is necessary to indicate that the derivative is part of a three-dimensional set-up in which the other variable, Q, was held constant. Such a derivative is called a partial derivative, denoted by using the symbol '∂' instead of 'd' for differentiation:

$$\frac{\mathrm{d}K}{\mathrm{d}L} \quad \rightarrow \quad \frac{\partial K}{\partial L} \qquad Q \text{ is fixed}$$

It may take some time to get used to treating variables as constants in one line, then treating them as variables again in the next line. You may find it useful to put the temporarily 'fixed variable' in square brackets in the first line before differentiation, and then drop the square brackets when differentiation is completed. For example,

$$K = \frac{[Q]}{5L} \quad \rightarrow \quad \frac{\partial K}{\partial L} = -\frac{Q}{5L^2}$$

WORKED EXAMPLE 7.2

PARTIAL DIFFERENTIATION: A FIRST EXAMPLE

Consider the function, $z = x + 2y + 4$. Use the following method to differentiate z partially with respect to (a) x and (b) y.

Method

(a) To differentiate z partially with respect to x, treat y as a constant, and then differentiate z w.r.t. x using the usual rules of differentiation as outlined in Chapter 6. To indicate partial differentiation is being used, the symbol '∂' is used instead of 'd'; that is,

$$\boxed{\frac{\partial z}{\partial x} \text{ denotes partial differentiation of } z \text{ w.r.t. } x}$$

$(\partial z/\partial x)$ means the partial derivative of z with respect to x.

$$z = x + 2[y] + 4 \quad \rightarrow \quad \frac{\partial z}{\partial x} = 1 + 0 + 0 = 1 \qquad \text{since } 2y \text{ is held constant and 4 is a constant}$$

(b) To differentiate z partially with respect to y, treat x as a constant, and then differentiate z w.r.t. y. In this case,

$$\boxed{\frac{\partial z}{\partial y} \text{ denotes partial differentiation of } z \text{ w.r.t. } y}$$

$z = [x] + 2y + 4 \quad \rightarrow \quad \frac{\partial z}{\partial y} = 0 + 2 + 0 = 2$ since x is held constant and 4 is a constant

Abbreviations: The first-order partial derivatives may be abbreviated as

$$\frac{\partial z}{\partial x} \equiv z_x, \quad \frac{\partial z}{\partial y} \equiv z_y$$

Some examples of first-order partial derivatives are outlined in Worked Example 7.3 where the normal rules for differentiation apply; power rule, chain rule, product rule, quotient rule, etc.

WORKED EXAMPLE 7.3

DETERMINING FIRST-ORDER PARTIAL DERIVATIVES

Find the first-order partial derivatives for each of the following functions.

(a) $z = 2x^2 + 3xy + 5$
(b) $Q = 10L^{0.7}K^{0.3}$
(c) $U = x^2y^5$

Solution

(a) To find $\partial z/\partial x$, differentiate partially w.r.t. x, therefore, treat y as if it were a constant (like 10, 12, any number!) and proceed with the following steps:

Step 1: Since we are differentiating w.r.t. x and treating y as if it were a constant, examine each term in turn, look for x in each term, then decide which rules of differentiation apply. In this example,

$\mathbf{2x^2}$**:** Use the power rule to differentiate this term.

$\mathbf{3[y]x}$: This term is simply a constant, $3y$ (since y is considered constant) multiplied by x; therefore, differentiation requires the power rule.

5: This is a constant term so its derivative is zero.

Step 2: Differentiate partially w.r.t. x:

$$z = 2x^2 + 3[y]x + 5$$

$$\frac{\partial z}{\partial x} = 2(2x) + 3y(1) + 0 \quad \text{differentiating partially w.r.t. } x$$

$$\frac{\partial z}{\partial x} = 4x + 3y$$

Now find $\partial z/\partial y$, the first-order partial derivative of z w.r.t. y (treat x as constant).

Step 1: Examine each term in turn, look for y in each term, and decide which rule for differentiation applies. In this example,

$\mathbf{2[x^2]}$**:** There is no y in this term, therefore treat it as a constant term, so its derivative is zero.

$3[x]y$: y is multiplied by $[3x]$, which is considered constant, so the power rule applies.

5: This is a constant term so its derivative is zero.

Step 2: Differentiate partially w.r.t. y

$$z = 2[x^2] + 3[x]y + 5$$

$$\frac{\partial z}{\partial y} = 0 + 3x(1) + 0 \quad \text{differentiating partially w.r.t. } y$$

$$\frac{\partial z}{\partial y} = 3x$$

(b) $Q = 10L^{0.7}K^{0.3}$

Find $\partial Q/\partial L$, differentiate partially w.r.t. L

Step 1: To differentiate Q partially w.r.t. L, treat K as a constant. Since there is only one term in Q, we are differentiating $L^{0.7}$ multiplied by a constant. Apply power rule.

Step 2: Differentiate partially w.r.t. L

$$Q = 10[K^{0.3}]L^{0.7}$$

$$\frac{\partial Q}{\partial L} = 10K^{0.3}(0.7L^{0.7-1})$$

$$= 7L^{-0.3}K^{0.3}$$

Find $\partial Q/\partial K$, differentiate partially w.r.t. K

Step 1: This time, L is treated as a constant, so we are differentiating a constant multiplied by $K^{0.3}$. The power rule applies.

Step 2: Differentiate partially w.r.t. K

$$Q = 10[L^{0.7}]K^{0.3}$$

$$\frac{\partial Q}{\partial K} = 10L^{0.7}(0.3K^{0.3-1})$$

$$= 3L^{0.7}K^{-0.7}$$

(c) $U = x^2y^5$. This problem is very similar to part (b).

Note: This is an equation of the utility function written in the form $U = f(x, y)$. U is the level of utility or satisfaction derived from consuming goods X and Y.

Find $\partial U/\partial x$, differentiate partially w.r.t. x

Step 1: Treat y as a constant. Since there is only one term in U, we are differentiating x^2 multiplied by a constant. Apply power rule.

Step 2: Differentiate partially w.r.t. x

$$U = [y^5]x^2$$

$$\frac{\partial U}{\partial x} = y^5(2x)$$

$$= 2xy^5$$

Find $\partial U/\partial y$, differentiate partially w.r.t. y

Step 1: This time x is treated as a constant so we are differentiating a constant multiplied by y^5. Power rule applies.

Step 2: Differentiate partially w.r.t. y

$$U = [x^2]y^5$$

$$\frac{\partial U}{\partial y} = x^2(5y^4)$$

$$= 5x^2y^4$$

Progress Exercises 7.1 First-order Partial Differentiation and Isoquants

Find the first-order partial derivatives of the following functions:

1. $z = x^2 + 2xy$

2. $z = 5x^2y^3$

3. $Q = \ln x + 3 \ln y$

4. $Q = 10L^{0.8}K^{0.2}$

5. $P = 120e^{0.8t}$

6. $U = 4x^2y^2$

7. $z = x^2(1 + 2y)$

8. $z = 5xy + \dfrac{y}{x}$

9. $Q = \ln 5x + 3x \ln y$

10. $Q = 10L^{0.8}K^{0.2} + 30 - 6L - 4K$

11. $C = 120(1 + e^{-0.8Q})$

12. Find Z_L, Z_K, Z_λ where $Z = L^{0.5}K^{0.5} - \lambda(50 - 2L - 3K)$

13. If $Q = L^{0.5}K^{0.5}$, show that

$$Q = L\frac{\partial Q}{\partial L} + K\frac{\partial Q}{\partial K}$$

14. A production function is given by the equation $Q = 2LK$.
(a) When production is fixed at $Q = 400$, write down the equation of the corresponding isoquant in the form $K = f(L)$.
(b) Calculate the values of K when the number of units of labour, $L = 5$; 10; 15; 20 on the isoquant given in (a).
Confirm that each of these points lie on the isoquant $Q = 400$.
(c) Graph the isoquant, $Q = 400$.

15. Repeat question 14 for the production function $Q = 2LK^{0.5}$ and production, $Q = 20$.

7.1.3 Second-order partial derivatives

To find the second-order partial derivatives, simply differentiate the first-order partial derivatives again (the same idea as in Chapter 6). The only problem is that there are several possibilities. If there are two first-order partial derivatives, $\partial z/\partial x$, $\partial z/\partial y$, each first-order partial derivative could be differentiated w.r.t. the same variable again, giving two straight second-order partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

Furthermore, each first-order partial derivative could also be differentiated w.r.t. the opposite variable giving two mixed second-order partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y}, \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x}$$

However, these two mixed second-order partial derivatives are equal for most functions met in practice; that is,

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$$

as is demonstrated in the following Worked Examples.

The **straight second-order partial derivatives** are calculated by differentiating the first-order partial derivatives again with respect to the same variable.

The **mixed second-order partial derivatives** are calculated by differentiating the first-order partial derivatives again with respect to the other variable.

The actual differentiation follows the same basic rules given in Chapter 6.

Abbreviations: The second-order partial derivatives may be abbreviated as

$$\frac{\partial^2 z}{\partial x^2} \equiv z_{xx}, \quad \frac{\partial^2 z}{\partial y^2} \equiv z_{yy}, \quad \frac{\partial^2 z}{\partial x\,\partial y} \equiv z_{yx}, \quad \frac{\partial^2 z}{\partial y\,\partial x} \equiv z_{xy}$$

WORKED EXAMPLE 7.4

DETERMINING SECOND-ORDER PARTIAL DERIVATIVES

(a) Find the second-order partial derivatives for the following functions:
- (i) $z = 2x^2 + 3xy + 5$
- (ii) $Q = 10L^{0.7}K^{0.3}$
- (iii) $U = x^2y^5$

(b) Show that the mixed second-order partial derivatives are equal.

Solution

To calculate the second-order derivatives (partial or otherwise), find the first-order derivatives, and simply differentiate again. The first-order derivatives for the above functions were calculated in Worked Example 7.3.

(a)

(i) Find the straight second-order partial derivatives of $z = 2x^2 + 3xy + 5$

To find $\partial^2 z/\partial x^2$,	To find $\partial^2 z/\partial y^2$,
1. Determine the first-order partial derivative; this has already been done in Worked Example 7.3: $\partial z/\partial x = 4x + 3y$	**1.** Determine the first-order partial derivative; this has already been done in Worked Example 7.3: $\partial z/\partial y = 3x$.
2. Differentiate partially w.r.t. x again.	**2.** Differentiate partially w.r.t. y again.

Step 1: The first term is x, the power rule applies. The second term is $3[y]$, considered a 'constant' term, so its partial derivative w.r.t. x is zero.

Step 2: Differentiate partially w.r.t. x

$$\frac{\partial z}{\partial x} = 4x + 3[y]$$

$$\frac{\partial^2 z}{\partial x^2} = 4(1) + 0 = 4$$

Step 1: The only term is $3[x]$, considered a 'constant' term, so its derivative w.r.t. y is zero.

Step 2: Differentiate partially w.r.t. y

$$\frac{\partial z}{\partial y} = 3[x]$$

$$\frac{\partial^2 z}{\partial y^2} = 0$$

Now find the mixed second-order partial derivatives, $\partial^2 z/\partial y\,\partial x$, $\partial^2 z/\partial x\,\partial y$.

To find $\partial^2 z/\partial y\,\partial x$

1. $\partial z/\partial x = 4x + 3y$, Worked Example 7.3
2. Now differentiate partially w.r.t. y.

Step 1: The second term is y, the power rule applies. The first term is $4[x]$, considered a 'constant' term, so its partial derivative w.r.t. y is zero.

Step 2: Differentiate partially w.r.t. y

$$\frac{\partial z}{\partial x} = 4[x] + 3y$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = 0 + 3(1) = 3$$

To find $\partial^2 z/\partial x\,\partial y$

1. $\partial z/\partial y = 3x$, Worked Example 7.3
2. Now differentiate partially w.r.t. x.

Step 1: The only term is $3x$ so differentiating partially w.r.t. x is the same as ordinary differentiation w.r.t. x.

Step 2: Differentiate partially w.r.t. x

$$\frac{\partial z}{\partial y} = 3x$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = 3(1) = 3$$

(ii) Find the straight second-order partial derivatives of $Q = 10L^{0.7}K^{0.3}$.

To find $\partial^2 Q/\partial L^2$

1. Determine the first-order partial derivative; this has been done in Worked Example 7.3: $\partial Q/\partial L = 7L^{-0.3}K^{0.3}$
2. Differentiate partially w.r.t. L again.

Step 1: The only term is $7L^{-0.3}K^{0.3}$ with $7[K^{0.3}]$ as a 'constant', multiplied by $L^{-0.3}$. Use the power rule.

To find $\partial^2 Q/\partial K^2$

1. Determine the first-order partial derivative; this has been done in Worked Example 7.3: $\partial Q/\partial K = 3L^{0.7}K^{-0.7}$
2. Differentiate partially w.r.t. K again.

Step 1: The only term is $3L^{0.7}K^{-0.7}$ with $3[L^{0.7}]$ as a 'constant', multiplied by $K^{-0.7}$. Use the power rule.

Step 2: Differentiate partially w.r.t. L.

$$\frac{\partial Q}{\partial L} = 7L^{-0.3}[K^{0.3}]$$

$$\frac{\partial^2 Q}{\partial L^2} = 7(-0.3L^{-0.3-1})K^{0.3}$$

$$= -2.1L^{-1.3}K^{0.3}$$

Step 2: Differentiate partially w.r.t. K.

$$\frac{\partial Q}{\partial K} = 3[L^{0.7}]K^{-0.7}$$

$$\frac{\partial^2 Q}{\partial K^2} = 3L^{0.7}(-0.7K^{-0.7-1})$$

$$= -2.1L^{0.7}K^{-1.7}$$

Now find the mixed second-order partial derivatives, $\partial^2 Q/\partial K\,\partial L$, $\partial^2 Q/\partial L\,\partial K$.

$$\frac{\partial^2 Q}{\partial K\,\partial L}$$

The first-order partial derivative, w.r.t. L is

$$\frac{\partial Q}{\partial L} = 7L^{-0.3}K^{0.3}$$

Now differentiate partially w.r.t. K.

Step 1: The only term is $7L^{-0.3}K^{0.3}$ with $7[L^{-0.3}]$ as a 'constant', multiplied by $K^{0.3}$. Use the power rule.

Step 2: Differentiate partially w.r.t. K

$$\frac{\partial Q}{\partial L} = 7[L^{-0.3}]K^{0.3}$$

$$\frac{\partial^2 Q}{\partial K\,\partial L} = 7L^{-0.3}(0.3K^{0.3-1})$$

$$= 2.1L^{-0.3}K^{-0.7}$$

$$\frac{\partial^2 Q}{\partial L\,\partial K}$$

The first-order partial derivative, w.r.t. K is

$$\frac{\partial Q}{\partial K} = 3L^{0.7}K^{-0.7}$$

Now differentiate partially w.r.t. L.

Step 1: The only term is $3L^{0.7}K^{-0.7}$ with $3[K^{-0.7}]$ as a 'constant', multiplied by $L^{0.7}$. Use the power rule.

Step 2: Differentiate partially w.r.t. L

$$\frac{\partial Q}{\partial K} = 3L^{0.7}[K^{-0.7}]$$

$$\frac{\partial^2 Q}{\partial L\,\partial K} = 3(0.7L^{0.7-1})K^{-0.7}$$

$$= 2.1L^{-0.3}K^{-0.7}$$

(iii) The straight second-order partial derivatives for $U = x^2y^5$ are outlined briefly, the comments being left to the reader.

$$\frac{\partial^2 U}{\partial x^2}$$

The first-order partial derivative w.r.t. x is

$$\frac{\partial U}{\partial x} = 2xy^5$$

Step 1: Which rule is required for differentiating partially w.r.t. x?

Step 2: Differentiate partially w.r.t. x.

$$\frac{\partial U}{\partial x} = 2x[y^5]$$

$$\frac{\partial^2 U}{\partial x^2} = 2(1)y^5 = 2y^5$$

$$\frac{\partial^2 U}{\partial y^2}$$

The first-order partial derivative w.r.t. y is

$$\frac{\partial U}{\partial y} = 5x^2y^4$$

Step 1: Which rule is required for differentiating partially w.r.t. y?

Step 2: Differentiate partially w.r.t. y.

$$\frac{\partial U}{\partial y} = 5[x^2]y^4$$

$$\frac{\partial^2 U}{\partial y^2} = 5x^2(4y^3) = 20x^2y^3$$

Now, find the mixed second-order partial derivatives $\partial^2 U/\partial y\,\partial x$, $\partial^2 U/\partial x\,\partial y$.

$$\frac{\partial U}{\partial x} = 2xy^5$$

Step 1: Which rule is required for differentiating partially w.r.t. y?

Step 2: Differentiate partially w.r.t. y

$$\frac{\partial U}{\partial x} = 2[x]y^5$$

$$\frac{\partial^2 U}{\partial y\,\partial x} = 2x(5y^4)$$

$$= 10xy^4$$

$$\frac{\partial U}{\partial y} = 5x^2y^4$$

Step 1: Which rule is required for differentiating partially w.r.t. x?

Step 2: Differentiate partially w.r.t. x

$$\frac{\partial U}{\partial y} = 5x^2[y^4]$$

$$\frac{\partial^2 U}{\partial x\,\partial y} = 5(2x)y^4$$

$$= 10xy^4$$

(b) In (i), (ii) and (iii) above it is clear that each pair of mixed second-order partial derivatives are equivalent. This is true for all the functions which are used in this text.

Progress Exercises 7.2 Second-order Partial Differentiation

Find the second-order partial derivatives for questions 1 to 12 from Progress Exercise 7.1, which are reproduced below.

1. $z = x^2 + 2xy$

2. $z = 5x^2y^3$

3. $Q = \ln x + 3\ln y$

4. $Q = 10L^{0.8}K^{0.2}$

5. $P = 120e^{0.8t}$

6. $U = 4x^2y^2$

7. $z = x^2(1 + 2y)$

8. $z = 5xy + \dfrac{y}{x}$

9. $Q = \ln 5x + 3x \ln y$

10. $Q = 10L^{0.8}K^{0.2} + 30 - 6L - 4K$

11. $C = 120(1 + e^{-0.8Q})$

12. $Z = L^{0.5}K^{0.5} - \lambda(50 - 2L - 3K)$

7.1.4 Differentials and small changes (incremental changes)

The rate of change of y w.r.t. x [where $y = f(x)$] has already been defined many times as $\Delta y/\Delta x$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta x} \quad \rightarrow \quad \Delta y = \left(\frac{\Delta y}{\Delta x}\right)\Delta x$$

and, as Δx becomes infinitesimal, these small changes, Δx, Δy are written as 'dx' and 'dy' respectively, so the differential of y may be written as

$$\boxed{dy = \left(\frac{dy}{dx}\right)dx} \tag{7.3}$$

where dy: differential of y derivative of y w.r.t. x dx: differential of x

❐ Incremental changes

Incremental changes are small changes in the dependent variable (y), which result from changes in the independent variable(s). The formula for incremental changes is derived from equation (7.3), where the differentials dx and dy are replaced by Δx and Δy, small changes, but not infinitesimally small changes in x and y.

$$\Delta y \cong \left(\frac{dy}{dx}\right)\Delta x \tag{7.4}$$

This formula, often referred to as 'the small changes' formula or 'incremental changes' formula, which gives the approximate change in y as a result of a small change in x, is illustrated graphically in Figure 7.4.

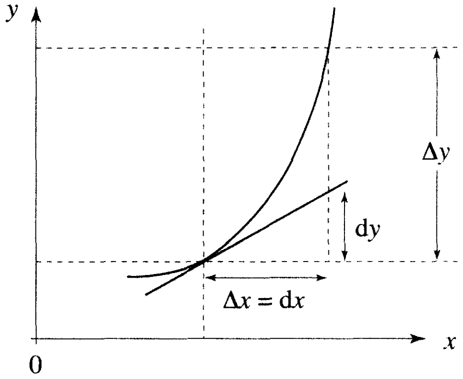


Figure 7.4 Differentials and small changes

WORKED EXAMPLE 7.5

DIFFERENTIALS FOR FUNCTIONS OF ONE VARIABLE

A supply function is given by the equation $P = Q^2$.

(a) Find the derivative of P.

(b) Find the differential of P.

(c) If Q is increased by 3%, use differentials to find the approximate change in P.

Solution

(a) The derivative of P simply involves ordinary differentiation as studied in Chapter 6:

$$P = Q^2$$

$$\frac{\mathrm{d}P}{\mathrm{d}Q} = 2Q$$

(b) Using equation (7.3), where $y \to P$ and $x \to Q$, the differential of P is defined as

$$\mathrm{d}P = \left(\frac{\mathrm{d}P}{\mathrm{d}Q}\right)\mathrm{d}Q$$

$$\mathrm{d}P = (2Q)\mathrm{d}Q$$

(c) The increase in Q is written as

$$\Delta Q = \frac{3}{100}Q$$

Note: 3% of Q is never just 3/100, it must be 3% of Q. If you are given a 3% increase in pay, would you be satisfied with a rise of 3/100 or would you prefer to get a rise of 3/100 × present pay?

Applying equation (7.4),

$$\Delta P \cong \left(\frac{\mathrm{d}P}{\mathrm{d}Q}\right)\Delta Q$$

$$\Delta P \cong 2Q\left(\frac{3Q}{100}\right) \cong \frac{2Q}{1}\frac{3Q}{100} \cong \frac{6}{100}Q^2 \cong \frac{6}{100}P$$

❐ Differentials for functions of two variables

The argument for the differential of a function of one variable may be extended to functions of two variables, so the total differential of $z = f(x, y)$, a function of two variables, is written as

$$\mathrm{d}z = \left(\frac{\partial f}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)\mathrm{d}y \qquad (7.5)$$

or for small changes (incremental changes),

$$\Delta z \cong \left(\frac{\partial f}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial y}\right)\Delta y \qquad (7.6)$$

WORKED EXAMPLE 7.6

DIFFERENTIALS AND INCREMENTAL CHANGES

A company's revenue is given by the equation

$$TR = 5W^2A^3$$

where W is the payment on wages and A is the advertising expenditure.

(a) Find the differential of *TR*.

(b) Use differentials to find the approximate percentage change if spending on
  (i) Wages is increased by 5%, no change in advertising expenditure
  (ii) Advertising expenditure decreases by 2%, no change in wages.
  (iii) Wages is increased by 5% and advertising expenditure is decreased by 2%

Solution

(a) Using equation (7.5), the differential of *TR* is calculated as

$$\mathrm{d}z = \left(\frac{\partial f}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)\mathrm{d}y$$

$$\text{then}\quad \mathrm{d}(TR) = \left(\frac{\partial TR}{\partial W}\right)\mathrm{d}W + \left(\frac{\partial TR}{\partial A}\right)\mathrm{d}A$$

$$= 5(2WA^3)\mathrm{d}W + (5W^2 3A^2)\mathrm{d}A$$

(b) (i) Wages increase by 5%, so $\Delta W = 5W/100$. The change in advertising expenditure is zero, so $\Delta A = 0$. Substituting these values into equation (7.6) for small changes,

$$\Delta(TR) \cong \left(\frac{\partial TR}{\partial W}\right)\Delta W + \left(\frac{\partial TR}{\partial A}\right)\Delta A$$

$$\cong 5(2WA^3)\Delta W + (15W^2A^2)\Delta A$$

$$\cong \frac{5(2WA^3)}{1} \times \frac{5W}{100} + 0 \qquad \text{as } \Delta A = 0$$

$$\cong \frac{5W^2A^3}{1}\,\frac{10}{100}$$

note, the numbers are not multiplied out, but $5W^2A^3 = TR$ is factored out since we require the change as a percentage of *TR*

$$\cong \frac{TR}{1}\,\frac{10}{100}$$

So *TR* has increased by 10%.

(ii) The change in *A* is $\Delta A = -(2A/100)$; a minus sign indicating a decrease in *A* and $\Delta W = 0$. Substitute these into equation (7.6) for small changes:

$$\Delta(TR) \cong \left(\frac{\partial TR}{\partial W}\right)\Delta W + \left(\frac{\partial TR}{\partial A}\right)\Delta A$$

$$\cong 5(2WA^3)\Delta W + (5W^2 3A^2)\Delta A$$

$$\cong 0 + \frac{3(5W^2A^2)}{1} \times \frac{-2A}{100} \qquad \text{as } \Delta W = 0$$

$$\cong \frac{5W^2A^3}{1} \times \frac{-6}{100}$$

note, the numbers are not multiplied out, but $5W^2A^3 = TR$ is factored out since we require the change as a percentage of *TR*

$$\cong \frac{TR}{1}\left(\frac{-6}{100}\right)$$

Total revenue has decreased by 6% as a result of a 2% decrease in advertising expenditure.

(iii) The change in W is $\Delta W = 5W/100$ and the change in A is $\Delta A = -(2A/100)$. Substitute these into equation (7.6) for small changes:

$$\begin{aligned}
\Delta(TR) &\cong \left(\frac{\partial TR}{\partial W}\right)\Delta W + \left(\frac{\partial TR}{\partial A}\right)\Delta A \\
&\cong 5(2WA^3)\Delta W + (15W^2A^2)\Delta A \\
&\cong \frac{5(2WA^3)}{1} \times \frac{5W}{100} + \frac{15W^2A^2}{1} \times \frac{-2A}{100} \\
&\cong \frac{5W^2A^3}{1}\frac{10}{100} - \frac{5W^2A^3}{1}\frac{6}{100} \qquad \text{factor out } 5W^2A^3 = TR, \text{ since we require the percentage change in } TR \\
&\cong \frac{TR}{1}\frac{10}{100} - \frac{TR}{1}\frac{6}{100} \\
&\cong TR\left(\frac{10}{100} - \frac{6}{100}\right) \cong TR\frac{4}{100}
\end{aligned}$$

TR has increased by approximately 4%.

❐ Summary

- **Functions of one variable:** $y = f(x)$

The **total derivative** is dy/dx.
The **differential** of y is

$$dy = \left(\frac{dy}{dx}\right)dx$$

where dy: differential of y derivative of y w.r.t. x dx: differential of x

and dy and dx are infinitesimally small.

- **Functions of two variables:** $z = f(x, y)$

The first-order partial derivatives are
$\partial z/\partial x$ is the partial derivative of z w.r.t. x, y treated as a constant (abbreviated to z_x).
$\partial z/\partial y$ is the partial derivative of z w.r.t. y, x treated as a constant (abbreviated to z_y).

The second-order partial derivatives are

$$\frac{\partial^2 z}{\partial x^2} \equiv z_{xx}, \quad \frac{\partial^2 z}{\partial y^2} \equiv z_{yy}, \quad \frac{\partial^2 z}{\partial y\,\partial x} \equiv z_{xy}, \quad \frac{\partial^2 z}{\partial x\,\partial y} \equiv z_{yx}$$

The total differential of z is given by

$$dz = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy$$

or for small changes $dx \to \Delta x$ etc., the **small (incremental) changes formula** is given by

$$\Delta z \cong \left(\frac{\partial f}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial y}\right)\Delta y$$

Progress Exercises 7.3 Differentials and Small Changes

Find the differential, dz, for questions 1–5:

1. $z = x^2$ **2.** $z = x^2 + y^2$ **3.** $z = 3e^x + y$

4. $z = 10L^{0.2}K^{0.6}$ **5.** $z = \ln x + 5 \ln y$

In questions 6–11, find the approximate percentage change which results from the indicated changes in the other variables.

6. $Q = L^{0.5}K^{0.5}$, L and K both increase by 3%.

7. $U = 5xy$, x and y both increase by 2%.

8. $TR = 50P^{0.6}A^{1.2}$, P decreases by 3% and A increases by 2%.

9. $TR = 125P^{0.5}A^{1.5}Q^{0.2}$, P decreases by 3%, A increases by 2%, Q is unchanged.

10. $TR = 125P^{0.5}A^{1.5}Q^{0.2}$, P, A and Q increase by 2%.

11. $Q = 200 - 2P + 0.02Y + 0.3P_s$, Y and P_s increase by 3%, P does not change, where P_s is the price of substitute products.

12. If $z = e^{x+y}$ show that $z = 0.5(z_x + z_y)$.

7.2 Applications of Partial Differentiation

In this section partial differentiation is applied to:

- Production functions, Cobb–Douglas production function in general form
- Returns to scale
- Utility functions
- Partial elasticities
- National income multipliers

We shall sketch each function, find marginal functions, rates of change and elasticities.

7.2.1 Production functions

At the end of this section you should be able to:

- Derive expressions for the marginal functions, MP_L and MP_K.
- Outline the relationship between marginal and average functions

- Derive the equation of and sketch the graph of an isoquant
- Derive the equation for the slope of an isoquant

In the previous section the general production function described output as a function of labour (L) and capital (K), that is,

$$Q = f(L, K)$$

A production function that is widely used in economic analysis is the Cobb–Douglas production function which is expressed in general form as

$$Q = AL^{\alpha}K^{\beta} \tag{7.7}$$

where A is a constant and $0 < \alpha < 1$, $0 < \beta < 1$, $L > 0, K > 0$. For example, a Cobb–Douglas production function in specific form could be written as

$$Q = 50L^{0.4}K^{0.6}$$

Table 7.4 Summary of marginal functions for $Q = AL^{\alpha}K^{\beta}$: $0 < \alpha < 1, 0 < \beta < 1$

Name	Partial derivative	Interpretation
Marginal product of labour	$Q_L =$ $MP_L = \dfrac{\partial Q}{\partial L} = A\alpha L^{\alpha-1}K^{\beta}$	$MP_L > 0$ With capital held constant, Q increases as labour input, L, increases.
	$Q_{LL} = \dfrac{\partial(MP_L)}{\partial L} =$ $\dfrac{\partial^2 Q}{\partial L^2} = (\alpha - 1)\dfrac{A\alpha L^{\alpha}K^{\beta}}{L^2}$ $= (\alpha - 1)\dfrac{\alpha Q}{L^2}$ Since $Q_{LL} < 0 \rightarrow$ curve concave to origin	$Q_{LL} < 0$ With capital held constant, Q increases, but at a decreasing rate; therefore MP_L decreases (its rate of change is negative) as labour input, L, increases. **Law of diminishing returns to labour**
Marginal product of capital	$Q_K =$ $MP_K = \dfrac{\partial Q}{\partial K} = A\beta L^{\alpha}K^{\beta-1}$	$MP_K > 0$ With labour held constant, Q increases as capital input, K, increases.
	$Q_{KK} = \dfrac{\partial(MP_K)}{\partial K} =$ $\dfrac{\partial^2 Q}{\partial K^2} = A\beta L^{\alpha}(\beta - 1)K^{\beta-2}$ $= (\beta - 1)\dfrac{\beta Q}{K^2}$ Since $Q_{KK} < 0 \rightarrow$ curve concave to origin.	$Q_{KK} < 0$ With labour held constant, Q increases but, at a decreasing rate; therefore MP_K decreases (its rate of change is negative) as capital input, K, increases. **Law of diminishing returns to capital**
	$Q_{KL} = Q_{LK} =$ $\dfrac{\partial^2 Q}{\partial K \partial L} = A\alpha\beta L^{\alpha-1}K^{\beta-1}$	$Q_{KL} > 0$ and $Q_{LK} > 0$ MP_L increases (its rate of change is positive) as capital input, K, increases and MP_K increases as L increases

☐ Marginal functions in general

The partial derivatives for the Cobb–Douglas production function are summarised in Table 7.4. The first derivatives are the marginal functions, referred to as marginal products. The second derivatives (the rate of change of the first derivatives) indicate whether the marginal product (MP_L, MP_K) is increasing or decreasing.

In Section 6.5, the law of diminishing returns to labour was outlined. The law of diminishing returns to capital reflects the change in output resulting from a change in capital, keeping labour constant.

A Cobb–Douglas production function exhibits diminishing returns to each factor. This is confirmed by the negative second derivatives:

Law of diminishing returns to labour: $Q_{LL} < 0$

Law of diminishing returns to capital: $Q_{KK} < 0$

☺ **Remember:** Negative second derivatives also indicate that the curves are concave towards the origin.

WORKED EXAMPLE 7.7

MP_L AND MP_K: INCREASING OR DECREASING?

Find the first and second derivatives for the production function, $Q = 10L^{0.5}K^{0.5}$. Use the second-order partial derivatives to determine whether the marginal functions (marginal products) are increasing or decreasing.

Solution

The second derivatives for the production function are calculated by differentiating the first derivatives again, as follows:

Q_{KK}	Q_{LL}	Q_{LK}
$Q = 10L^{0.5}K^{0.5}$	$Q = 10L^{0.5}K^{0.5}$	$Q = 10L^{0.5}K^{0.5}$
$Q_K = 5L^{0.5}K^{-0.5}$	$Q_L = 5L^{-0.5}K^{0.5}$	$Q_K = 5L^{0.5}K^{-0.5}$
$Q_{KK} = 5(-0.5)L^{0.5}K^{-0.5-1}$	$Q_{LL} = 5(-0.5)L^{-0.5-1}K^{0.5}$	$Q_{LK} = 5(0.5)L^{0.5-1}K^{-0.5}$
$= -2.5L^{0.5}K^{-1.5}$	$= -2.5L^{-1.5}K^{0.5}$	$= 2.5L^{-0.5}K^{-0.5}$
Negative: MP_K (or Q_K) decreases as K input increases. Law of diminishing returns to capital (short-run law of production)	Negative: MP_L (or Q_L) decreases as L input increases. Law of diminishing returns to labour (short-run law of production)	Positive: MP_K increases as L input increases.
Note: MP_K is positive.	**Note:** MP_L is positive.	

❐ The relationship between marginal and average functions

In Chapter 6, the APL (average product of labour) was defined as total output divided by the number of units of labour employed. Similarly, the APK (the average product of capital) is defined as total output divided by the number of units of capital. The relationships between the APL and the MP_L and between the APK and the MP_K are summarised in Table 7.5 for the Cobb–Douglas production function, $Q = AL^{\alpha}K^{\beta}$.

Table 7.5 Average and marginal functions for the production function, $Q = AL^{\alpha}K^{\beta}$

	Average function	Marginal function	Comment
Labour	$Q = AL^{\alpha}K^{\beta}$ $APL = \frac{Q}{L}$ $= \frac{AL^{\alpha}K^{\beta}}{L}$ $= AL^{\alpha-1}K^{\beta}$	$Q = AL^{\alpha}K^{\beta}$ $MP_L = \frac{\partial Q}{\partial L}$ $= A\alpha L^{\alpha-1}K^{\beta}$	$MP_L = \alpha(AL^{\alpha-1}K^{\beta})$ $= \alpha(APL)$ $\Rightarrow MP_L < APL$ since $0 < \alpha < 1$
Capital	$APK = \frac{Q}{K}$ $= \frac{AL^{\alpha}K^{\beta}}{K}$ $= AL^{\alpha}K^{\beta-1}$	$MP_K = \frac{\partial Q}{\partial K}$ $= A\beta L^{\alpha}K^{\beta-1}$	$MP_K = \beta(AL^{\alpha}K^{\beta-1})$ $= \beta(APK)$ $\Rightarrow MP_K < APK$ since $0 < \beta < 1$

❐ Production conditions

Normally, a producer would desire productivity to increase as the amount of each input increases, indicated by positive marginal functions. However, the rate of increase usually slows down as the amount of inputs become progressively larger, indicated by negative second derivatives. In practice, a firm produces output over a certain range of the production function as outlined by the following production conditions.

Conditions for using labour	Conditions for using capital
$MP_L = \frac{\partial Q}{\partial L}$ is positive	$MP_K = \frac{\partial Q}{\partial K}$ is positive
$\frac{\mathrm{d}(MP_L)}{\mathrm{d}L} = \frac{\partial^2 Q}{\partial L^2} < 0$	$\frac{\mathrm{d}(MP_K)}{\mathrm{d}K} = \frac{\partial^2 Q}{\partial K^2} < 0$
$MP_L < APL$	$MP_K < APK$

❐ Graphical representation of production functions: isoquants

In Section 7.1 a production function $Q = \mathrm{f}(L, K)$ was graphically represented by a two-dimensional graph known as an isoquant, $K = f(L)$. To plot the isoquant as a two-dimensional graph, quantity was fixed at a constant value, and then K was expressed in terms of L. The isoquant gives all the combinations of L and K for which Q (production) has the same fixed value. The graphs of four isoquants were plotted in Figures 7.2 and 7.3.

❐ The slope of an isoquant (*MRTS*)

The slope $\mathrm{d}K/\mathrm{d}L$ may be derived directly from the equation of the isoquant, as in Worked Example 7.1. This slope is called the **marginal rate of technical substitution** (*MRTS*).

At any point ($L = L_0$, $K = K_0$) on an isoquant, the value of the slope is a measure of the decrease in capital for each unit increase in labour, that is, the number of units of capital which would be replaced (substituted) when labour increases by one unit, while still maintaining the same output, Q.

$\left.\frac{\mathrm{d}K}{\mathrm{d}L}\right|_{L_0 K_0}$ is a short way of denoting the value of slope at (L_0, K_0).

The isoquant in Figure 7.5 exhibits a **diminishing marginal rate of technical substitution**, that is, $\frac{\Delta K}{\Delta L}$ or the rate at which the amount of capital decreases for each unit increase in labour gets smaller (diminishes) as L increases.

☺ **Remember:** Do not mix up the marginal products: MP_L, MP_K and the $MRTS = \mathrm{d}K/\mathrm{d}L$.

❐ Slope of an isoquant is the ratio of the marginal products

The slope of an isoquant may be expressed in terms of the marginal products of labour and capital by means of the total differential

$$\mathrm{d}Q = \left(\frac{\partial Q}{\partial L}\right)\mathrm{d}L + \left(\frac{\partial Q}{\partial K}\right)\mathrm{d}K \tag{7.8}$$

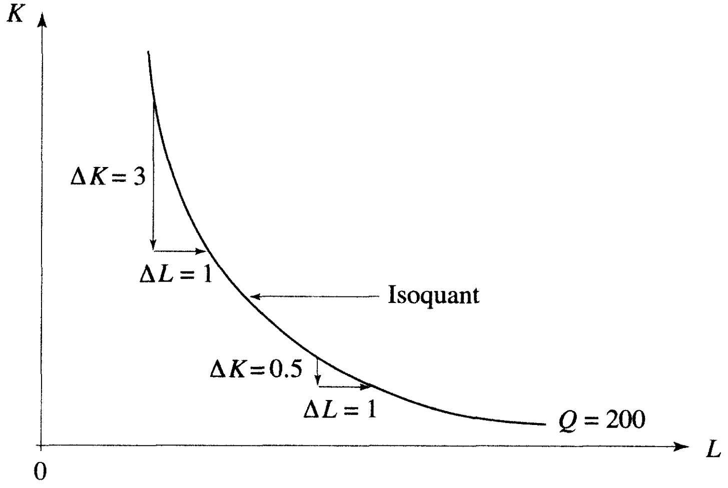


Figure 7.5 *MRTS* = slope of isoquant, slope is diminishing

Along any isoquant, $dQ = 0$, therefore substituting $dQ = 0$ into equation (7.8), an expression for dK/dL may be written as

$$0 = \left(\frac{\partial Q}{\partial L}\right)dL + \left(\frac{\partial Q}{\partial K}\right)dK$$

$$-\left(\frac{\partial Q}{\partial K}\right)dK = \left(\frac{\partial Q}{\partial L}\right)dL$$

$$-Q_K\, dK = Q_L\, dL$$

$$\frac{-Q_K\, dK}{dL} = \frac{Q_L}{1}$$

$$\frac{dK}{dL} = -\frac{Q_L}{Q_K} \tag{7.9}$$

As the partial derivatives Q_L and Q_K are the marginal product of labour and marginal product of capital respectively, usually denoted by MP_L and MP_K, dK/dL may also be written as

$$\frac{dK}{dL} = -\frac{MP_L}{MP_K} \tag{7.10}$$

WORKED EXAMPLE 7.8

SLOPE OF AN ISOQUANT IN TERMS OF MP_L, MP_K

A production function is given by the equation $Q = 10L^{0.5}K^{0.5}$.

(a) Graph the isoquants for $Q = 50$ and $Q = 70$.

(b) Derive expressions for the marginal product of labour and the marginal product of capital.

(c) Deduce an expression for the slope of the isoquants, dK/dL. For $Q = 50$, calculate the $MRTS$ at $L = 1$ and give a verbal interpretation of your answer.

Solution

(a) Substitute $Q = 50$ into the equation for the production function and solve for $K = f(L)$.

$$50 = 10L^{0.5}K^{0.5}$$

$$5 = L^{0.5}K^{0.5}$$

$$\frac{5}{L^{0.5}} = K^{0.5}$$

$$\frac{25}{L} = K \quad \text{square each side of the equation}$$

$$\text{or } K = \frac{25}{L}$$

Similarly, the equation of the isoquant for $Q = 70$ is $K = 49/L$.

A series of points are calculated for each isoquant, as shown in Table 7.6. The graphs are plotted in Figure 7.6.

(b) The marginal product of labour:

$$MP_L = \frac{\partial Q}{\partial L} = 10(0.5L^{0.5-1})K^{0.5} = 5L^{-0.5}K^{0.5}$$

The marginal product of capital:

$$MP_K = \frac{\partial Q}{\partial K} = 10L^{0.5}(0.5K^{0.5-1}) = 5L^{0.5}K^{-0.5}$$

(c) To derive an expression for $\mathrm{d}K/\mathrm{d}L$ or $MRTS$, start by writing out an expression for the total differential as given in equation (7.8) which is rewritten here:

$$\mathrm{d}Q = \left(\frac{\partial Q}{\partial L}\right)\mathrm{d}L + \left(\frac{\partial Q}{\partial K}\right)\mathrm{d}K$$

As Q is constant along a given isoquant, then $\mathrm{d}Q = 0$. Substitute $\mathrm{d}Q = 0$ and the partial derivatives MP_L and MP_K into equation (7.8), then solve for $\mathrm{d}K/\mathrm{d}L$ as follows:

$$0 = (5L^{-0.5}K^{0.5})\mathrm{d}L + (5L^{0.5}K^{-0.5})\mathrm{d}K$$

$$-(5L^{0.5}K^{-0.5})\mathrm{d}K = (5L^{-0.5}K^{0.5})\mathrm{d}L$$

$$\frac{-(5L^{0.5}K^{-0.5})\mathrm{d}K}{\mathrm{d}L} = \frac{(5L^{-0.5}K^{0.5})}{1} \quad \text{divide both sides by } \mathrm{d}L$$

Table 7.6 Points for plotting the isoquants $K = 25/L$ and $K = 49/L$

L	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.00**	**1.10**	**1.20**	**1.30**	**1.40**
$K = 25/L$	250	125	83	63	50	42	36	31	28	25	23	21	19	18
$K = 49/L$	490	245	163	123	98	82	70	61	54	49	45	41	38	35

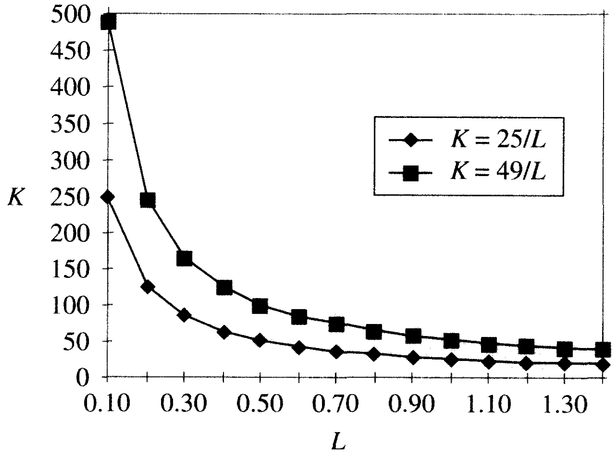


Figure 7.6 Isoquants

$$\frac{\mathrm{d}K}{\mathrm{d}L} = -\frac{5L^{-0.5}K^{0.5}}{5L^{0.5}K^{-0.5}} \qquad \text{divide both sides by } -(5L^{0.5}K^{-0.5})$$

$$\frac{\mathrm{d}K}{\mathrm{d}L} = -\frac{\cancel{5}L^{-0.5-0.5}K^{0.5-(-0.5)}}{\cancel{5}}$$

simplify using the rules for indices

$$\frac{\mathrm{d}K}{\mathrm{d}L} = -\frac{L^{-1}K^{1}}{1} = -\frac{K}{L} \qquad \text{simplify using the rules for indices}$$

Note: Exactly the same answer is obtained by using equation (7.10) directly. (However, formulae are easily forgotten, so you should always be able to work from basics.)

$$\frac{\mathrm{d}K}{\mathrm{d}L} = -\frac{MP_L}{MP_K} = -\frac{5L^{-0.5}K^{0.5}}{5L^{0.5}K^{-0.5}} = -\frac{K}{L}$$

At $L = 1$, the *MRTS* or

$$\frac{\mathrm{d}K}{\mathrm{d}L} = -\frac{K}{L} = -\frac{25}{1} = -25$$

This means that the producer is willing to give up 25 units of capital to receive one more unit of labour and still maintain production, $Q = 50$.

☐ The *MRTS* in reduced form for a Cobb–Douglas production function

Given the general Cobb–Douglas production function, $Q = AL^{\alpha}K^{\beta}$, then

$$MRTS\colon \frac{\mathrm{d}K}{\mathrm{d}L} = -\frac{\partial Q/\partial L}{\partial Q/\partial K} = -\frac{MP_L}{MP_K} = -\frac{A\alpha L^{\alpha-1}K^{\beta}}{A\beta L^{\alpha}K^{\beta-1}}$$

$$= -\frac{\alpha K^{\beta}K^{-\beta+1}}{\beta L^{\alpha}L^{-\alpha+1}} \qquad \text{using rules for indices}$$

$$= -\frac{\alpha K}{\beta L}$$

The *MRTS* in reduced form is expressed as:

$$MRTS = -\frac{\alpha}{\beta}\frac{K}{L} \tag{7.11}$$

Progress Exercises 7.4 Partial Differentiation and Production Functions

1. A firm has a production function $Q = 10L^{0.5}K^{0.5}$.
   (a) Derive expressions for MP_L and MP_K.
   (b) Prove that

$$Q = L\frac{\partial Q}{\partial L} + K\frac{\partial Q}{\partial K}$$

2. Find all the first and second partial derivatives for the production function

$$Q = 80L^{0.5}K^{0.2}.$$

   (a) Give a verbal description of each derivative.
   (b) Are the marginal functions increasing or decreasing? Give reasons for your answer.
   (c) Evaluate MP_L and MP_K when $Q = 80$ and $L = 2, 5, 10$.
   Do these calculations confirm your answer in part (b)?
3. A firm's production function is given by the equation $Q = 80L^{0.5}K^{0.2}$.
   (a) Derive expressions for MP_L and APL. Show that $MP_L < APL$.
   (b) Derive expressions for MP_K and APK. Show that $MP_K < APK$.
4. A firm producing keyboards has a production function $Q = 10LK^{0.5}$ keyboards per hour.
   (a) If production is set at 80 keyboards ($Q = 80$), write down the equation of the corresponding isoquant in the form $K = f(L)$. Calculate K when $L = 2, 4, 6$. Graph the isoquant.
   (b) Derive expressions for MP_L and MP_K. Evaluate MP_L and MP_K when $L = 2, 4, 6$.
   (c) Use the results in (b) to evaluate $MRTS = \mathrm{d}K/\mathrm{d}L$ when $L = 2, 4, 6$. Comment on the results.
5. (See question 2. $Q = 80L^{0.5}K^{0.2}$.)
   (a) Derive an expression for $MRTS = \mathrm{d}K/\mathrm{d}L$ in terms of MP_L and MP_K.
   (b) Write down the equation of the isoquant $Q = 320$.
   Evaluate the $MRTS$ on the isoquant $Q = 320$ at $L = 4, 9, 16$. Do your calculations confirm the law of diminishing marginal rate of technical substitution?
6. Confirm that

$$MRTS = -\frac{\alpha K}{\beta L}$$

   for the production function $Q = 25L^{0.6}K^{0.5}$.
7. A manager proposes that a firm increases capital by 5% and decreases labour by 3%. If the firm's production function is $Q = 80L^{0.5}K^{0.2}$, use partial differentiation to find the approximate change in output, Q.
8. Given the Cobb–Douglas production function $Q = 50L^{0.3}K^{0.5}$, calculate and comment on the values of the APL and the MP_L at $L = 2, 4, 6$ when K remains constant at $K = 10$.

7.2.2 Returns to scale

If both inputs (L, K) in the Cobb–Douglas production function are changed by the same proportion (for example, each input is doubled or each input is reduced by one-fifth etc.) we can easily determine the proportionate change in output, Q. For any constant, λ, if L is replaced by λL and K is replaced by λK, then the proportionate change in Q may be determined from the equation of the production function.

Given the Cobb–Douglas production function

$$Q_1 = AL^{\alpha}K^{\beta} \tag{7.12}$$

then, replacing L by λL and K by λK in equation (7.12), the new output, Q_2 is given by

$$\begin{aligned} Q_2 &= A(\lambda L)^{\alpha}(\lambda K)^{\beta} \\ &= A\lambda^{\alpha}L^{\alpha}\lambda^{\beta}K^{\beta} \\ &= \lambda^{\alpha+\beta}AL^{\alpha}K^{\beta} \\ &= \lambda^{\alpha+\beta}Q_1 \end{aligned} \tag{7.13}$$

- If $\alpha + \beta = 1$, then the proportionate change in output is the same as the proportionate change in each input. This is described as **constant returns to scale**. For example, if $\lambda = 2$. then $Q_2 = 2^1 Q_1 = 2Q_1$.
- If $\alpha + \beta < 1$, then the proportionate change in output is less than the proportionate change in each input. This is described as **decreasing returns to scale**. For example, if $\lambda = 2$, $\alpha + \beta = 0.8$, then $Q_2 = 2^{0.8}Q_1 = 1.7Q_1$.
- If $\alpha + \beta > 1$, then the proportionate change in output is greater than the proportionate change in each input. This is described as **increasing returns to scale**. For example, if $\lambda = 2$. $\alpha + \beta = 1.3$, then $Q_2 = 2^{1.3}Q_1 = 2.5Q_1$.

Note: Returns to scale are known as the long-run laws of production whereas the law of diminishing returns to labour or capital is known as the short-run law of production.

☐ Homogeneous functions of degree *r*

In general, a Cobb–Douglas production function is described as homogeneous, order r, if

$$f(\lambda L, \lambda K) = \lambda^r f(L, K)$$

where $r = (\alpha + \beta)$.

WORKED EXAMPLE 7.9

CONSTANT, INCREASING AND DECREASING RETURNS TO SCALE

Given the Cobb–Douglas production functions

$$Q = 50L^{0.3}K^{0.5} \tag{7.14}$$

$$Q = 50L^{0.4}K^{0.6} \tag{7.15}$$

$$Q = 50L^{0.5}K^{0.6} \tag{7.16}$$

(i) Calculate the level of output when $L = 10$, $K = 15$.

(ii) Calculate the level of output when both inputs double, $L = 20$, $K = 30$. Comment on the returns to scale.

Solution

The level of output is calculated by substituting the appropriate values of L and K into each production function. The calculations are

Production function	(i) $L = 10, K = 15$	(ii) $L = 20, K = 30$ (inputs doubled)	**Comment**
$Q = 50L^{0.3}K^{0.5}$	$Q_1 = 50L^{0.3}K^{0.5}$ $= 50(10)^{0.3}(15)^{0.5}$ $= 50(2.0)(3.9)$ $= 390$	$Q_2 = 50(L)^{0.3}(K)^{0.5}$ $= 50(20)^{0.3}(30)^{0.5}$ $= 50(2.46)(5.48)$ $= 674$ $< 390 \times 2$, i.e. $< Q_1 \times 2$	Production function exhibits **decreasing returns to scale.** $(\alpha + \beta) < 1$
$Q = 50L^{0.4}K^{0.6}$	$Q_1 = 50L^{0.4}K^{0.6}$ $= 50(10)^{0.4}(15)^{0.6}$ $= 50(2.51)(5.08)$ $= 637$	$Q_2 = 50L^{0.4}K^{0.6}$ $= 50(20)^{0.4}(30)^{0.6}$ $= 50(3.31)(7.70)$ $= 1274$ $= 637 \times 2$, i.e. $= Q_1 \times 2$	Production function exhibits **constant returns to scale.** $(\alpha + \beta) = 1$
$Q = 50L^{0.5}K^{0.6}$	$Q_1 = 50L^{0.5}K^{0.6}$ $= 50(10)^{0.5}(15)^{0.6}$ $= 50(3.2)(5.1)$ $= 816$	$Q_2 = 50L^{0.5}K^{0.6}$ $= 50(20)^{0.5}(30)^{0.6}$ $= 50(4.47)(7.69)$ $= 1719$ $> 816 \times 2$, i.e. $> Q_1 \times 2$	Production function exhibits **increasing returns to scale.** $(\alpha + \beta) > 1$

❐ Incremental changes

If both labour and capital change simultaneously by small but different proportions or amounts, then the total change in output can be determined by the 'small changes' formula which was expressed as equation (7.6). In this case, with $Q = \mathrm{f}(L, K)$, equation (7.6) is modified to become

$$\Delta Q \cong \left(\frac{\partial Q}{\partial L}\right)\Delta L + \left(\frac{\partial Q}{\partial K}\right)\Delta K \qquad \textbf{(7.17)}$$

7.2.3 Utility functions

At the end of this section you should be able to:

- Define marginal utilities
- Derive the equation for an indifference curve and sketch its graph
- Derive the slope of an indifference curve in terms of marginal utilities

The analysis of utility functions parallels that of production functions in the previous section. A utility function describes utility as a function of the goods consumed and may be written in general form as

$$U = f(x, y)$$

A utility function that is widely used in economic analysis is the Cobb–Douglas utility function which is expressed in general form as

$$U = Ax^{\alpha}y^{\beta} \qquad \textbf{(7.18)}$$

where A is a constant and $0 < \alpha < 1$, $0 < \beta < 1$, $x > 0$, $y > 0$. x and y are the quantities of goods X and Y consumed. For example, a Cobb–Douglas utility function in specific form could be written as

$$U = 10x^{0.3}y^{0.5}$$

❐ Marginal utility

In Chapter 6, utility was expressed as a function of one variable, x. Hence, marginal utility was defined as the derivative of total utility w.r.t. x,

$$U = f(x), \text{ then } \frac{dU}{dx} = U_x = MU_x$$

In this section, where $U = f(x, y)$, the marginal utilities are defined as the partial derivatives:

$$\frac{\partial U}{\partial x} = U_x = MU_x \text{ is the marginal utility w.r.t. good } X \text{, and}$$

$$\frac{\partial U}{\partial y} = U_y = MU_y \text{ is the marginal utility w.r.t. good } Y$$

❐ Graphical representation of utility functions

The graphical representation of utility functions is similar to that of production functions. A utility function $U = f(x, y)$ can be represented by a series of two-dimensional graphs known as **indifference curves** (equal utility), $y = f(x)$. To plot an indifference curve, utility is fixed at a constant value, then y is expressed in terms of x. An indifference curve gives all combinations of x and y for which utility, U, has the same value.

An isoquant is a combination of inputs L and K which when used give a firm the same level of output.
An indifference curve is a combination of goods X and Y which when consumed give the consumer the same level of utility

❐ Slope of an indifference curve

The slope of an indifference curve dy/dx is called the **marginal rate of substitution** (MRS), and may be expressed in terms of the marginal utilities of good X and good Y. The expression is derived from the total differential,

$$dU = \left(\frac{\partial U}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y}\right)dy \tag{7.19}$$

Since U is constant along a given indifference curve, then $dU = 0$. Substituting $dU = 0$ into equation (7.19), an expression for dy/dx may be written as

$$0 = \left(\frac{\partial U}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y}\right)dy$$

$$-\left(\frac{\partial U}{\partial y}\right)dy = \left(\frac{\partial U}{\partial x}\right)dx$$

$$-U_y dy = U_x dx$$

$$\frac{-U_y dy}{dx} = \frac{U_x}{1}$$

$$\frac{dy}{dx} = -\frac{U_x}{U_y} \tag{7.20}$$

Conventionally, $U_x \equiv MU_x$ and $U_y \equiv MU_y$; hence, equation (7.20) can be rewritten as

$$\frac{dy}{dx} = -\frac{MU_x}{MU_y} \tag{7.21}$$

Thus, the slope of an indifference curve, *MRS*, is equal to the ratio of the marginal utility of good X to the marginal utility of good Y.

The value of the slope of an indifference curve at any given point $(x = x_0,\ y = y_0)$, $dy/dx|_{\text{at } x_0,\, y_0}$, is the number of units by which good Y decreases when good X increases by one unit, while maintaining the same level of utility. The *MRS* is similar to the *MRTS* for isoquants. The *MRS* is illustrated in Figure 7.7. The indifference curve in Figure 7.7

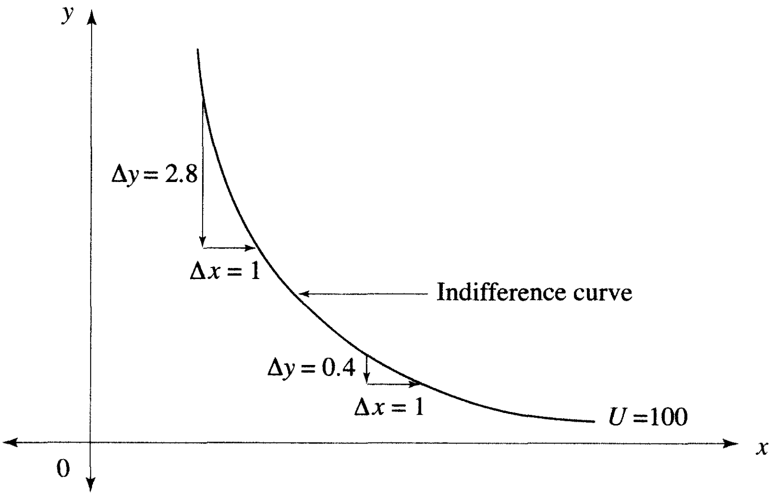


Figure 7.7 *MRS* = slope of an indifference curve, slope is diminishing

exhibits a **diminishing marginal rate of substitution**, that is, $\left(\frac{\Delta y}{\Delta x}\right)$ or the rate at which the amount of good Y decreases for each unit increase in good X becomes smaller (diminishes) as x increases.

☺ **Remember:** Do not mix up marginal utilities, $\mathrm{d}U/\mathrm{d}x = MU_x$, $\mathrm{d}U/\mathrm{d}y = MU_y$ and the MRS, $\mathrm{d}y/\mathrm{d}x$.

WORKED EXAMPLE 7.10

INDIFFERENCE CURVES AND SLOPE

A utility function is given by the equation $U = 8x^{0.5}y^{0.5}$.

(a) Graph the indifference curves for $U = 40$, $U = 48$ and $U = 56$.
(b) Derive expressions for the marginal utility of good X and good Y.
(c) Deduce an expression for the slope of the indifference curves, $\mathrm{d}y/\mathrm{d}x$.

Solution

(a) Substitute $U = 40$ into the equation for the utility function and solve for $y = \mathrm{f}(x)$:

$$40 = 8x^{0.5}y^{0.5}$$

$$5 = x^{0.5}y^{0.5}$$

$$\frac{5}{x^{0.5}} = y^{0.5}$$

$$\frac{25}{x} = y \quad \text{square each side of the equation}$$

$$\text{or } y = \frac{25}{x}$$

Similarly, the equations of the indifference curves for $U = 48$ is $y = 36/x$ and $U = 56$ is $y = 49/x$. A series of points are calculated for each indifference curve, as shown in Table 7.7. The graphs are plotted in Figure 7.8.

(b) The marginal utility of good X is: $MU_x = \partial U/\partial x = 8(0.5x^{0.5-1})y^{0.5} = 4x^{-0.5}y^{0.5}$. The marginal utility of good Y is: $MU_y = \partial U/\partial y = 8x^{0.5}(0.5y^{0.5-1}) = 4x^{0.5}y^{-0.5}$.

Table 7.7 Points for plotting indifference curves

x	1	2	3	4	5	6
$y = 25/x$	25	12.5	8.3	6.3	5	4.2
$y = 36/x$	36	18	12	9	7.2	6
$y = 49/x$	49	24.5	16.3	12.3	9.8	8.2

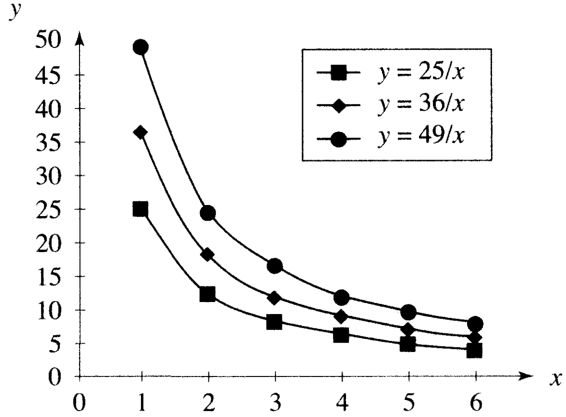


Figure 7.8 Indifference curves, $U = f(x, y)$

(c) To find an expression for *MRS*, substitute the expressions for the marginal utility of each good into equation (7.21):

$$\frac{dy}{dx} = -\frac{MU_x}{MU_y} = -\frac{4x^{-0.5}y^{0.5}}{4x^{0.5}y^{-0.5}} = -x^{-1}y = -\frac{y}{x}$$

❒ Incremental changes

If the quantities of both products X and Y, are changed by small but different amounts or proportions, then the resulting overall change in utility may be calculated by means of the incremental changes formula (7.6). Economic theory and the analysis of consumer behaviour argues that consumers will always prefer more to less, thereby increasing their overall utility.

Progress Exercises 7.5 Returns to Scale: Utility and Production Functions

1. A consumer has a utility function $U = 10x^{0.2}y^{0.8}$, where x is the number of cups of tea consumed, y is the number of glasses of brandy.
   (a) Derive expressions for U_x and U_y
   (b) Prove that
$$U = x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}$$
2. A utility function is given as
$$U = \frac{500}{y} - 10 \ln x$$
   (a) Write down the equation of the indifference curve $U = 20$ in the form $y = f(x)$.
   (b) Evaluate y when $x = 2, 10, 25$. Plot the indifference curve.
   (c) Derive an expression for the slope of the indifference curve $U = 20$ by differentiating the equation of the indifference curve directly.
   Calculate the slope of the indifference curve. Hence evaluate the slope at $x = 2, 10, 25$. Do your calculations confirm the law of diminishing marginal rate of substitution?

3. (See question 2)

$$U = \frac{500}{y} - 10\ln x$$

(a) Derive expressions for U_x and U_y.

(b) Derive an expression for the *MRS*, dy/dx, the slope of an indifference curve, in terms of the marginal utilities.

Your answers should agree with those in question 2.

4. Given the Cobb–Douglas utility function $U = 25x^{0.5}y^{0.2}$,

(a) Derive expression for the marginal utilities.

(b) Derive an expression for the slope of the indifference curve in terms of the marginal utilities.

(c) Confirm that the slope of the indifference curve satisfies the equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\alpha y}{\beta x}$$

5. The consumer in question 1 plans to decrease her consumption of cups of tea by 5% and decrease consumption of glasses of brandy by 8%. Use partial differentiation to find the approximate change in utility, U.

6. Determine whether each of the following production functions have increasing, decreasing or constant returns to scale.

(a) $Q = 10K^{0.5}L^{0.5}$ (b) $Q = 10K^{0.7}L^{0.6}$ (c) $Q = 10K^{0.3}L^{0.5}$

7. Given the Cobb–Douglas production function, $Q = L^{0.7}K^{0.6}$,

(a) Calculate Q when $L = 50$ and $K = 200$.

(b) Calculate Q when both inputs are increased by 10%. Compare your answer when Q in part (a) is increased by 10%. What do these calculations imply about the level of returns to scale?

(c) If L increases by 5% and K increases by 10%, use the incremental changes formula to find the approximate increase in Q.

7.2.4 Partial elasticities

☐ Partial elasticities of demand

In Section 6.7 price elasticity of demand was defined as the responsiveness of quantity demanded to changes in the price of the good itself. For the demand function $Q = \mathrm{f}(P)$, price elasticity of demand was expressed as

$$\varepsilon_{\mathrm{d}} = \frac{\mathrm{d}Q}{\mathrm{d}P}\cdot\frac{P}{Q}$$

We now, however, want to examine price elasticity of demand and other demand elasticities for a more general demand function of the form

$$Q_{\mathrm{A}} = \mathrm{f}(P_{\mathrm{A}}, Y, P_{\mathrm{B}}\ldots)$$

where Q_{A} = quantity demanded of good A
P_{A} = price of good A
Y = consumer income
P_{B} = price of another good B

The responsiveness of the quantity demanded of good A to a change in any of these three variables is now examined.

- **Price elasticity of demand**

This measures the percentage change in Q_A w.r.t. P_A keeping Y and P_B constant. It is given by

$$\varepsilon_d = \frac{\partial Q_A}{\partial P_A}\frac{P_A}{Q_A} \tag{7.22}$$

$\varepsilon_d = (\partial Q_A/\partial P_A)(P_A/Q_A)$ and $\varepsilon_d = dQ/dP \cdot P/Q$ are similar except that 'd' is replaced by '∂' since demand is a function of several variables.

Income elasticity of demand: This measures the percentage change in Q_A w.r.t. Y keeping P_A and P_B constant. It is given by

$$\varepsilon_Y = \frac{\partial Q_A}{\partial Y}\frac{Y}{Q_A} \tag{7.23}$$

Cross-price elasticity of demand: This measures the percentage change in Q_A w.r.t. P_B keeping P_A and Y constant. It is given by

$$\varepsilon_c = \frac{\partial Q_A}{\partial P_B}\frac{P_B}{Q_A} \tag{7.24}$$

WORKED EXAMPLE 7.11

PARTIAL ELASTICITIES OF DEMAND

The demand function for good A is given by

$$Q_A = 100 - 2P_A + 0.2Y + 0.3P_B$$

Find the price, income and cross-price elasticities of demand at $P_A = 6$, $Y = 500$, $P_B = 10$.

Solution

The respective elasticity values are:

Price elasticity of demand	**Income elasticity of demand**	**Cross-price elasticity of demand**
Differentiate the demand function partially w.r.t. P_A. $\frac{\partial Q_A}{\partial P_A} = -2$ then use equation (7.22) $\varepsilon_d = \frac{\partial Q_A}{\partial P_A}\frac{P_A}{Q_A^*}$ $= -2\frac{6}{191} = -0.06$	Differentiate the demand function partially w.r.t. Y. $\frac{\partial Q_A}{\partial Y} = 0.2$ then use equation (7.23) $\varepsilon_Y = \frac{\partial Q_A}{\partial Y}\frac{Y}{Q_A}$ $= 0.2\frac{500}{191} = 0.52$	Differentiate the demand function partially w.r.t. P_B. $\frac{\partial Q_A}{\partial P_B} = 0.3$ then use equation (7.24) $\varepsilon_c = \frac{\partial Q_A}{\partial P_B}\frac{P_B}{Q_A}$ $= 0.3\frac{10}{191} = 0.02$

* Since at $P_A = 6$, $Y = 500$, $P_B = 10$, $Q_A = 100 - 2(6) + 0.2(500) + 0.3(10) = 191$

☐ Partial elasticity of labour and capital

Partial elasticity w.r.t. labour is defined as the proportionate change in output (Q) resulting from a proportionate change in labour input (L) when capital is held constant:

$$\varepsilon_{QL} = \frac{\partial Q}{\partial L}\frac{L}{Q} \tag{7.25}$$

Note: The partial elasticity w.r.t. labour may also be expressed as the ratio of the marginal product of labour to the average product of labour:

$$\varepsilon_{QL} = \frac{\partial Q}{\partial L}\frac{L}{Q} = \frac{(\partial Q/\partial L)}{(Q/L)} = \frac{MP_L}{APL} \tag{7.26}$$

For the Cobb–Douglas production function, $Q = AL^{\alpha}K^{\beta}$,

$$\varepsilon_{QL} = \frac{\alpha AL^{\alpha-1}K^{\beta}}{1}\frac{L}{Q} = \frac{\alpha AL^{\alpha}K^{\beta}}{Q} = \frac{\alpha Q}{Q} = \alpha \tag{7.27}$$

Similarly, **the partial elasticity w.r.t. capital** is defined as the proportionate change in output (Q) resulting from a proportionate change in capital input (K) when labour is held constant:

$$\varepsilon_{QK} = \frac{\partial Q}{\partial K}\frac{K}{Q} \tag{7.28}$$

Note: The partial elasticity w.r.t. capital may also be expressed as the ratio of the marginal product of capital to the average product of capital:

$$\varepsilon_{QK} = \frac{\partial Q}{\partial K}\frac{K}{Q} = \frac{(\partial Q/\partial K)}{(Q/K)} = \frac{MP_K}{APK} \tag{7.29}$$

For the Cobb–Douglas production function, $Q = AL^{\alpha}K^{\beta}$,

$$\varepsilon_{QK} = \frac{\beta AL^{\alpha}K^{\beta-1}}{1}\frac{K}{Q} = \frac{\beta AL^{\alpha}K^{\beta}}{Q} = \frac{\beta Q}{Q} = \beta \tag{7.30}$$

WORKED EXAMPLE 7.12

PARTIAL ELASTICITIES OF LABOUR AND CAPITAL

Derive the partial elasticities w.r.t. labour and capital for the production function $Q = 10L^{0.5}K^{0.5}$.

Solution

The partial elasticities are derived as follows:

Partial elasticity w.r.t. labour	**Partial elasticity w.r.t. capital**
$\varepsilon_{QL} = \frac{\partial Q}{\partial L}\frac{L}{Q}$	$\varepsilon_{QK} = \frac{\partial Q}{\partial K}\frac{K}{Q}$

Substitute the partial derivative,

$$\varepsilon_{QL} = \frac{10(0.5L^{-0.5})K^{0.5}}{1}\frac{L}{Q}$$
$$= \frac{0.5(10L^{0.5}K^{0.5})}{Q} = \frac{0.5(Q)}{Q}$$
$$= 0.5$$
$$= \alpha$$

The answer agrees with equation (7.27)

Substitute the partial derivative,

$$\varepsilon_{QK} = \frac{10L^{0.5}(0.5K^{-0.5})}{1}\frac{K}{Q}$$
$$= \frac{0.5(10L^{0.5}K^{0.5})}{Q} = \frac{0.5(Q)}{Q}$$
$$= 0.5$$
$$= \beta$$

The answer agrees with equation (7.30)

7.2.5 The multipliers for the linear national income model

In Chapter 3 the linear national income model for a three-sector economy was given as

$$Y = C + I + G$$
$$C = C_0 + bY_d$$
$$T = tY$$

The equilibrium level of income in its reduced form was derived as

$$Y_e = \frac{C_0 + I_0 + G_0}{1 - b(1 - t)}$$

Partial differentiation may be used to determine the effect on the equilibrium level of income of very small changes in any one of the variables: C, I, G or T.

WORKED EXAMPLE 7.13

USE PARTIAL DERIVATIVES TO DERIVE EXPRESSIONS FOR VARIOUS MULTIPLIERS

The national income model for a three-sector economy is given as

$$Y = C + I + G, \qquad C = C_0 + bY_d, \qquad T = tY$$

and equilibrium national income is given by the equation

$$Y_e = \frac{C_0 + I_0 + G_0}{1 - b(1 - t)}$$

Use partial derivatives to derive expressions for the change in Y_e which result from very small changes in (a) investment, I, only, (b) government expenditure, G, only, and (c) taxation rate, t, only.

Solution

(a) The rate of change of Y_e w.r.t. I is the first partial derivative of Y_e w.r.t. I.

Note: Since the level of private investment is changing, write I instead of I_0.

$$Y_e = \frac{C_0 + I + G_0}{1 - b(1-t)}$$

$$\frac{\partial Y_e}{\partial I} = \frac{1}{1 - b(1-t)}$$

This partial derivative is known as the investment multiplier.

(b) The rate of change of Y_e w.r.t. G is the first partial derivative of Y_e w.r.t. G.

Note: Since the level of government expenditure is changing, write G instead of G_0.

$$Y_e = \frac{C_0 + I_0 + G}{1 - b(1-t)}$$

$$\frac{\partial Y_e}{\partial G} = \frac{1}{1 - b(1-t)}$$

This partial derivative is known as the government expenditure multiplier.

(c) The rate of change of Y_e w.r.t. the tax rate, t, is the first partial derivative of Y_e w.r.t. t.

$$Y_e = \frac{C_0 + I_0 + G_0}{1 - b(1-t)} = (C_0 + I_0 + G_0)(1 - b + bt)^{-1}$$

$$\frac{\partial Y_e}{\partial t} = (C_0 + I_0 + G_0)[(-1)(1 - b + bt)^{-2}(b)]$$

differentiating by the chain rule

$$= \frac{-b(C_0 + I_0 + G_0)}{(1 - b(1-t))^2}$$

$$= -\frac{bY_e}{1 - b(1-t)}$$

This partial derivative is known as the income tax rate multiplier.

The multipliers are summarised:

$$\frac{\partial Y_e}{\partial I} = \frac{1}{1 - b(1-t)} \quad \text{the investment multiplier}$$

$$\frac{\partial Y_e}{\partial G} = \frac{1}{1 - b(1-t)} \quad \text{the government expenditure multiplier}$$

$$\frac{\partial Y_e}{\partial t} = -\frac{bY_e}{1 - b(1-t)} \quad \text{the income tax rate multiplier}$$

Progress Exercises 7.6 Partial Elasticities, National Income Model Multipliers

1. The demand function for good X is given by

$$Q_X = 520 - 20P_X + 0.6Y + 2.9P_Y$$

Find the price, income and cross-price elasticities of demand at $P_X = 10$, $Y = 700$, $P_Y = 21$.

2. Given a Cobb–Douglas production function $Q = 250L^{0.6}K^{0.2}$,
   (a) Derive expressions for the average and marginal products of labour.
   (b) Derive the partial elasticity w.r.t. labour.
   (c) Confirm that the partial elasticity is equal to the ratio of the marginal to average product.
3. The national income model for a three-sector economy is given by $Y = C + I + G$, where $C = 120 + 0.6Y_d$, $T = 0.25Y_d$, $G = 200$ and $I = 120$.
   (a) Calculate the equilibrium level of national income.
   (b) Use partial derivatives to evaluate the following multipliers:
      (i) investment, (ii) government expenditure, (iii) income-tax rate.
4. Derive the partial elasticities w.r.t. labour and capital for the production functions I: $Q = 10L^{0.6}K^{0.4}$, II: $Q = 180L^{0.4}K^{0.3}$.
   (a) Use the definition of elasticity to find the approximate change in output Q in
      (i) I, when labour increases by 3%, (ii) II, when capital increases by 5%.
   (b) Use partial derivatives to calculate the approximate percentage change in Q in
      (i) I, when L increases by 3%, (ii) II, when K increases by 5%.
   Compare the answers in (a) and (b).
5. (See question 2) For the Cobb–Douglas production function $Q = 250L^{0.6}K^{0.2}$,
   (a) Derive expressions for the average and marginal product of capital.
   (b) Derive the partial elasticity w.r.t. capital.
   (c) Confirm that the partial elasticity is equal to the ratio of the marginal to average product.

7.3 Unconstrained Optimisation

At the end of this section you should be able to:

- Locate the optimum points for functions of two variables
- Calculate maximum profit for a firm producing two goods
- Show how firms use price discrimination in order to maximise profits

7.3.1 Find the optimum points for functions of two variables

❐ Optimisation of functions of one variable revisited

The methods for finding optimum points (maxima, minima), or points of inflection, for functions of two variables are simply extensions of those used for finding the maxima, minima and points of inflection for functions of one variable which are covered in Chapter 6. If you recall, we used a three-step method; that is, for the function, $y = f(x)$, the steps are:

Step 1: Find the first and second derivatives: dy/dx, d^2y/dx^2.

Step 2: Equate the first derivative to zero, $dy/dx = 0$, and solve this equation to find the x-coordinate of the potential turning point(s).

Step 2a: If required, substitute the x-coordinate of the turning point(s) into the equation of the curve to find the corresponding y-coordinate.

Step 3: Use the second derivatives to determine the nature of the turning point(s). Evaluate d^2y/dx^2 at each turning point.

If $\left.\dfrac{d^2y}{dx^2}\right|_{\text{at turning point}} > 0$ the point is a minimum

If $\left.\dfrac{d^2y}{dx^2}\right|_{\text{at turning point}} < 0$ the point is a maximum

If $\left.\dfrac{d^2y}{dx^2}\right|_{\text{at turning point}} = 0$ the point may be a point of inflection

❑ Optimisation of functions of two variables: method

In order to optimise functions of two variables, such as $z = f(x, y)$, each of the above three steps are extended. Given the function, $z = f(x, y)$, the method is as follows:

Step 1: Find the first and second derivatives:

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x\, \partial y}$$

Step 2: Equate the first derivatives to zero, then solve these equations to find the x-coordinate and the y-coordinate of the turning point(s):

$$\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial z}{\partial y} = 0$$

Step 2a: If required, find z for the x- and y-coordinates of the turning point(s).

Step 3: Use the second derivatives to determine the nature of the turning points. Evaluate all second derivatives at the x- and y-coordinates of the turning points.

Extension: As the number of dimensions increases (and consequently the number of derivatives), so also do the number of conditions which must be satisfied to confirm the nature of the optimum point. We define a new term, Δ, where

$$\Delta = \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 \qquad \textbf{(7.31)}$$

The point is a minimum if $\partial^2 z/\partial x^2 > 0$ and $\partial^2 z/\partial y^2 > 0$ and provided $\Delta > 0$.

The point is a maximum if $\partial^2 z/\partial x^2 < 0$ and $\partial^2 z/\partial y^2 < 0$ and provided $\Delta > 0$.

The point is a point of inflection if both second derivatives have the same sign but $\Delta < 0$.

The point is a saddle point if the second derivatives have different signs and $\Delta < 0$.

If $\Delta = 0$ then there is no conclusion.

A saddle point is a point which is a maximum when viewed along one axis but a minimum when viewed along the other. In fact, it looks like a saddle; hence its name. Figure 7.9 shows (a) a maximum, (b) a minimum and (c) a saddle point in three dimensions.

At this stage it might seem that there is a confusing amount of conditions to remember; however, if you view these conditions as extensions of the methods of Chapter 6 they will soon fall into place, especially with practice at working through the examples.

Note: In this text it is possible to classify all given functions according to the second-order conditions outlined above. However, for those functions for which these conditions fail, then higher-order derivatives are required, but this is beyond the scope of this text.

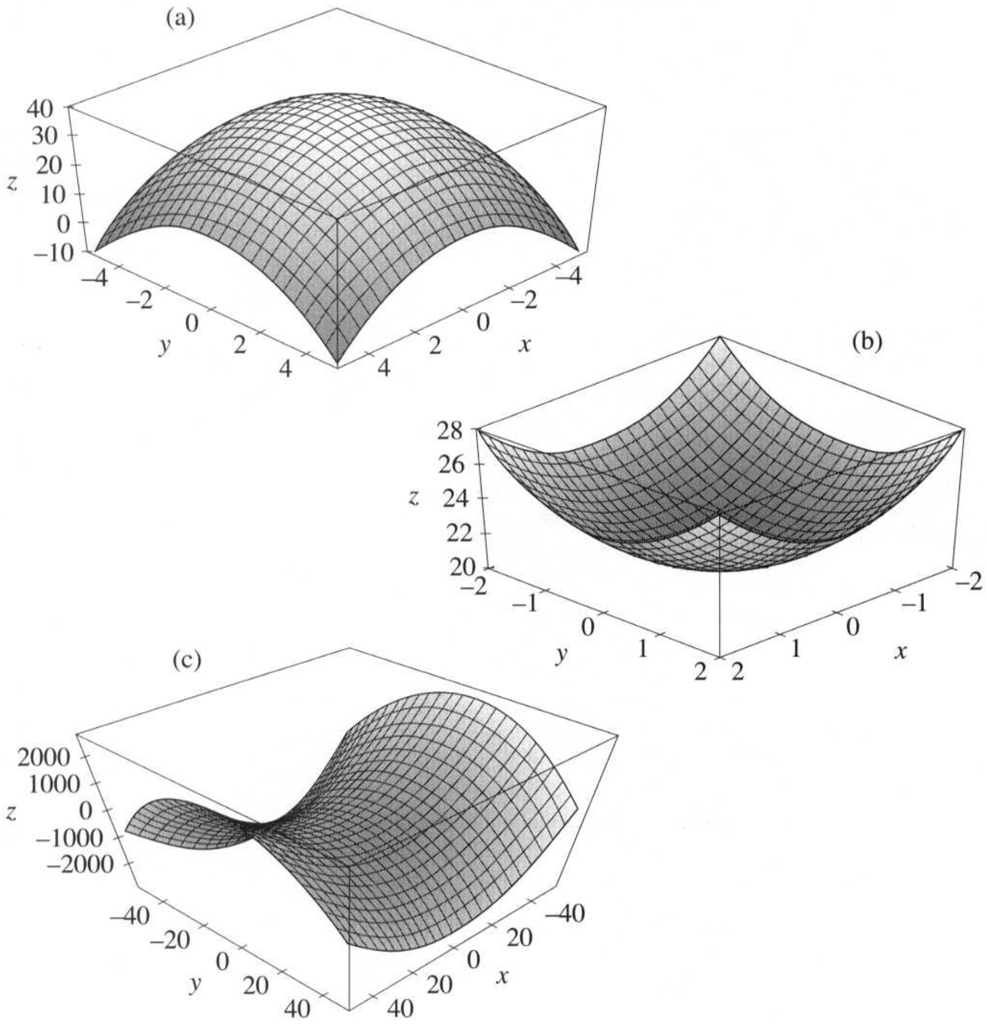


Figure 7.9 (a) $z = -x^2 - y^2 + 40$ has a maximum point at $x = 0$, $y = 0$, $z = 40$. (b) $z = x^2 + y^2 + 20$ has a minimum point at $x = 0$, $y = 0$, $z = 20$. (c) $z = x^2 - y^2 - 4x + 4y$: a saddle point at $x = 2$, $y = 2$, $z = 0$.

WORKED EXAMPLE 7.14

OPTIMUM POINTS FOR FUNCTIONS OF TWO VARIABLES

Given that $z = x^2 + y^2 + 20$,

(a) Find the optimum point.
(b) Determine the nature of the optimum point.

Solution

Work through the method for finding the optimum point for functions of two variables.

Step 1: Find the derivatives. The first and second derivatives are $z_x = 2x$, $z_y = 2y$, $z_{xx} = 2$, $z_{yy} = 2$, $z_{xy} = 0$.
Step 2: Equate the first derivatives to zero, and solve for the x- and y-coordinates of the optimum point:

$$\begin{aligned} z_x &= 2x \qquad & z_y &= 2y \\ 0 &= 2x \qquad & 0 &= 2y \end{aligned}$$

The only solution to each of these equations is $x = 0$ and $y = 0$.
Step 2a: Find the z-coordinate of the optimum point (we need three coordinates for three dimensions). Substituting $x = 0$ and $y = 0$ into the equation for z gives $z = 0^2 + 0^2 + 20 = 20$; therefore, there is an optimum point at $x = 0$, $y = 0$, $z = 20$.
Step 3: Use the second derivatives to determine the nature of the turning point. The two straight second derivatives are constants and positive:

$$z_{xx} = 2 > 0 \quad \text{and} \quad z_{yy} = 2 > 0$$

Therefore this point is a minimum provided $\Delta > 0$, which is calculated as

$$\begin{aligned} \Delta &= (z_{xx})(z_{yy}) - (z_{xy})^2 \\ &= (2)(2) - (0)^2 = 4 > 0 \end{aligned}$$

Therefore a minimum point exists at $x = 0$, $y = 0$, $z = 20$. This is illustrated in Figure 7.9(b).

Progress Exercises 7.7 Unconstrained Optimisation

Find and classify the stationary points for the functions given in questions 1 to 6.

1. $z = x^2 + y^2 - 10x - 12y$
2. $z = 20x - x^2 - y^2 + 8y$
3. $F(x, y) = 2y^2 + 2xy + x^2 - 16x - 20y$
4. $C = 5x^2 + 2y^2 - 4xy - 36x$
5. $Q = 5\ln(L) + 2\ln(K) - 0.1L - 0.4K$
6. $U = x^3 + y^3 - 27x - 12y + 210$

7.3.2 Total revenue maximisation and profit maximisation

The method of optimisation of functions of two variables is now used to examine more practical problems in economics and business. For example, we might want to calculate the maximum total revenue or maximum profit for a multi-product monopolist or a perfectly competitive firm.

WORKED EXAMPLE 7.15

MONOPOLIST MAXIMISING TOTAL REVENUE FOR TWO GOODS

A monopolist produces two goods, X and Y; the demand function for each good is given as

$$P_X = 36 - 3x \qquad \text{and} \qquad P_Y = 56 - 4y$$

(a) Write down the equation for the monopolist's total revenue.
(b) Determine the quantities of each good which must be sold so that total revenue is maximised. What is the maximum revenue?

Solution

(a) Total revenue = price × quantity for each good produced, that is,

$$\begin{aligned} TR &= P_X x + P_Y y \\ &= (36 - 3x)x + (56 - 4y)y \\ &= 36x - 3x^2 + 56y - 4y^2 \end{aligned}$$

(b) Work through each step of the optimisation method:

Step 1: Find derivatives

$$\frac{\partial TR}{\partial x} \text{ or } TR_x = 36 - 6x$$

$$TR_y = 56 - 8y$$

$$TR_{xx} = -6$$

$$TR_{yy} = -8$$

$$TR_{xy} = 0$$

Step 2: Solve

$TR_x = 0$, therefore

$$0 = 36 - 6x$$

$$6x = 36$$

$$x = 6$$

and $TR_y = 0$, therefore

$$0 = 56 - 8y$$

$$8y = 56$$

$$y = 7$$

Step 2a: Find TR when $x = 6$ and $y = 7$.

$$\begin{aligned} TR &= 36(6) - 3(6)^2 \\ &\quad + 56(7) - 4(7)^2 \\ &= 304 \end{aligned}$$

Step 3: Use second-order derivatives to decide the nature of the turning point.

$$TR_{xx} = -6$$

$$TR_{yy} = -8$$

$$TR_{xy} = 0$$

$$\left.\begin{aligned} TR_{xx} &= -6 \\ TR_{yy} &= -8 \end{aligned}\right\} \text{both}$$

negative so possible maximum, but need to calculate Δ, so

$$\begin{aligned} \Delta &= (TR_{xx})(TR_{yy}) \\ &\quad - (TR_{xy})^2 \\ &= (-6)(-8) - (0)^2 \\ &= 48 > 0 \end{aligned}$$

So a maximum is confirmed.

Maximum revenue is £304 when $x = 6$ and $y = 7$.

When we are given the firm's total cost function along with the demand functions, we may use unconstrained optimisation to find the quantities of each good which must be produced and sold in order to maximise profit.

WORKED EXAMPLE 7.16

MAXIMISE PROFIT FOR A MULTI-PRODUCT FIRM

A perfectly competitive firm produces two goods, X and Y, which are sold at £54 and £52 per unit respectively. The firm has a total cost function given by

$$TC = 3x^2 + 3xy + 2y^2 - 100$$

Find the quantities of each good which must be produced and sold in order to maximise profits. What is the maximum profit?

Solution

A perfectly competitive firm charges the same price irrespective of the number of units sold; therefore

$$\begin{aligned} \text{Total revenue} &= \text{revenue from good } X + \text{revenue from good } Y \\ &= P_X x + P_Y y \\ &= 54x + 52y \end{aligned}$$

Profit = total revenue − total cost, therefore

$$\begin{aligned} \pi &= TR - TC \\ &= 54x + 52y - 3x^2 - 3xy - 2y^2 + 100 \end{aligned}$$

Now work through the three steps for optimisation:

Step 1: Find the first- and second-order partial derivatives:

$$\pi_x = 54 - 6x - 3y, \quad \pi_y = 52 - 3x - 4y$$

$$\pi_{xx} = -6, \quad \pi_{yy} = -4, \quad \pi_{xy} = -3$$

Step 2: Equate the first-order partial derivatives to zero and solve.

(1) $\pi_x = 0 \rightarrow 0 = 54 - 6x - 3y$

(2) $\pi_y = 0 \rightarrow 0 = 52 - 3x - 4y$

Solve (1) and (2) for x and y:

(1) × 1 $0 = 54 - 6x - 3y$

(2) × 2 $0 = 104 - 6x - 8y$

Subtract $0 = -50 + 5y$ therefore $y = 10$

Solve for x by substituting $y = 10$ into either (1) or (2); therefore $x = 4$.

Step 3: Use second-order conditions to determine the type of optimum point at $x = 4$, $y = 10$:

$$\pi_{xx} = -6, \qquad \pi_{yy} = -4, \qquad \pi_{xy} = -3$$

$$\Delta = (\pi_{xx})(\pi_{yy}) - (\pi_{xy})^2 = (-6)(-4) - (-3)^2 = 15$$

Both straight second-order partial derivatives are negative, indicating a possible maximum. Since Δ is positive, then this confirms that the point is a maximum. Maximum profits are 468 when $y = 10$ and $x = 4$.

7.3.3 Price discrimination

Price discrimination for a monopolist producing a single good, X, which is sold in two separate markets, market 1 and market 2 (each with a different demand function), was introduced in Worked Example 6.24. In this section the analysis of profit and revenue for such a monopolist is continued with more general methods.

☐ To find the prices which should be charged in each market to maximise profit

Start by writing down the equation for the profit function:

$\pi = TR - TC$ where

$TR = (TR)_1 + (TR)_2$	TR is the sum of the total revenues for each market
$= P_1x_1 + P_2x_2$	P_1 and x_1 are the price and quantity in market 1 P_2 and x_2 are the price and quantity in market 2
$x = x_1 + x_2$	the total quantity sold is the sum of the quantities sold in each market

(In this section refer to the quantities sold in each market as x_1 and x_2 instead of Q_1 and Q_2.)

To find the conditions for maximum profit, work through the steps for optimisation:

Step 1: Find the first- and second-order partial derivatives:

$$\frac{\partial \pi}{\partial x_1} = \frac{\partial TR_1}{\partial x_1} - \frac{\partial TC}{\partial x_1} \quad \rightarrow \quad \pi_{x_1} = TR_{x_1} - TC_{x_1} = MR_1 - MC_1$$

$$\frac{\partial \pi}{\partial x_2} = \frac{\partial TR_2}{\partial x_2} - \frac{\partial TC}{\partial x_2} \quad \rightarrow \quad \pi_{x_2} = TR_{x_2} - TC_{x_2} = MR_2 - MC_2$$

$$\pi_{x_1x_1} = TR_{x_1x_1} - TC_{x_1x_1}; \quad \pi_{x_2x_2} = TR_{x_2x_2} - TC_{x_2x_2}; \quad \pi_{x_1x_2} = TR_{x_1x_2} - TC_{x_1x_2}$$

Step 2: Equate the first-order partial derivatives to zero and solve:

$$\pi_{x_1} = TR_{x_1} - TC_{x_1} \quad \rightarrow \quad MR_1 - MC_1 = 0 \quad \rightarrow \quad MR_1 = MC_1$$

$$\pi_{x_2} = TR_{x_2} - TC_{x_2} \quad \rightarrow \quad MR_2 - MC_2 = 0 \quad \rightarrow \quad MR_2 = MC_2$$

In other words, $MR = MC$ for maximum profit for each market.

However, since the single good X is produced by one firm, the cost function is the same for each market, consequently the marginal cost function is the same for each market. Therefore

$$MC_1 = MC_2 = MC$$

Therefore the first-order condition for maximum profit is

$$MR_1 = MR_2 = MC \tag{7.32}$$

However, the second-order conditions must be used to classify the point as a maximum, minimum or otherwise.

WORKED EXAMPLE 7.17

MONOPOLIST: PRICE AND NON-PRICE DISCRIMINATION

A monopolist produces a single good X but sells it in two separate markets. The demand function for each market is

$$P_1 = 50 - 4x_1 \quad \text{and} \quad P_2 = 80 - 3x_2$$

where P_1 and x_1, P_2 and x_2 are the price and quantity in markets 1 and 2 respectively. The cost function is $TC = 120 + 8x$, where $x = x_1 + x_2$.

(a) Find the price and quantity of the good in each market which maximises profit.

(b) Determine the price elasticity of demand for each market.

(c) Find the price and quantity of the good in each market which maximises profit when the monopolist does not use price discrimination, that is, $P_1 = P_2$.

Solution

It is now left as an exercise to the reader to use the more general methods outlined above to obtain the same answers for part (a) as those obtained in Worked Example 6.24(a). Part (c) is the same as in Worked Example 6.24(b).

(b) Elasticity in each market: $\varepsilon_1 = \dfrac{\mathrm{d}x_1}{\mathrm{d}P_1}\dfrac{P_1}{x_1}$ and $\varepsilon_2 = \dfrac{\mathrm{d}x_2}{\mathrm{d}P_2}\dfrac{P_2}{x_2}$

(You may find it convenient to use $\varepsilon = P/(P-a)$; the elasticity for linear functions, $P = a - bQ$ derived in Chapter 6.) The derivatives are obtained from the equations of the respective demand functions:

Market 1

$$\begin{aligned}
\varepsilon_1 &= \frac{\mathrm{d}x_1}{\mathrm{d}P_1}\frac{P_1}{x_1}\\
&= \frac{1}{(\mathrm{d}P_1/\mathrm{d}x_1)}\frac{P_1}{x_1}\\
&= \frac{1}{-4}\frac{P_1}{x_1}\\
&= -\frac{1}{4}\left(\frac{29}{5.25}\right) = -1.4
\end{aligned}$$

Market 2

$$\begin{aligned}
\varepsilon_2 &= \frac{\mathrm{d}x_2}{\mathrm{d}P_2}\frac{P_2}{x_2}\\
&= \frac{1}{(\mathrm{d}P_2/\mathrm{d}x_2)}\frac{P_2}{x_2}\\
&= \frac{1}{-3}\frac{P_2}{x_2}\\
&= -\frac{1}{3}\left(\frac{44}{12}\right) = -1.2
\end{aligned}$$

When $P_2 > P_1$, the demand elasticities $|\varepsilon_2| < |\varepsilon_1|$. Therefore, the higher price is charged in the market with the most inelastic demand (where demand is less responsive to price changes).

Progress Exercises 7.8 Unconstrained Optimisation: Applications

1. A utility function is given by the equation $U = x - 2x^2 + xy + 40y - y^2$, where x is the number of units of good X and y is the number of units of good Y consumed. Determine the values for x and y which maximise U. (Confirm that the point is a maximum.)

2. Given the demand functions for two goods $P_X = 320 - 4x + 2y$ and $P_Y = 106 + 2x - 20y$,

(a) Write down the equation for overall total revenue.

(b) Find the values of P_X, x, P_Y, y for which revenue is optimised. Hence calculate the maximum total revenue.

3. (See question 2) A firm producing two goods has demand functions as given in question 2 and a total cost function $TC = 4xy + 40x + 26y$.

(a) Write down the equation of the profit function.

(b) Determine the values of x and y for which profit is optimised. Hence calculate the maximum profit. Also calculate total revenue and the prices charged when profit is maximised.

4. A firm sells two related products whose demand functions are given as $Q_1 = 190 - 4P_1 - P_2$ and $Q_2 = 120 - 2P_1 - 3P_2$, where P_1, Q_1, P_2, Q_2 are the prices and quantities of goods 1 and 2 respectively.

(a) Write the equation of the overall total revenue in terms of P_1 and P_2 only.

(b) Determine the values of P_1, Q_1, P_2, Q_2 for which revenue is maximised.

(c) Calculate the maximum revenue.

5. Show that the total cost function $TC = 4Q_1Q_2 + 10Q_1 + 15Q_2$ has no economically meaningful optimum point.

6. A firm has two separate plants, each producing the same product. The cost functions for each plant are given by the equations

$$C_1(x) = 400 + \frac{x^2}{20} - 10x \quad \text{and} \quad C_2(y) = 400 - y + \frac{y^3}{147}$$

where x and y are the number of units produced in each plant. If overall costs are $C = C_1(x) + C_2(y)$, determine the number of units which should be manufactured in each plant to minimise costs.

7. The profit function for two goods is given by the equation

$$\pi = 527 - 2x^2 - 3y^2 - 2xy + 20x + 20y$$

Determine the values of x and y for which profit is maximised.

8. The total cost and demand functions for two goods are given by the equations

$$TC = 10 + 4x + 6y; \quad P_x = 8 - 2x; \quad P_y = 20 - 0.5y$$

where x is the number of units of the first good, y is the number of units of the second good.

(a) Write down the equations for (i) total revenue and (ii) profit.

(b) Determine the number of units of each good which should be sold to maximise revenue. Calculate the maximum revenue.

(c) Determine the number of units of each good which should be bought and sold to maximise profits. Calculate the maximum profit.

(d) Calculate the price elasticity of demand at maximum revenue and maximum profit for each good. Comment.

9. A monopolist can charge different prices in each of two markets whose demand and total cost functions are given as

$$P_X = 80 - 2.5x \quad P_Y = 125 - 10y \quad TC = 200 + 5(x + y)$$

(a) Calculate the maximum profit with price discrimination.

(b) Calculate the maximum profit with no price discrimination.

(c) Calculate the price elasticity of demand in each market when profits are maximised with price discrimination.

7.4 Constrained Optimisation and Lagrange Multipliers

At the end of this section you should be able to:

- Understand the meaning of a constrained maximum or minimum
- Use Lagrange multipliers to find the maximum or minimum values subject to a constraint
- Find maximum utility subject to a budget constraint
- Find maximum output subject to a cost constraint
- Find minimum cost subject to a production constraint.

In the previous section, unconstrained optimum values were calculated for different economic variables such as total revenue and profit. However, in most real-life situations, consumers and firms face financial and production constraints as there are limits on the availability of labour, capital and raw materials and, therefore, on the total level of output and income produced within an economy. We are still interested in finding optimum values, but now these optimum values will be subject to constraints.

7.4.1 What is a constrained maximum or minimum?

In this text we deal with linear constraints (see Section 2.7). For example, a budget constraint for two goods is usually given by an equation of the form:

expenditure = limit **or** expenditure ≤ limit **or** expenditure ≥ limit

$$xP_X + yP_Y = M \quad \text{or} \quad \begin{matrix} xP_X + yP_Y < M \\ xP_X + yP_Y \le M \end{matrix} \quad \text{or} \quad \begin{matrix} xP_X + yP_Y > M \\ xP_X + yP_Y \ge M \end{matrix}$$

where M is the monetary limit. In this text only equality constraints are used.

In Worked Example 7.15 total revenue for goods X and Y was maximised. It was found that the maximum revenue was £304 when $x = 6$ and $y = 7$. However, there was no limit on the number of units of each good which could be sold. Suppose a constraint is imposed, that only allows £80 worth of goods to be produced and sold where each unit of X costs £5 while each unit of Y costs £10. The problem is then reduced to finding how many units of each good must be produced and sold so that revenue is maximised. This means that the firm faces a budget constraint which is given by the equation

$$P_X \times x + P_Y \times y = 80$$

$$5x + 10y = 80$$

Without resorting to further mathematics, we can see that the number of units of each good ($x = 6$ and $y = 7$), for which revenue is a maximum when no constraints were imposed, can no longer be used, since, $TR = 5(6) + 10(7) = 100$ exceeds the limit of 80. The problem we are now faced with is how to find the combination of x and y which will yield the maximun possible revenue subject to a given constraint.

7.4.2 Finding the constrained extrema with Lagrange multipliers

To find the optimum values of a function $z = f(x, y)$ subject to a constraint, $ax + by = M$, we define the **Lagrangian function**, L, where

$$L = L(x, y, \lambda) = f(x, y) + \lambda(M - ax - by) \tag{7.33}$$

where λ is called a **Lagrange multiplier** and is treated as a variable. L is the sum of the original function to be optimised and $\lambda \times (\text{constraint} = 0)$. The optimum value of λ must be determined in addition to the optimum values of the variables x and y.

The method for finding the optimum value(s) of L is the same as in the previous section, but extended slightly (again!), since the problem consists of three independent variables, x, y and λ and, therefore, three first-order derivatives, L_x, L_y, L_λ. The method of Lagrange multipliers is demonstrated in the following Worked Examples.

WORKED EXAMPLE 7.18

MAXIMISING TOTAL REVENUE SUBJECT TO A BUDGET CONSTRAINT

The total revenue function for two goods is given by the equation

$$TR = 36x - 3x^2 + 56y - 4y^2$$

Find the number of units of each good which must be sold if profit is to be maximised when the firm is subject to a budget constraint, $5x + 10y = 80$.

Solution

Define the Lagrangian: $L = 36x - 3x^2 + 56y - 4y^2 + \lambda(80 - 5x - 10y)$
Now use the methods in Section 7.3 to find the values of x, y and λ for which revenue is maximised.

Step 1: Find the first-order partial derivatives:

$$L_x = 36 - 6x - 5\lambda \quad L_y = 56 - 8y - 10\lambda \quad L_\lambda = 80 - 5x - 10y$$

Step 2: Equate the first-order partial derivatives to zero and solve for x, y and λ. It is usually more convenient to eliminate λ from the first two equations and then solve for x and y.

(1) $36 - 6x - 5\lambda = 0$

(2) $56 - 8y - 10\lambda = 0$

(3) $80 - 5x - 10y = 0$

Solve as follows:

(1) $\times -2 \rightarrow -72 + 12x + 10\lambda = 0$
(2) $\rightarrow \underline{56 - 8y - 10\lambda = 0}$
(4) add $\rightarrow -16 + 12x - 8y = 0$

Now use equations (3) and (4) to solve for x and y.

(5) $\rightarrow$ (3)/5 $\rightarrow 16 - x - 2y = 0$
(6) $\rightarrow$ (4)/ -4 $\rightarrow \underline{4 - 3x + 2y = 0}$
add $20 - 4x + 0 = 0 \rightarrow x = 20/4 = 5$

Substitute $x = 5$ into any equation containing x and y such as (5), and solve for y:

$$y = (16 - x)/2 = 11/2 = 5.5$$

Therefore $y = 5.5$ and $x = 5$ are the quantities of goods X and Y for which revenue is maximised. Maximum total revenue is

$$TR = 36(5) - 3(5)^2 + 56(5.5) - 4(5.5)^2 = 292$$

which is less than the value for maximum revenue (£304) for the unconstrained maximum calculated in Worked Example 7.15. Therefore, the constraint is a limitation. See Figure 7.10 and Table 7.8.

Table 7.8 *TR* evaluated at selected points on the constraint: $80 = 5x + 8y$ ($y = 8 - 0.5x$)

x	0	2	4	5	6	8	10	12	16
$y = 8 - 0.5x$	8	7	6	5.5	5	4	3	2	0
TR	192	256	288	292	288	256	192	96	−192
$5x + 8y =$	80	80	80	80	80	80	80	80	80

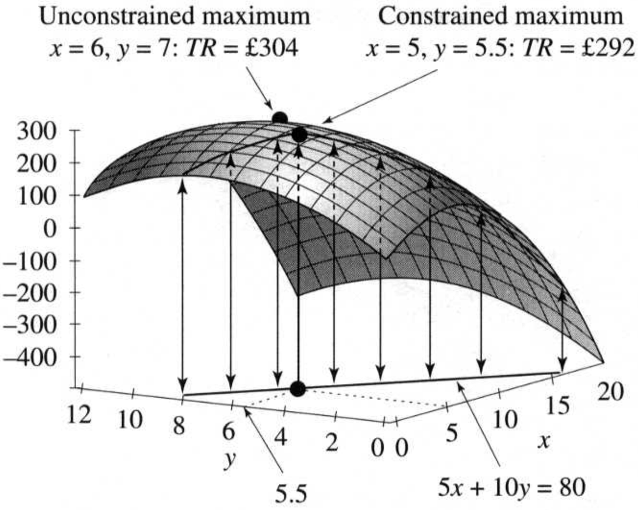


Figure 7.10 Constrained maximum: $TR = 36x - 3x^2 + 56y - 4y^2$, subject to $5x + 10y = 80$

Note 1: The values of x and y which optimise L are also the values of x and y which optimise the original function, $f(x, y)$ subject to the constraint. A formal mathematical proof is beyond the scope of this text, but is available in the Bibliography at the end of this book.

Note 2: The second-order conditions are more complex when three independent variables x, y, λ are involved. So, yet again, texts which cover formal proofs are listed in the Bibliography. However, for the usual economic/business problems presented in this text, it is quite safe to assume that the method of Lagrange multipliers yields a maximum for a maximisation problem and a minimum for a minimisation problem.

In the remaining worked examples the methods will be the same as those used in Worked Example 7.18; however, the equations, $L_x = 0$, $L_y = 0$, $L_\lambda = 0$ may not be three linear equations in three unknowns. The following worked examples will suggest various methods for solving three non-linear equations in three unknowns.

7.4.3 Maximum utility subject to a budget constraint

In this section maximum utility subject to a constraint is found by using the standard method of Lagrange multipliers. We then demonstrate the general condition

$$-\frac{U_x}{U_y} = -\frac{P_X}{P_Y}$$

which is stated in economic textbooks for finding maximum utility subject to a budget constraint. This condition is derived by Lagrange multipliers and the standard graphical methods of economics.

WORKED EXAMPLE 7.19

LAGRANGE MULTIPLIERS AND UTILITY MAXIMISATION

Use Lagrange multipliers to find the maximum utility for the utility function $U = 5xy$, when subject to a budget of £30, where the price of each unit of X is £5 and each unit of Y is £1.

Solution

The equation of the budget constraint is

$$xP_X + yP_Y = M$$
$$x(5) + y(1) = 30$$
$$5x + y = 30$$

The equation of the Lagrangian is therefore

$$L = 5xy + \lambda(30 - 5x - y)$$

Step 1: Find the first-order partial derivatives:

$$L_x = 5y - 5\lambda \qquad L_y = 5x - \lambda \qquad L_\lambda = 30 - 5x - y$$

Step 2: Equate the first-order derivatives to zero and solve

(1) $5y - 5\lambda = 0$ or $5y = 5\lambda$

(2) $5x - \lambda = 0$ or $5x = \lambda$

(3) $30 - 5x - y = 0$

To solve, first eliminate λ from equations (1) and (2).

(1) → $5y = 5\lambda$
(2) × 5 → $\underline{25x = 5\lambda}$
(4) subtract → $5y - 25x = 0$

Now use equations (3) and (4) to solve for x and y.

(5) → (4)/5 → $y - 5x = 0$
(6) → (3) rearranged → $\underline{y + 5x = 30}$
add $2y + 0 = 30$ → $y = 15$

Substituting $y = 15$ into equation (5) gives $x = 3$; therefore, maximum utility occurs when $x = 3$, $y = 15$. The level of maximum utility is $U = 5(3)(15) = 225$. From equation (1), $\lambda = 15$.

❐ Graphical analysis for locating maximum utility, $-(U_x/U_y) = -(P_X/P_Y)$

Figure 7.11 illustrates three indifference curves, each representing three different levels of utility. If there is no limit on the available quantities of goods X and Y that can be consumed,

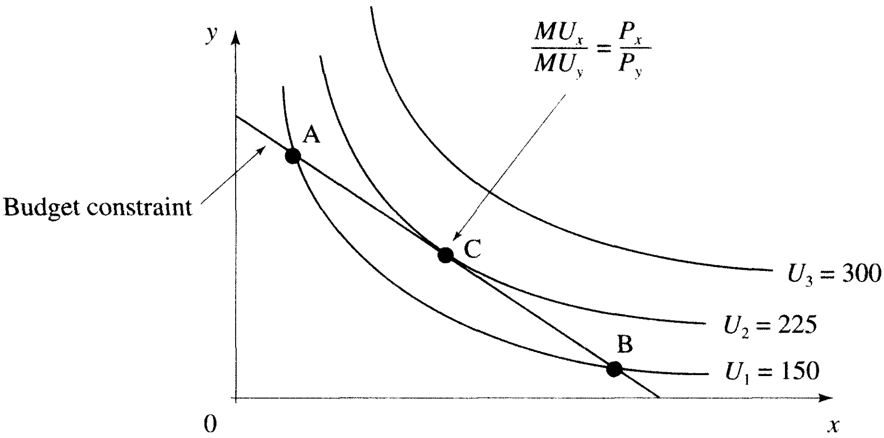


Figure 7.11 Utility subject to a budget constraint

then utility will increase indefinitely. There will be no limiting maximum value. If, however, a budget constraint is imposed, then the quantities consumed of each good, x and y, are limited. The budget constraint is also illustrated in Figure 7.11. The equation of the budget constraint is

$$xP_X + yP_Y = M$$

$$y = \frac{M}{P_Y} - \frac{P_X}{P_Y}x \tag{7.34}$$

The point of intersection of any indifference curve with the budget line gives (x, y), the quantities of goods X and Y respectively, which satisfy the budget constraint. However, utility increases as the indifference curves move further from the origin. Therefore the maximum possible utility occurs at the point where the 'highest' indifference curve is tangent to the budget line. Point C in Figure 7.11 is such a point.

Therefore the quantities of x and y for which utility is a maximum are given by the coordinates of point C at which the indifference curve is tangent to the budget constraint. From Chapter 6, the slope of the curve at a point was shown to be the same as the slope of the tangent at that point. Hence, equating the slope of the indifference curve and the slope of the budget constraint gives

$$-\frac{U_x}{U_y} = -\frac{P_X}{P_Y} \tag{7.35}$$

where the slope of the indifference curve is $-(U_x/U_y)$, (this was derived in Section 7.2) and the slope of the budget line is $-(P_X/P_Y)$ (read slope directly from equation (7.34)). Equation (7.35) may also be written as

$$\frac{MU_x}{MU_y} = \frac{P_X}{P_Y} \tag{7.36}$$

where the minus signs are cancelled and U_x is replaced by the alternative representation, MU_x etc.

WORKED EXAMPLE 7.20

USE LAGRANGE MULTIPLIERS TO DERIVE THE IDENTITY $U_x/U_y = P_X/P_Y$

Use Lagrange multipliers to show that a utility function $U(x, y)$ which is subject to a budget constraint $xP_X + yP_Y = M$ is maximised when $U_x/U_y = P_X/P_Y$. (Ignore negative signs.)

Solution

The Lagrangian is $L = U(x, y) + \lambda(M - xP_X - yP_Y)$. To find the maximum (or any optimum point!) equate the first-order partial derivatives to zero.

(1) $L_x = U_x - \lambda P_X \quad \rightarrow \quad 0 = U_x - \lambda P_X$ therefore $U_x = \lambda P_X$

(2) $L_y = U_y - \lambda P_Y \quad \rightarrow \quad 0 = U_y - \lambda P_Y$ therefore $U_y = \lambda P_Y$

(3) $L_\lambda = M - xP_X - yP_Y$ therefore $M = xP_X + yP_Y$

If we divide U_x by U_y, then from equations (1) and (2) we may write

$$\frac{U_x}{U_y} = \frac{\cancel{\lambda} P_X}{\cancel{\lambda} P_Y} = \frac{P_X}{P_Y}$$

This is the same as equation (7.35) as required. The budget equation, expressed as equation (3) above, must also be satisfied. So the Lagrange multiplier method produces exactly the same results as the graphical analysis.

☐ Interpretation of the Lagrange multiplier λ

The Lagrange multiplier method is a more systematic method for finding constrained maximum or minimum points. In addition, the value of the Lagrange multiplier, λ, has many applications. For example, the value of λ gives the change in the maximum or minimum value of the function being optimised for each unit change in the constraint. To demonstrate, Worked Example 7.19 is repeated in Worked Example 7.21, but with the budget constraint being increased from £30 to £31, that is, by one unit.

WORKED EXAMPLE 7.21

MEANING OF λ

Given the utility function $U = 5xy$, find the change in the maximum level of utility when the budget constraint $5x + y = 30$ is increased by one unit.

Solution

This utility function and budget constraint were given in Worked Example 7.19 where utility was maximised at $x = 3$, $y = 15$ and maximum utility, $U = 225$.

When the budget constraint is increased by 1 unit its equation becomes

$$5x + y = 31$$

Therefore the equation of the Lagrangian is

$$L = 5xy + \lambda(31 - 5x - y)$$

Step 1: Find the first-order partial derivatives:

$$L_x = 5y - 5\lambda \qquad L_y = 5x - \lambda \qquad L_\lambda = 31 - 5x - y$$

Step 2: Equate the first-order partial derivatives to zero and solve:

$$\begin{array}{lll} (1) & 0 = 5y - 5\lambda & \text{or} \quad 5y = 5\lambda \\ (2) & 0 = 5x - \lambda & \text{or} \quad 5x = \lambda \\ (3) & 0 = 31 - 5x - y & \end{array}$$

Now solve, first eliminate λ from equations (1) and (2):

$$\begin{array}{lcr} (1)\text{ rewritten} & \rightarrow & 5y = 5\lambda \\ (2) \times 5 & \rightarrow & \underline{25x = 5\lambda} \\ (4) \quad \text{subtract} & & 5y - 25x = 0 \end{array}$$

Now use equations (3) and (4) to solve for x and y:

$$\begin{array}{llllll} (5) & \rightarrow & (4)/5 & \rightarrow & y - 5x = 0 & \\ (6) & \rightarrow & (3)\text{ rewritten} & \rightarrow & \underline{y + 5x = 31} & \\ & \text{add} & & & 2y + 0 = 31 & \rightarrow \quad y = 15.5 \end{array}$$

Substituting $y = 15.5$ into equation (5) gives $x = 3.1$; therefore utility is at a maximum when $y = 15.5$ and $x = 3.1$. The maximum utility is $U = 5(3.1)(15.5) = 240.25$.

From equation (1), $\lambda = 15.5$. U has increased by $(240.25 - 225) = 15.25$. The increase in U is approximately equal to λ.

If $\lambda > 0$, the optimum value of utility will increase by λ for each unit increase in the constraint and decrease by λ for each unit reduction in the constraint, $xP_X + yP_Y = M$.

	Change in M	**Change in U**
$\lambda > 0$	+1 unit	λ approximately
	−1 unit	$-\lambda$ approximately
$\lambda < 0$	+1 unit	$-\lambda$ approximately
	−1 unit	λ approximately

7.4.4 Production functions

The method of Lagrange multipliers may also be used to find the maximum level of output subject to constraints on the availability of labour, L, capital, K and, fixed cost expenditure (C).

WORKED EXAMPLE 7.22

MAXIMISE OUTPUT SUBJECT TO A COST CONSTRAINT

Given a Cobb–Douglas production function $Q = L^{0.3}K^{0.7}$, subject to a cost constraint of £150 with the price of labour $w = 3$ and the price of capital, $r = 15$.

(a) Find the values of L and K for which production is maximised.
(b) What is the maximum level of production possible subject to the constraint?
(c) What is the maximum level of production possible with no constraint?

Solution

(a) First, write down the equation for the cost constraint:

$$wL + rK = C$$

$$3L + 15K = 150$$

Therefore the Lagrangian is given by the equation

$$L = L^{0.3}K^{0.7} + \lambda(150 - 3L - 15K)$$

Now use the usual method for finding the optimum point (steps 1 and 2 only).

Step 1: Find the first-order partial derivatives:

$$L_L = 0.3L^{-0.7}K^{0.7} - 3\lambda$$

$$L_K = 0.7L^{0.3}K^{-0.3} - 15\lambda$$

$$L_\lambda = 150 - 3L - 15K$$

Step 2: Equate the first-order partial derivatives to zero and solve:

(1) $0 = 0.3L^{-0.7}K^{0.7} - 3\lambda$

(2) $0 = 0.7L^{0.3}K^{-0.3} - 15\lambda$

(3) $0 = 150 - 3L - 15K$

Again, we suggest that you eliminate λ from equations (1) and (2):

$$\begin{array}{lll} (1)\times 5 & \rightarrow & 0 = 1.5L^{-0.7}K^{0.7} - 15\lambda \\ (2)\times 1 & \rightarrow & \underline{0 = 0.7L^{0.3}K^{-0.3} - 15\lambda} \\ & \text{Subtract} & 0 = 1.5L^{-0.7}K^{0.7} - 0.7L^{0.3}K^{-0.3} \end{array}$$

and now simplify. There are several ways to simplify, and the following is just one way.

$$1.5L^{-0.7}K^{0.7} = 0.7L^{0.3}K^{-0.3}$$

$$\frac{1.5L^{-0.7}K^{0.7}}{0.7L^{0.3}K^{-0.3}} = 1 \qquad \text{dividing both sides by } 0.7L^{0.3}K^{-0.3}$$

$$\frac{1.5K^{0.7-(-0.3)}}{0.7L^{0.3-(-0.7)}} = 1 \qquad \text{dividing numbers with the same base, subtract the indices}$$

$$\frac{1.5K}{0.7L} = 1 \qquad \text{simplify the indices}$$

(4) $1.5K = 0.7L$ or $15K = 7L$ multiplying both sides by 10 to avoid decimals

We now have two equations in L and K only. Now solve for L and K.

(5) → (4) rearranged $7L - 15K = 0$

(6) → (3) rearranged $\underline{3L + 15K = 150}$

add $10L = 150$ → $L = 15$

Substituting $L = 15$ into equation (5) gives $7(15) - 15K = 0$ or $K = 7$. Therefore, production is maximised when $L = 15$ and $K = 7$. Check that these values satisfy the cost constraint:

$$3L + 15K = 150 \rightarrow 3(15) + 15(7) = 150 \rightarrow 45 + 105 = 150 \text{ TRUE}$$

(b) The maximum level of production is found by substituting these values back into the equation for the production function:

$$Q = L^{0.3}K^{0.7} = 15^{0.3}7^{0.7} = (2.2533)(3.9045) = 8.798 \text{ using the calculator}$$

(c) If there were no limits on L and K, then there would be no limit on the level of production, since $L^{0.3}$ increases indefinitely as L increases and, so also does $K^{0.7}$ as K increases. (Try some values on the calculator!)

7.4.5 Minimising cost subject to a production constraint

Again, the Lagrange multiplier method is used to find the minimum cost subject to a production constraint.

WORKED EXAMPLE 7.23

MINIMISE COSTS SUBJECT TO A PRODUCTION CONSTRAINT

A production function is $Q = 12L^{0.5}K^{0.5}$. Find the values of L and K which minimise costs when labour costs £25 per unit (wage rate), capital costs £50 per unit (rent) and when the production constraint is 240 units of output.

Solution

The equation of the cost function is

$$wL + rK = C$$

$$25L + 50K = C$$

This is the function which is to be minimised subject to the production constraint. The Lagrangian is

$$L = 25L + 50K + \lambda(240 - 12L^{0.5}K^{0.5})$$

Now work through the method for finding optimum points:

Step 1: Find the first-order partial derivatives:

$$L_L = 25 - \lambda 12(0.5L^{0.5-1}K^{0.5}) = 25 - \lambda 6L^{-0.5}K^{0.5}$$

$$L_K = 50 - \lambda 12(0.5L^{0.5}K^{0.5-1}) = 50 - \lambda 6L^{0.5}K^{-0.5}$$

$$L_\lambda = 240 - 12L^{0.5}K^{0.5}$$

Step 2: Equate the first-order partial derivatives to zero and solve:

(1) $0 = 25 - \lambda 6L^{-0.5}K^{0.5} \rightarrow 25 = \lambda 6L^{-0.5}K^{0.5}$

(2) $0 = 50 - \lambda 6L^{0.5}K^{-0.5} \rightarrow 50 = \lambda 6L^{0.5}K^{-0.5}$

(3) $0 = 240 - 12L^{0.5}K^{0.5} \rightarrow 240 = 12L^{0.5}K^{0.5}$

As usual, eliminate λ from equations (1) and (2). Divide the corresponding sides of each equation:

$$\frac{(1)}{(2)} \rightarrow \frac{25}{50} = \frac{\lambda 6L^{-0.5}K^{0.5}}{\lambda 6L^{0.5}K^{-0.5}}$$

$$\frac{1}{2} = \frac{L^{-0.5}K^{0.5}}{L^{0.5}K^{-0.5}}$$

$$\frac{1}{2} = \frac{K}{L}$$

$$\frac{L}{2} = K$$

Substitute $K = L/2$ into equation (3):

$$240 = 12L^{0.5}\left(\frac{L}{2}\right)^{0.5}$$

$$240 = \frac{12}{\sqrt{2}}L$$

$$L = \frac{\sqrt{2}(240)}{12}$$

$$L = 28.28$$

and $K = 14.14$. Therefore $C = 25L + 50K = 25(28.28) + 50(14.14) = 1414$.

Progress Exercises 7.9 Constrained Optimisation in General

1. Find the values of x and y for which the utility function $U = x^{0.2}y^{0.8}$ is maximised, subject to the budget constraint, $180 = 4x + 2y$ (assume second-order conditions are satisfied).

2. (See question 1) Given $U = x^{0.2}y^{0.8}$,
   (a) Write down the equations for the marginal utilities.
   (b) Show that $U_x/U_y = P_X/P_Y$ when $U = x^{0.2}y^{0.8}$ is maximised, subject to the budget constraint, $xP_X + yP_Y = M$.
3. A student is restricted to a monthly budget constraint of £200 which is allocated to two activities, bridge and skating. His utility function is given by the equation $U = 5x^{0.2}y^{0.8}$, where x and y represent the number of hours spent on skating and bridge respectively. Skating costs £5 per hour, while bridge costs £2 per hour.
   (a) Derive expressions for MU_x and MU_y.
   (b) Solve the simultaneous equations
   $$\frac{MU_x}{MU_y} = \frac{P_X}{P_Y} \quad \text{and} \quad xP_X + yP_Y = M$$
   (c) Use Lagrange multipliers to determine the number of hours that should be spent on each activity to maximise utility.
   Comment on the answers in (b) and (c).
4. The production function for a garage which services cars and trucks is, $Q = 15L^{2/3}K^{1/3}$. If each labour unit used costs £5, while each unit of capital costs £3, find the number of units of capital and labour which should be used to maximise productivity subject to the constraint $LP_L + KP_K = 450$.
5. Determine the values of K and L which minimise the cost function $C = 3K + 5L$ subject to the production constraint, $250 = 10K^{0.5}L^{0.5}$.

   Note: This is rather different from the usual 'maximise the production function subject to the constraint'.
6. A firm has a production function $Q = 15L^{1/5}K^{1/3}$, where L and K are the number of units of labour and capital. Find the maximum level of production, subject to the constraint $K + 4L = 20$.
7. A monopolist has a total cost function $TC = 10x + 6$, where $x = x_1 + x_2$, the sum of the number of units of goods 1 and 2 produced respectively. The product may be sold in two separate markets, in which the demand functions are given by the equations
   $$P_1 = 50 - 5x_1 \quad \text{and} \quad P_2 = 30 - 2x_2$$
   (a) Write down the equation of the profit function. Hence determine the price which should be charged in each market to maximise profits. Calculate the maximum profit.
   (b) Determine the price elasticity of demand in each market. Comment.
8. A school has a utility function $U = 5\ln(x_A) + 10\ln(x_B)$, for two services, A (sports) and B (remedial teaching). Find the maximum level of utility if sports (A) costs £200 per unit, while remedial teaching (B) costs £400 per unit. The school's budget limit is £3600.
9. A firm's production function is $Q = 50L^{0.6}K^{0.4}$, subject to the cost constraint $8L + 4K = 400$.
   (a) Deduce the price of a unit of labour and capital from the constraint equation.
   (b) Determine the levels of L and K at which production is maximised.
   (c) Show that $MP_L/MP_K = P_L/P_K$ when production is maximised.
10. A firm manufactures a good J (jam) from raw materials X (strawberries) and Y (blackberries). The quantity of jam (in jars) manufactured per day is $Q = 20x^{0.2}y^{0.8}$. If

each unit of X (strawberries) costs £20 and each unit of Y (blackberries) costs £2, but total daily expenditure is restricted to £1450,

(a) Write down the equation of the constraint.

(b) Use the method of Lagrange to determine the number of units of strawberries and blackberries which should be purchased to maximise production subject to the constraint. Calculate the maximum production.

(c) Show that $MP_x/MP_y = P_X/P_Y$ when production is maximised.

7.5 Summary

❑ Function of one variable $y = f(x)$

The **derivative** is dy/dx.

The **differential** of y is

$$dy = \left(\frac{dy}{dx}\right)dx$$

where dy: differential of y derivative of y w.r.t. x dx: differential of x

and dy and dx are infinitesimally small.

❑ Function of two variables: $z = f(x, y)$

The first-order partial derivatives are

$\partial z/\partial x$ is the partial derivative of z w.r.t. x, y treated as a constant (abbreviated to z_x)

$\partial z/\partial y$ is the partial derivative of z w.r.t. y, x treated as a constant (abbreviated to z_y)

The second-order partial derivatives are

$$\frac{\partial^2 z}{\partial x^2} \equiv z_{xx}, \quad \frac{\partial^2 z}{\partial y^2} \equiv z_{yy}, \quad \frac{\partial^2 z}{\partial y\, \partial x} \equiv z_{xy}, \quad \frac{\partial^2 z}{\partial x\, \partial y} \equiv z_{yx}$$

The total differential of z is given by

$$dz = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy$$

or for small changes $dx \to \Delta x$ etc., the **small (incremental) changes formula** is given by

$$\Delta z \cong \left(\frac{\partial f}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial y}\right)\Delta y$$

❒ Unconstrained optimisation

Step 1: Find the first and second derivatives

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^2 z}{\partial x\,\partial y}$$

Step 2: First-order conditions:

At stationary point $\partial z/\partial x = 0$, $\partial z/\partial y = 0$
Solve these two equations for the x- and y-coordinates of the turning point(s).

Step 2a: If required, find z for the x- and y-coordinates of the turning point(s).

Step 3: Second-order conditions:

The point is a minimum if $\partial^2 z/\partial x^2 > 0$ and $\partial^2 z/\partial y^2 > 0$ and provided $\Delta > 0$.
The point is a maximum if $\partial^2 z/\partial x^2 < 0$ and $\partial^2 z/\partial y^2 < 0$ and provided $\Delta > 0$.
The point is a point of inflection if both second derivatives have the same sign but $\Delta < 0$.
The point is a saddle point if the second derivatives have different signs and $\Delta < 0$, where

$$\Delta = \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2$$

❒ Constrained optimisation: Lagrange multipliers

Given a function $z = \mathrm{f}(x, y)$ to be optimised, subject to a constraint, $M - ax - by = 0$, the Lagrangian (L), is defined as $L = \mathrm{f}(x, y) + \lambda(M - ax - by)$
Solve the equations

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

for the values of x, y, and λ which optimise L.
Second-order conditions are not required in this text.

❒ Applications

Production: Cobb–Douglas production: $Q = AL^{\alpha}K^{\beta}$:

1. Represent production functions in two-dimensional diagrams (isoquants) by fixing the value of Q, then express $K = \mathrm{f}(L)$.
2. The slope $\mathrm{d}K/\mathrm{d}L$ is called the **marginal rate of technical substitution** ($MRTS$).
3. Slope is also expressed as

$$\text{(i)}\ \frac{\mathrm{d}K}{\mathrm{d}L} = -\frac{MP_L}{MP_K}, \quad \text{(ii)}\ MRTS = \frac{\alpha}{\beta}\frac{K}{L}$$

4. Returns to scale. When L and K are each replaced by λL and λK respectively in $Q_1 = AL^{\alpha}K^{\beta}$ then $Q_2 = A(\lambda L)^{\alpha}(\lambda K)^{\beta} = \lambda^{\alpha+\beta}AL^{\alpha}K^{\beta} = \lambda^{\alpha+\beta}Q_1$

If $\alpha + \beta = 1$: constant returns to scale, i.e. $Q_2 = \lambda Q_1$
If $\alpha + \beta < 1$: decreasing returns to scale, i.e. $Q_2 < \lambda Q_1$
If $\alpha + \beta > 1$: increasing returns to scale, i.e. $Q_2 > \lambda Q_1$

(all these rules apply to a Cobb–Douglas function).

☐ Partial elasticity

1. Price elasticity of demand:

$$\varepsilon_d = \frac{\partial Q_A}{\partial P_A}\frac{P_A}{Q_A}$$

2. Income elasticity of demand:

$$\varepsilon_Y = \frac{\partial Q_A}{\partial Y}\frac{Y}{Q_A}$$

3. Cross-price elasticity of demand:

$$\varepsilon_c = \frac{\partial Q_A}{\partial P_B}\frac{P_B}{Q_A}$$

4. Elasticity w.r.t. labour:

$$\varepsilon_{QL} = \frac{\partial Q}{\partial L}\frac{L}{Q}$$

Note:

$$\varepsilon_{QL} = \frac{\partial Q}{\partial L}\frac{L}{Q} = \frac{\partial Q/\partial L}{Q/L} = \frac{MP_L}{APL}$$

5. Elasticity w.r.t. capital:

$$\varepsilon_{QK} = \frac{\partial Q}{\partial K}\frac{K}{Q} \quad \text{and} \quad \varepsilon_{QK} = \frac{\partial Q}{\partial K}\frac{K}{Q} = \frac{\partial Q/\partial K}{Q/K} = \frac{MP_K}{APK}$$

☐ National income model multipliers

$$\frac{\partial Y_e}{\partial I} = \frac{1}{1-b(1-t)}: \quad \text{the investment multiplier}$$

$$\frac{\partial Y_e}{\partial G} = \frac{1}{1-b(1-t)}: \quad \text{the government expenditure multiplier}$$

$$\frac{\partial Y_e}{\partial t} = -\frac{bY_e}{1-b(1-t)}: \quad \text{the income tax rate multiplier}$$

Test Exercises 7

1. (a) Determine the first and second partial derivatives of the following:

(i) $U = x^3 + 2y^3$ (ii) $Z = x^2y + 10xy$ (iii) $Q = 100L^{0.4}K^{0.6}$

2. A production function $Q = 100L^{0.4}K^{0.6}$ relates output, Q, to the number of labour units, L, and capital units K.

(a) Derive the equation for the marginal and the average products of labour and capital.

(b) Prove the identity

$$Q = L\frac{\partial Q}{\partial L} + K\frac{\partial Q}{\partial K}$$

(c) Use partial derivatives to calculate the approximate percentage change in Q when L increases by 6% while K decreases by 4%.

3. The satisfaction derived from activities X (watching TV) and Y (playing golf) is given by the utility function: $U = 4x^{0.5}y^{0.5}$.

(a) Show that when $U = 40$, the equation of the corresponding indifference curve is $y = 100/x$.

(b) Plot the indifference curve for $0 < x < 5$ and derive the equation for the slope of the indifference curve. Use the second derivative to determine the curvature algebraically.

4. (See question 3) $U = 4x^{0.5}y^{0.5}$

(a) Derive the equations for

(i) the marginal utility w.r.t. x ($MU_x = \partial U/\partial x$) and (ii) the marginal utility w.r.t. y ($MU_y = \partial U/\partial y$).

Show that the slope of the indifference curve may be expressed as

$$\frac{dy}{dx} = -\frac{MU_x}{MU_y}$$

5. (See question 3) $U = 4x^{0.5}y^{0.5}$

(a) Explain the terms (i) constant, (ii) increasing and (iii) decreasing returns to scale

(b) Calculate the level of utility when the hours spent at each activity are

(i) 4 hours of TV and 6 hours of golf (ii) 2 hours of TV and 3 hours of golf.

Do the results indicate that the function exhibits increasing returns to scale?

(c) Use the definitions of returns to scale to show that the utility function exhibits constant returns to scale.

6. Given a Cobb–Douglas production function $Q = AL^{\alpha}K^{\beta}$, define MP_L (the marginal product of labour), APL (the average product of labour), ε_{QL} (the partial elasticity w.r.t. labour). Show that $\varepsilon_{QL} = MP_L/APL$.

7. Given a Cobb–Douglas production function $Q = 200L^{0.4}K^{0.8}$,

(a) Determine the price elasticity w.r.t. labour: the price elasticity w.r.t. capital.

(b) Determine the maximum level of production subject to the constraint $L + 4K = 300$.

8. A utility function $U = 100x^{0.8}y^{0.2}$ relates level of benefit to the number of glasses of lemon tea and wine consumed, respectively.

(a) Use partial derivatives to calculate the overall increase in utility when consumption of wine increases by 5% and consumption of tea increases by 8%,

(b) If consumption of both beverages increase by 50% calculate the overall increase in utility. What type of returns to scale is exhibited by this function. Give reasons.

9. (See question 8) A budget constraint of £40 is imposed, hence the equation of the constraint is $x + 4y = 40$. Find the number of units of each beverage which should be consumed to maximise utility.

CHAPTER 8

Integration and Applications

At the end of this chapter you should understand the term 'integration' and its applications in economics and business covered in the following sections:

8.1 Integration as the reverse of differentiation
8.2 The power rule for integration
8.3 Integration of the natural exponential function
8.4 Integration by algebraic substitution
8.5 The definite integral and the area under a curve
8.6 Consumer and producer surplus
8.7 First-order differential equations and applications
8.8 Differential equations for limited and unlimited growth
8.9 Summary

8.1 Integration as the Reverse of Differentiation

You have already encountered many examples of mathematical operations which reverse each other. One example is multiplication then division by the same number, or vice versa:

$$N \times x \div x = N \text{ such as } 345 \times 20 \div 20 = 345$$

Another example is taking the antilog of the log of any number (to the same base) or vice versa:

$$\text{antilog}\,[\log(10^N)] = N \text{ such as antilog}[\log(50)] = 50 \text{ or } 10^{\log(50)} = 50$$

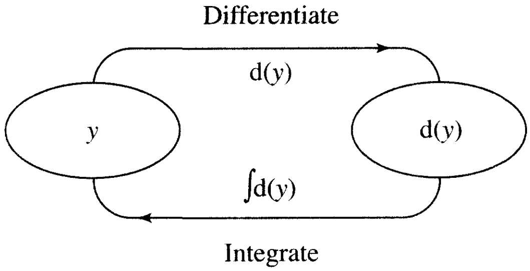


Figure 8.1 Integration reverses differentiation

A third example is to integrate the derivative (w.r.t. the same variable), or vice versa; in other words, when a function is differentiated then integrated, you will end up with the same function back again. This statement is written

$$\int [\mathrm{d(function)}] = \text{function} \qquad \textbf{(8.1)}$$

$$\mathrm{d}\left[\int (\text{function})\right] = \text{function} \qquad \textbf{(8.2)}$$

where $\int$ indicates integration and d indicates differentiation (Figure 8.1).

Or, when the independent variable is stated explicitly, e.g. $y = \mathrm{f}(x) + c$, the operation 'integration w.r.t. x' is denoted by enclosing the function to be integrated between the integration symbol $\int$ and $\mathrm{d}x$. In general, the function obtained by integrating $\mathrm{f}(x)$ is symbolised by $\mathrm{F}(x)$ in this text (Figure 8.2).

Note the inclusion of the constant c. The derivative of a constant term is zero, so when integrating (thereby reversing differentiation), allowance must be made for the possibility that the function (which was differentiated) may have contained a non-zero constant term. This definition of integration as the operation which reverses differentiation allows us to deduce the rules for integration, such as the power rule.

Note: As always, rigorous mathematical analysis and proofs are beyond the scope of the text. The examples and exercises in this chapter deal with functions which are continuous over the interval of integration. If you require a rigorous treatment of this topic, see Chapter 13 of Binmore or Chapter 9 of Hatton; both books are listed in the bibliography.

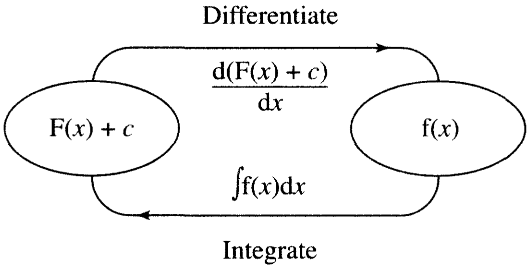


Figure 8.2

8.2 The Power Rule for Integration

At the end of this section you should be able to:

- Integrate power functions, x^n
- Integrate the sum or difference of several power functions
- Integrate any linear function raised to a power

❐ Deduce the power rule for integration

Following from Section 8.1, the integral of $f(x) = x$ is the function which is differentiated to give x. The derivative of $\frac{1}{2}x^2 + c$ is x since

$$\boxed{\frac{d}{dx}\left(\frac{x^2}{2}+c\right)=\frac{2x}{2}+0=x \quad \text{therefore} \quad \int x\,dx=\frac{x^2}{2}+c}$$

See Figure 8.3.

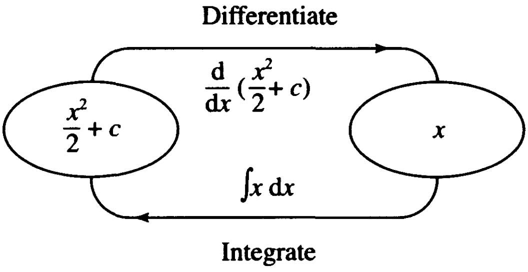


Figure 8.3

This example and the following examples demonstrate how the power rule for integration is deduced.

	$y = F(x) + c$	$\frac{d[F(x)+c]}{dx} = f(x)$	$\int f(x)dx = F(x) + c$
Example (a)	$y=\frac{x^2}{2}+c$	$\frac{dy}{dx}=\frac{2x}{2}=x$	$\int x\,dx=\frac{x^2}{2}+c$
Example (b)	$y=\frac{x^3}{3}+c$	$\frac{dy}{dx}=x^2$	$\int x^2\,dx=\frac{x^3}{3}+c$
Example (c)	$y=\frac{x^4}{4}+c$	$\frac{dy}{dx}=x^3$	$\int x^3\,dx=\frac{x^4}{4}+c$
Example (d)	$y=\frac{x^{n+1}}{n+1}+c$	$\frac{dy}{dx}=x^n$	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+c$

☺ **Example (d)** is the power rule for integration:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad \textbf{(8.3)}$$

The power rule is very simple to apply:

Step 1: Always start by simplifying where possible. If you do not simplify, you will give yourself a lot of extra work, or worse still, it may not be possible to integrate the unsimplified function at all.
Step 2: Write the power function 'on line', so that you may state the power n clearly.
Step 3: Apply the power rule for integration:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

WORKED EXAMPLE 8.1

USING THE POWER RULE FOR INTEGRATION

Integrate the following functions:

(a) x^4 (b) $\frac{x^4}{x^2}$ (c) $\frac{1}{x^4}$ (d) $\sqrt{x}$ (e) $\frac{1}{\sqrt{x^3}}$

Solution

	Step 1 Simplify, if possible	**Step 2** State n and $n+1$	**Step 3** Apply power rule $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$	Simplify the answer if possible
(a) x^4	x^4	$n = 4$: $n + 1 = 5$	$\int x^4 \, dx = \frac{x^5}{5} + c$	none possible
(b) $\frac{x^4}{x^2}$	$\frac{x^4}{x^2} = x^{4-2} = x^2$	$n = 2$: $n + 1 = 3$	$\int x^2 \, dx = \frac{x^3}{3} + c$	none possible
(c) $\frac{1}{x^4}$	$\frac{1}{x^4} = x^{-4}$	$n = -4$: $n + 1 = -3$	$\int x^{-4} \, dx = \frac{x^{-3}}{-3} + c$	$\frac{x^{-3}}{-3} = -\frac{x^{-3}}{3}$ $= -\frac{1}{3x^3} + c$
(d) $\sqrt{x}$	$\sqrt{x} = x^{0.5}$	$n = 0.5$: $n + 1 = 1.5$	$\int x^{0.5} \, dx = \frac{x^{1.5}}{1.5} + c$	none possible
(e) $\frac{1}{\sqrt{x^3}}$	$\frac{1}{\sqrt{x^3}} = \frac{1}{(x^3)^{0.5}}$ $= x^{-1.5}$	$n = -1.5$: $n + 1 = -0$	$\int x^{-0.5} \, dx = \frac{x^{-0.5}}{-0.5} + c$	$-\frac{x^{-0.5}}{0.5} = -2x^{-0.5}$ $= -\frac{2}{x^{0.5}} + c$

☐ The minus one exception to the power rule

The power rule applies for any value of n except $n = -1$. If $n = -1$ then $n + 1 = -1 + 1 = 0$, therefore, according to the power rule, we have

$$\int x^{-1}\,\mathrm{d}x = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = ?$$

Division by zero is not defined, so the power rule cannot be used in this case. The rule for integrating x^{-1} is

$$\boxed{\int \frac{1}{x}\,\mathrm{d}x = \int x^{-1}\,\mathrm{d}x = \ln|x| + c} \qquad \textbf{(8.4)}$$

where $|x|$ means the absolute value of x (Figure 8.4). The use of $|x|$ will be important in the section on definite integration. In this section, on indefinite integration, we simply write

$$\int \frac{1}{x}\,\mathrm{d}x = \ln(x) + c$$

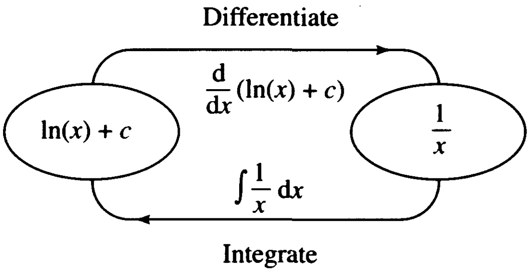


Figure 8.4

☐ The integral of a constant term

The integral of a constant term (K) is deduced as follows:

$$\int K\,\mathrm{d}x = \int Kx^0\,\mathrm{d}x \quad \text{since } x^0 = 1$$

Now apply the power rule. When $n = 0 \rightarrow n + 1 = 0 + 1 = 1$, therefore

$$\int Kx^0\,\mathrm{d}x = K\int x^0\,\mathrm{d}x = K\frac{x^1}{1} + c = Kx + c$$

$$\boxed{\int K\,\mathrm{d}x = Kx + c} \qquad \textbf{(8.5)}$$

In other words, the integral of a constant is the constant multiplied by the independent variable.

❒ Working rules for integration

The following working rules (similar to those used in differentiation) apply when integrating sums and differences of several functions.

- As in differentiation, the sum or difference of several power functions is integrated by integrating each function separately:

$$\int \{f(x) + g(x)\}\,dx = \int f(x)dx + \int g(x)dx \tag{8.6}$$

- As in differentiation, the integral of a constant multiplied by a variable term is the constant multiplied by the integral of the variable term:

$$\int Kf(x)dx = K\int f(x)dx \tag{8.7}$$

WORKED EXAMPLE 8.2

INTEGRATING SUMS AND DIFFERENCES, CONSTANT MULTIPLIED BY VARIABLE TERM

Integrate (a) $x^2 + \dfrac{1}{x^2}$ and (b) $5x - 3x^2$

Solution

(a)

Step 1: Simplify, $x^2 + \dfrac{1}{x^2} = x^2 + x^{-2}$

Step 2: State n hence $n + 1$ for each term

for x^2 we have $n = 2 \quad \rightarrow \quad n + 1 = 3$

for x^{-2} we have $n = -2 \quad \rightarrow \quad n + 1 = -2 + 1 = -1$

Step 3: Apply the power rule to obtain

$$\int (x^2 + x^{-2})\,dx = \frac{x^3}{3} + \left(\frac{x^{-1}}{-1}\right) = \frac{x^3}{3} - \frac{x^{-1}}{1} = \frac{x^3}{3} - \frac{1}{x} + c$$

(b) The steps are worked through very briefly; explain each step to yourself:

$$\begin{aligned}\int (5x - 3x^2)dx &= \int 5x\,dx - \int 3x^2\,dx \\ &= 5\int x\,dx - 3\int x^2\,dx \\ &= 5\frac{x^2}{2} - 3\frac{x^3}{3} = 2.5x^2 - x^3 + c\end{aligned}$$

Note 1: You can easily check that your integration is correct. According to equation (8.2), $d[\int(\text{function})] = \text{function}$. So differentiate your answer and you should end up with the function you originally set out to integrate.

Note 2: In the examples so far we have integrated functions of x w.r.t. x. The same rules follow for any other variable; integrate functions of Q w.r.t. Q, functions of P w.r.t. P etc. Now for some more general worked examples.

WORKED EXAMPLE 8.3

INTEGRATING MORE GENERAL FUNCTIONS

Integrate the following: (a) $\dfrac{x + 5x^2}{x^2}$ and (b) $(Q + 3)\sqrt{Q}$

Solution

(a)

Step 1: Simplify, $\dfrac{x + 5x^2}{x^2} = \dfrac{x}{x^2} + \dfrac{5x^2}{x^2} = \dfrac{1}{x} + 5$

Step 2: State n for each term. $1/x = x^{-1} \rightarrow n = -1$, so this is the exception to the power rule. Integration of $1/x$ is given by equation (8.4). The term 5 is a constant; from equation (8.5) we have that integrating a constant gives the constant multiplied by the variable.

Step 3: Write out the integration, $\int(1/x + 5)dx = \ln|x| + 5x + c$. This is the answer; it cannot be further simplified.

(b) Since this is a function of Q, integration is w.r.t. Q. Determine $\int(Q + 3)\sqrt{Q}\,dQ$.

Step 1: Simplify

$$\begin{aligned}(Q+3)\sqrt{Q} &= (Q+3)(Q)^{0.5}\\ &= (Q+3)Q^{0.5}\\ &= Q \times Q^{0.5} + 3 \times Q^{0.5}\\ &= Q^{1.5} + 3Q^{0.5} \qquad \text{since } Q^1Q^{0.5} = Q^{1.5}\end{aligned}$$

Step 2: State n hence $n + 1$ for each term.

for $Q^{1.5}$ we have $n = 1.5 \quad \rightarrow \quad n + 1 = 2.5$

for $Q^{0.5}$ we have $n = 0.5 \quad \rightarrow \quad n + 1 = 1$

Step 3: Apply the power rule to obtain

$$\int(Q+3)\sqrt{Q}\,dQ = \int(Q^{1.5} + 3 \times Q^{0.5})\,dQ = \frac{Q^{2.5}}{2.5} + 3 \times \frac{Q^{1.5}}{1.5} + c$$

Simplify the answer; in this case the numbers can be evaluated:

$$\frac{Q^{2.5}}{2.5} + 3 \times \frac{Q^{1.5}}{1.5} = 0.4Q^{2.5} + 2Q^{1.5} + c$$

Progress Exercises 8.1 Integration by the Power Rule

Integrate each of the following

1. $x + x^3 + x^{3.5}$

2. $\dfrac{1}{\sqrt{x^5}}$

3. $\dfrac{x}{\sqrt{x^5}}$

4. $\dfrac{x + \sqrt{x}}{x}$

5. $2x + \dfrac{2}{x}$

6. $2x + \dfrac{1}{2x}$

7. $x\sqrt{x} + 3$

8. $5x + \sqrt{5x}$

9. $x(x - 3)^2$

10. $\dfrac{x + 7}{x}$

11. $x^2\left(1 + \dfrac{1}{x^2}\right)$

12. $Q(Q - \sqrt{Q})$

13. $t\left(1 + \dfrac{2}{t}\right)$

14. $4 + \dfrac{4}{t}$

15. $5 + \dfrac{1}{Q}$

16. $\dfrac{2Q - 5}{Q}$

17. $Q(20 - 0.5Q)$

18. $\dfrac{Q + 4}{Q^2}$

19. $\dfrac{1}{5Q} - 5$

20. $\dfrac{1}{0.2}\left(Y - \dfrac{Y}{2}\right)$

8.3 Integration of the Natural Exponential Function

In Chapter 4 it was stated that the base e in the exponential function e^x is the natural base for describing growth and decay for all types of system. The number e is also a natural number in calculus. It is the only function which does not change when differentiated or integrated (Figure 8.5).

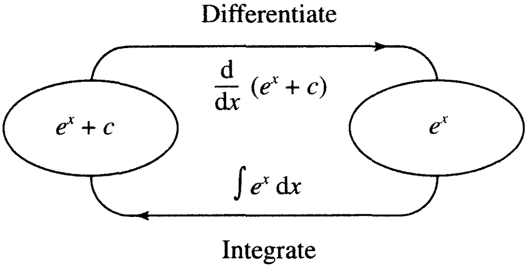


Figure 8.5 Integration of e^x

Therefore

$$\int e^x\,dx = e^x + c \tag{8.8}$$

WORKED EXAMPLE 8.4

INTEGRATING FUNCTIONS CONTAINING e^x

Integrate (a) $3e^x + 5$ and (b) $\dfrac{e^t + e^{2t}}{e^t}$

Solution

Follow the usual procedure for integration, simplifying first if necessary. State and apply the appropriate rule for integration. The solutions are as follows:

(a)

$3e^x$ is 3 multiplied by the standard function, which is integrated according to equation (8.8). 5 is a constant term, which is integrated according to equation (8.5) to give $5x$

$$\text{Therefore} \int (3e^x + 5)\,dx$$

$$= 3\int e^x dx + \int 5\,dx$$

$$= 3e^x + 5x + c$$

(b)

Here we must simplify first:

$$\frac{e^t + e^{2t}}{e^t} = \frac{e^t}{e^t} + \frac{e^{2t}}{e^t} \quad \text{dividing each by } e^t$$

$$= 1 + e^{2t}e^{-t}$$

$$= 1 + e^t \quad \text{rule 1, indices}$$

This simplifies to the sum of two terms: 1 is a constant term, integrated according equation (8.5) and e^t is a standard function; use equation (8.8) to obtain

$$\int \frac{e^t + e^{2t}}{e^t} dt = \int (1 + e^t)\,dt = 1 \times t + e^t + c$$

$$= t + e^t + c$$

The independent variable is x in part (a), so the integration must be w.r.t. x: the independent variable is t in part (b), so the integration must be w.r.t. t.

8.4 Integration by Algebraic Substitution

At the end of this section you should be able to:

- Integrate functions of linear functions by substitution
- Integrate certain products by substitution

8.4.1 Using substitution to integrate functions of linear functions

Compare the functions on the left of the table with those on the right.

☺ **Remember,** a function is a rule for operating on x.

Function of x, f(x)	**Function of a linear function, f($5x - 2$)**
x^{10}, x raised to the power 10	$(5x-2)^{10}$, $(5x-2)$ raised to the power 10
$\frac{1}{x}$, 1 divided by x	$\frac{1}{(5x-2)}$, 1 divided by $(5x-2)$
e^x, e raised to the power x	$e^{(5x-2)}$, e raised to the power $(5x-2)$

- The functions of x on the left are standard functions whose integrals are known.
- The functions on the right are of exactly the same type as those on the left, except that each x has been replaced by $(5x - 2)$, a linear function of x. These are called functions of linear functions; here they are functions of $(5x - 2)$.

To integrate these functions of linear functions, consider the effect of replacing the linear function by a single variable, say u. The substitution of u for $(5x - 2)$ would reduce each one to a standard function in u. However, there is just one further adjustment necessary before integration of the standard function of u may proceed, as demonstrated in the following worked example.

WORKED EXAMPLE 8.5

INTEGRATING FUNCTIONS OF LINEAR FUNCTIONS BY SUBSTITUTION

Use the method of substitution to integrate the following:

(a) $\int (5x-2)^{10}\,dx$ (b) $\int \frac{1}{5x-2}\,dx$ (c) $\int e^{5x-2}\,dx$

Solution

(a) Substitute u for the linear function $(5x - 2)$:

$$\int (5x-2)^{10}\,dx \rightarrow \int (u)^{10}\,\underbrace{dx}_{\downarrow}$$

a du is required in order to integrate functions of u

☺ *This symbol indicates integration* $\longrightarrow \int(\cdots)\,du$

du indicates that only functions of u will be integrated

To deduce an expression for dx in terms of du, use the substitution equation $u = (5x - 2)$ as outlined below.

Method

Substitution equation

$$u = 5x - 2$$

Differentiate w.r.t. x

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 5 \rightarrow \mathrm{d}u = 5\,\mathrm{d}x$$

Multiply both sides by $\frac{1}{5}$

$$\frac{1}{5}\mathrm{d}u = \frac{1}{\cancel{5}}\frac{\cancel{5}\,\mathrm{d}x}{1} = \mathrm{d}x \rightarrow \mathrm{d}x = \frac{1}{5}\,\mathrm{d}u$$

Now, replace $\mathrm{d}x$ by $\frac{1}{5}\,\mathrm{d}u$

$$\int u^{10}\,\mathrm{d}x \rightarrow \int u^{10}\left(\frac{1}{5}\right)\mathrm{d}u$$

Factor out the constant $\frac{1}{5}$ and integrate

$$\frac{1}{5}\int u^{10}\,\mathrm{d}u = \frac{1}{5}\,\frac{u^{11}}{11} + c$$

Write u back in terms of x

$$= \frac{(5x-2)^{11}}{55} + c$$

The solutions to parts (a), (b) and (c) are written briefly as follows.

(a) $\int (5x-2)^{10}\mathrm{d}x$	(b) $\int \frac{1}{(5x-2)}\,\mathrm{d}x$	(c) $\int e^{5x-2}\mathrm{d}x$
Step 1 Let $u = 5x - 2$, so the integral becomes $\int (u)^{10}\mathrm{d}x$, and $\mathrm{d}x$ needs to be expressed in terms of $\mathrm{d}u$	**Step 1** Let $u = 5x - 2$, so the integral becomes $\int u^{-1}\mathrm{d}x$, and $\mathrm{d}x$ needs to be expressed in terms of $\mathrm{d}u$	**Step 1** Let $u = 5x - 2$, so the integral becomes $\int e^{u}\mathrm{d}x$, and $\mathrm{d}x$ needs to be expressed in terms of $\mathrm{d}u$
Step 2 Use the substitution equation to deduce an expression for $\mathrm{d}x$ in terms of $\mathrm{d}u$ by differentiating w.r.t. x, then solving for $\mathrm{d}x$. In this problem $\mathrm{d}x = \frac{1}{5}\mathrm{d}u$	**Step 2** Use the substitution equation to deduce an expression for $\mathrm{d}x$ in terms of $\mathrm{d}u$ by differentiating w.r.t. x, then solving for $\mathrm{d}x$. In this problem $\mathrm{d}x = \frac{1}{5}\mathrm{d}u$	**Step 2** Use the substitution equation to deduce an expression for $\mathrm{d}x$ in terms of $\mathrm{d}u$ by differentiating w.r.t. x, then solving for $\mathrm{d}x$. In this problem $\mathrm{d}x = \frac{1}{5}\mathrm{d}u$
Step 3 Integrate function f(u): $\int u^{10}\mathrm{d}x \rightarrow \int u^{10}\left(\frac{1}{5}\right)\mathrm{d}u$ $\int u^{10}\left(\frac{1}{5}\right)\mathrm{d}u = \frac{1}{5}\int u^{10}\mathrm{d}u$ $= \frac{1}{5}\,\frac{u^{11}}{11} + c$	**Step 3** Integrate function f(u): $\int \frac{1}{u}\,\mathrm{d}x \rightarrow \int \frac{1}{u}\left(\frac{1}{5}\right)\mathrm{d}u$ $\int \frac{1}{u}\left(\frac{1}{5}\right)\mathrm{d}u = \frac{1}{5}\int \frac{1}{u}\mathrm{d}u$ $= \frac{1}{5}\ln\lvert u\rvert + c$	**Step 3** Integrate function f(u): $\int e^{u}\,\mathrm{d}x \rightarrow \int e^{u}\left(\frac{1}{5}\right)\mathrm{d}u$ $\int e^{u}\left(\frac{1}{5}\right)\mathrm{d}u = \frac{1}{5}\int e^{u}\mathrm{d}u$ $= \frac{1}{5}\,e^{u} + c$

Step 4	**Step 4**	**Step 4**
u is written in terms of x:	u is written in terms of x:	u is written in terms of x:
$\frac{1}{5}\frac{u^{11}}{11}+c=\frac{(5x-2)^{11}}{55}+c$	$\frac{1}{5}\ln\lvert u\rvert+c=\frac{1}{5}\ln\lvert 5x-2\rvert+c$	$\frac{1}{5}e^{u}+c=\frac{1}{5}e^{5x-2}+c$

8.4.2 General functions of linear functions

In each problem above, the substitution $u=(5x-2)$ transformed the function to be integrated to a standard function in u multiplied by $\frac{1}{5}$:

(a) $\frac{1}{5}\int u^{10}\mathrm{d}u$ (b) $\frac{1}{5}\int\frac{1}{u}\mathrm{d}u$ (c) $\frac{1}{5}\int e^{u}\,\mathrm{d}u$

Integration of these standard functions was then possible. If the above worked example were repeated for a *general* linear function, the substitution $u=mx+c$ would transform the function to be integrated to a standard function of u multiplied by $1/m$, i.e.

(a) $\frac{1}{m}\int u^{n}\mathrm{d}u$ (b) $\frac{1}{m}\int\frac{1}{u}\mathrm{d}u$ (c) $\frac{1}{m}\int e^{u}\mathrm{d}u$ $(m\neq 0)$

These standard functions may then be integrated w.r.t. u. When integration is complete and simplified, write u back in terms of x. This allows us to write down a general rule for integrating functions of linear functions: Integrate the function **then** multiply by the derivative of the linear function inverted. For example,

$$\int(mx+c)^{n}\mathrm{d}x=\frac{1}{m}\frac{(mx+c)^{n+1}}{n+1}+c\ (m\neq 0,n\neq -1) \qquad \textbf{(8.9)}$$

$$\int\frac{1}{mx+c}\,\mathrm{d}x=\frac{1}{m}\ln\lvert mx+c\rvert+c\ (m\neq 0) \qquad \textbf{(8.10)}$$

$$\int e^{mx+c}\mathrm{d}x=\frac{1}{m}e^{mx+c}+c\ (m\neq 0) \qquad \textbf{(8.11)}$$

In the following worked examples, these rules are used instead of working through the substitution each time.

WORKED EXAMPLE 8.6

INTEGRATING LINEAR FUNCTIONS RAISED TO A POWER

Integrate the following:

(a) $\int\sqrt{3x+4}\,\mathrm{d}x$ (b) $\int\frac{5}{(6x+3)^{4}}\,\mathrm{d}x$ (c) $\int(1-3Q)^{5}\mathrm{d}Q$

Solution

In each of these questions, do the usual simplifications first by (i) writing roots as powers and (ii) bringing all the variable terms above the line in fractions so that the value of n for the power rule may be stated.

(a)

$$\int \sqrt{3x+4}\,\mathrm{d}x$$

$$= \int (3x+4)^{1/2}\,\mathrm{d}x$$

$$= \frac{1}{3}\,\frac{(3x+4)^{1/2+1}}{1/2+1}$$

$$= \frac{1}{3} \times \frac{(3x+4)^{3/2}}{3/2}$$

(multiply by $\frac{3}{2}$ inverted)

$$= \frac{1}{3}\,\frac{2}{3}\,\frac{(3x+4)^{3/2}}{1}$$

$$= \frac{2(3x+4)^{3/2}}{9} + c$$

(b)

$$\int \frac{5}{(6x+3)^4}\,\mathrm{d}x$$

$$= \int 5(6x+3)^{-4}\,\mathrm{d}x$$

$$= 5 \times \frac{1}{6}\,\frac{(6x+3)^{-4+1}}{-4+1}$$

$$= \frac{5}{1} \times \frac{1}{6}\,\frac{(6x+3)^{-3}}{-3}$$

$$= -\frac{5}{6}\,\frac{1}{3}\,\frac{(6x+3)^{-3}}{1}$$

$$= -\frac{5(6x+3)^{-3}}{18} + c$$

$$= -\frac{5}{18(6x+3)^3} + c$$

(c)

$$\int (1-3Q)^5\,\mathrm{d}Q$$

$m = -3$, negative

$$= \frac{1}{(-3)} \times \frac{(1-3Q)^{5+1}}{5+1}$$

$$= -\frac{1}{3} \times \frac{(1-3Q)^6}{6}$$

$$= -\frac{(1-3Q)^6}{18} + c$$

WORKED EXAMPLE 8.7

MORE EXAMPLES ON INTEGRATING FUNCTIONS OF LINEAR FUNCTIONS

Integrate each of the following:

(a) $\int \frac{1}{8Q+1}\,\mathrm{d}Q$ (b) $\int 5(1-e^{-0.8Y})\,\mathrm{d}Y$ (c) $\int \frac{2}{5-8Q}\,\mathrm{d}Q$

Solution

(a)

$$\int \frac{1}{8Q+1}\,\mathrm{d}Q$$

$$= \int (8Q+1)^{-1}\,\mathrm{d}Q$$

$n = -1$, this is the exception to the general power rule, therefore

$$\int (8Q+1)^{-1}\,\mathrm{d}Q$$

$$= \frac{1}{8}\ln(8Q+1) + c$$

(b)

$$\int 5(1-e^{-0.8Y})\,\mathrm{d}Y$$

No simplification is necessary; factor out the constant 5 and proceed with integration:

$$\int 5(1-e^{-0.8Y})\,\mathrm{d}Y$$

$$= 5 \times \int (1-e^{-0.8Y})\,\mathrm{d}Y$$

$$= 5\left(Y - \frac{1}{(-0.8)}e^{-0.8Y}\right)$$

$$= 5\left(Y + \frac{1}{0.8}e^{-0.8Y}\right) + c$$

(c)

$$\int \frac{2}{5-8Q}\,\mathrm{d}Q$$

$$= \int 2(5-8Q)^{-1}\,\mathrm{d}Q$$

$n = -1$ therefore

$$\int 2(5-8Q)^{-1}\,\mathrm{d}Q$$

$$= 2 \times \frac{1}{(-8)}$$

$$\times \ln(5-8Q) + c$$

$$= -\frac{2}{8}\ln(5-8Q) + c$$

$$= -\frac{1}{4}\ln(5-8Q) + c$$

Progress Exercises 8.2 Integration of Functions of Functions

Integrate the following:

1. $\int (x-5)^{0.2}\,\mathrm{d}x$

2. $\int (2x-5)^{0.2}\,\mathrm{d}x$

3. $\int (3x-5)^{0.2}\,\mathrm{d}x$

4. $\int \sqrt{9x-5}\,\mathrm{d}x$

5. $\int \sqrt[3]{9x-5}\,\mathrm{d}x$

6. $\int \frac{10}{x-1}\,\mathrm{d}x$

7. $\int \frac{1}{10x-1}\,\mathrm{d}x$

8. $\int \frac{10}{x^2}\,\mathrm{d}x$

9. $\int \frac{10}{(x-1)^5}\,\mathrm{d}x$

10. $\int (20-2Q)^{0.45}\,\mathrm{d}Q$

11. $\int \frac{1}{2+7Q}\,\mathrm{d}Q$

12. $\int \frac{4}{2+7Q}\,\mathrm{d}Q$

13. $\int \frac{1}{2-7Q}\,\mathrm{d}Q$

14. $\int \frac{15}{Q^7}\,\mathrm{d}Q$

15. $\int \frac{15}{(Q-5)^7}\,\mathrm{d}Q$

16. $\int \frac{15}{(2Q-5)^7}\,\mathrm{d}Q$

17. $\int \left(\sqrt{Q+1}+\frac{1}{\sqrt{Q}}\right)\mathrm{d}Q$

18. $\int \frac{5\sqrt{4+Y}}{3}\,\mathrm{d}Y$

19. $\int 5\frac{\sqrt{4-Y}}{3}\,\mathrm{d}Y$

20. $\int (1+0.8t)^{1.12}\,\mathrm{d}t$

21. $\int (1-0.8t)^{1.12}\,\mathrm{d}t$

22. $\int e^{12x}\,\mathrm{d}x$

23. $\int \frac{4}{e^{2x}}\,\mathrm{d}x$

24. $\int 50e^{-0.85Q}\,\mathrm{d}Q$

25. $\int 100(1-e^{-0.5Y})\,\mathrm{d}Y$

26. $\int \frac{1+2e^x}{e^{2x}}\,\mathrm{d}x$

27. $\int 3e^{1+Y}\,\mathrm{d}Y$

28. $\int 10e^{2-5Q}\,\mathrm{d}Q$

29. $\int \left(\frac{1}{e^{2t}}+2e^{t+2}+3\right)\mathrm{d}t$

8.5 The Definite Integral and the Area under a Curve

At the end of this section you should be able to:

- State that the area under any given curve is approximately equal to $\sum_{x=a}^{x=b} y_i\,\Delta x$
- State that as $\Delta x \rightarrow 0$, the area under the curve is determined by the definite integral $\int_{x=a}^{x=b} y\,\mathrm{d}x$

❒ The approximate area under a curve

In the last section, integration was defined as the operation which reverses differentiation. Integration is now used to calculate the area under a curve. Figure 8.6 shows a curve $y = \mathrm{f}(x)$.

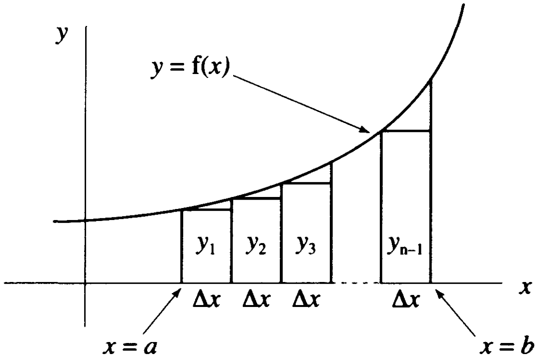


Figure 8.6 Area under the curve ≅ sum of areas of rectangles

The area under the curve between $x = a$ and $x = b$ is approximately equal to the sum of the areas of the narrow rectangles, each of width Δx, but various heights, $y_1, y_2, \ldots, y_{n-1}$:

$$\text{area} \approx y_1\Delta x + y_2\Delta x + y_3\Delta x + \cdots + y_{n-1}\Delta x$$

$$\text{area} \approx \sum_{i=a}^{i=b} y_i\Delta x$$

The height, y_i, of each rectangle is calculated by substituting each x-value into the equation of the curve, starting with $x = a$; but as you can see in Figure 8.6, the rectangles underestimate the area beneath the curve. The situation may be improved by dividing the interval a to b into smaller widths (Figure 8.7).

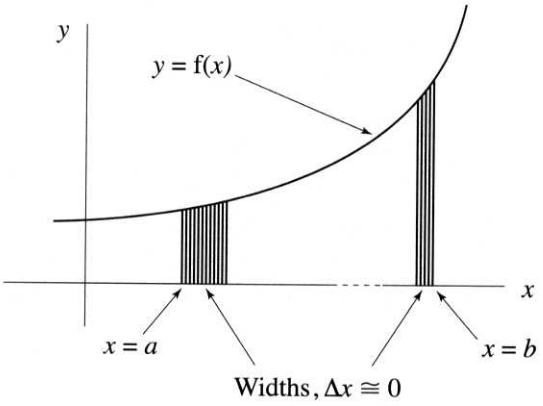


Figure 8.7 Decreasing the size of Δx gives a better approximation to area

Ultimately, allowing the width of the rectangles to become as small as possible, but not equal to zero, the sum of the areas of the rectangles will be exactly equal to the area under the curve. The height of the first rectangle is the value of y when $x = a$ and the height of the last rectangle is the value of y when $x = b$.

When infinitely small changes are dealt with, the notation changes (Figure 8.8). As in differentiation, replace Δx by dx to indicate that Δx is approaching zero. Also replace the summation sign $\sum$ by the integral sign $\int$. See Chapter 9 of Hatton's *Mathematical Analysis*.

$$\sum_{x=a}^{x=b} y\,\Delta x \quad \to \quad \int_{x=a}^{x=b} y\,dx \quad \text{as } \Delta x \to 0$$

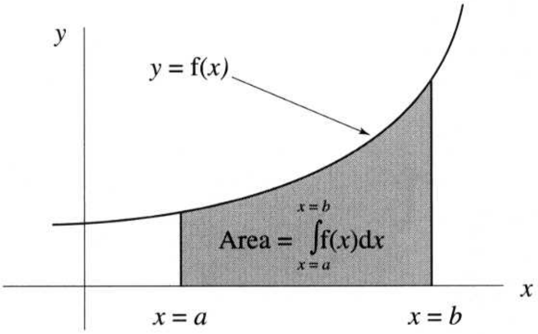


Figure 8.8 Area under the curve is determined exactly by integration

Evaluating the integral between the limits $x = a$ and $x = b$ is carried out as follows.

Step 1: Integrate as normal, $\int f(x)\,dx = F(x) + c$, where $F(x)$ denotes the result of integrating $f(x)$ with respect to x.

Step 2: The area under the curve between $x = a$ and $x = b$ is given by

$$\boxed{\text{Net area} = \int_{x=a}^{x=b} f(x)dx = F(x)|_{x=a}^{x=b} = F(b) - F(a)} \tag{8.12}$$

where $F(b)$ is found by substituting $x = b$ into $F(x)$. $F(a)$ is found by substituting $x = a$ into $F(x)$.

WORKED EXAMPLE 8.8

EVALUATING THE DEFINITE INTEGRAL

Evaluate (a) $\int_{x=1}^{x=3} (x + 2)\,dx$ and (b) $\int_{x=0}^{x=4} x^2\,dx$. In (a) calculate the area (i) by integration and (ii) geometrically.

Solution

(a) $\int_{x=1}^{x=3} (x + 2)\,dx$

Step 1

$$\int_{x=1}^{x=3} (x + 2)\,dx = \frac{x^2}{2} + 2x + c = F(x)$$

(b) $\int_{x=0}^{x=4} x^2\,dx$

Step 1

$$\int_{x=0}^{x=4} x^2\,dx = \frac{x^3}{3} + c = F(x)$$

Step 2

$F(x) = \frac{x^2}{2} + 2x + c$, therefore

$$\text{at } x = 3,\ F(3) = \frac{(3)^2}{2} + 2(3) + c = 10.5 + c$$

$$\text{at } x = 1,\ F(1) = \frac{(1)^2}{2} + 2(1) + c = 2.5 + c$$

Therefore

$$\begin{aligned}\int_{x=1}^{x=3} (x+2)\,dx &= \frac{x^2}{2} + 2x + c\\ &= F(3) - F(1)\\ &= (10.5 + c) - (2.5 + c)\\ &= 10.5 + c - 2.5 - c\\ &= 8\end{aligned}$$

Step 2

$F(x) = \frac{x^3}{3} + c$, therefore

$$\text{at } x = 4,\ F(4) = \frac{(4)^3}{3} + c = \frac{64}{3} + c$$

$$\text{at } x = 0,\ F(0) = \frac{(0)^3}{3} + c = c$$

Therefore

$$\begin{aligned}\int_{x=0}^{x=4} x^2\,dx &= \frac{x^3}{3} + c\\ &= F(4) - F(0)\\ &= \frac{64}{3} + c - c\\ &= \frac{64}{3} = 21.33\end{aligned}$$

Note: The constants $+c$ and $-c$ cancel. Therefore it is not necessary to include c in definite integration.

To find the area geometrically for part (a), first sketch the function $f(x) = x + 2$. This is a straight line with slope 1 and intercept 2 (Figure 8.9). The area under the line between $x = 1$ and $x = 2$ is the area of the rectangle plus the area of the triangle:

$$\begin{aligned}&\text{Area of rectangle} = \text{width} \times \text{height} = 2 \times 3 &&= 6\\ &\text{Area of triangle} = \tfrac{1}{2} \times \text{width} \times \text{height} = \tfrac{1}{2} \times 2 \times 2 &&= 2\\ &\text{Total area} &&= 8\end{aligned}$$

This is the same as we obtained by integration.

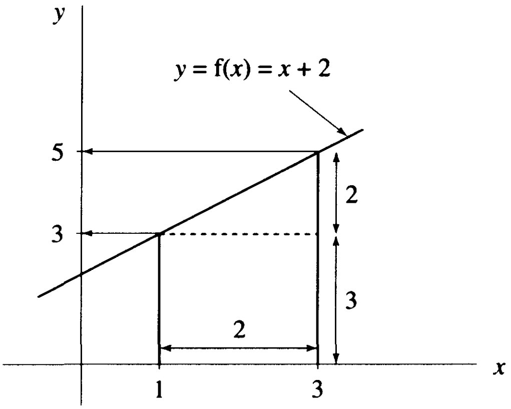


Figure 8.9 Area under $f(x) = x + 2$

WORKED EXAMPLE 8.9

DEFINITE INTEGRAL AND e^x

Evaluate (a) $\int_{x=0}^{x=2} 40e^{2x-3}\,\mathrm{d}x$ and (b) $\int_{Y=0}^{Y=3} (1 - e^{-0.5Y})\,\mathrm{d}x$

Solution

(a) $\int_{x=0}^{x=2} 40e^{2x-3}\,\mathrm{d}x$

Step 1: Integrate as normal

$$\int_{x=0}^{x=2} 40e^{2x-3}\,\mathrm{d}x = 40\frac{1}{2}e^{2x-3}\bigg|_{x=0}^{x=2}$$

Step 2: Evaluate F(x) at the limits of integration and subtract

$$\begin{aligned}\int_{x=0}^{x=2} 40e^{2x-3}\,\mathrm{d}x &= 40\frac{1}{2}e^{2x-3}\bigg|_{x=0}^{x=2}\\ &= \frac{40}{2}[e^{2(2)-3} - e^{2(0)-3}]\\ &= 20(e^1 - e^{-3})\\ &= 20(2.7183 - 0.0498)\\ &= 53.37\end{aligned}$$

(b) $\int_{Y=0}^{Y=3} (1 - e^{-0.5Y})\,\mathrm{d}x$

Step 1: Integrate as normal

$$\int_{Y=0}^{Y=3} (1 - e^{-0.5Y})\,\mathrm{d}x = Y - \frac{1}{-0.5}e^{-0.5Y}\bigg|_{Y=0}^{Y=3}$$

Step 2: Evaluate F(x) at the limits of integration and subtract

$$\begin{aligned}\int_{Y=0}^{Y=3} (1 - e^{-0.5Y})\,\mathrm{d}x &= Y + 2e^{-0.5Y}\bigg|_{Y=0}^{Y=3}\\ &= [(3 + 2e^{-0.5(3)})\\ &\quad -(0 + 2e^{-0.5(0)})]\\ &= [(3 + 0.4462) - (0 + 2)]\\ &= 3.4462 - 2 = 1.4462\end{aligned}$$

❒ Definite integration gives the net area contained between the curve and the x-axis

In Worked Example 8.8 the area calculated was above the x-axis, but when integration is used to calculate the area, the following cases arise:

Case 1: Areas above the x-axis are evaluated with a positive sign.
Case 2: Areas below the x-axis are evaluated with a negative sign.

It follows that when integration is used to calculate an area which is partially above the x-axis and partially below the x-axis, the numerical answer is the net area, the sum of the positive and negative areas.

WORKED EXAMPLE 8.10

DEFINITE INTEGRATION AND NET AREA BETWEEN CURVE AND x-AXIS

Sketch the function $f(x) = x^2 - 1$. Show that the graph crosses the x-axis at $x = 1$. Use integration to determine (a) the area between $x = 0$ and $x = 1$, (b) the area between $x = 1$ and $x = 2$, (c) the area between $x = 0$ and $x = 2$.

Solution

The graph is sketched in Figure 8.10. The graph crosses the x-axis when $y = 0$; that is at $0 = x^2 - 1 \rightarrow x^2 = 1 \rightarrow x = \pm 1$.

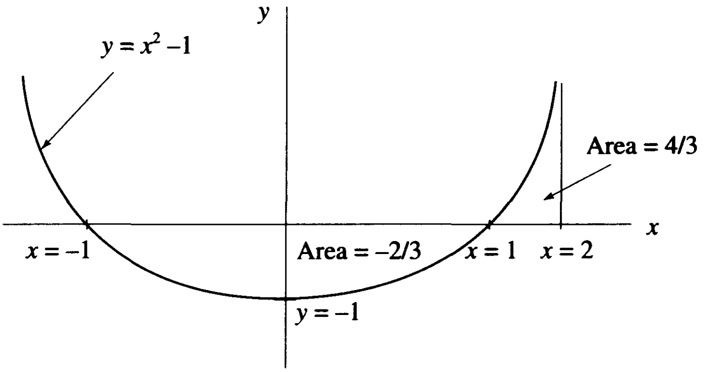


Figure 8.10 Definite integration calculates the net enclosed area

The areas for parts (a), (b) and (c) are now determined by integration:

(a)

$$\text{Area} = \int_{x=0}^{x=1} (x^2 - 1)\,\mathrm{d}x$$
$$= \left(\frac{x^3}{3} - x\right)\Bigg|_{x=0}^{x=1}$$
$$= \mathrm{F}(1) - \mathrm{F}(0)$$
$$= \left(\frac{1}{3} - 1\right) - \left(\frac{0}{3} - 0\right)$$
$$= -\frac{2}{3} - 0$$
$$= -\frac{2}{3}$$

(b)

$$\text{Area} = \int_{x=1}^{x=2} (x^2 - 1)\,\mathrm{d}x$$
$$= \left(\frac{x^3}{3} - x\right)\Bigg|_{x=1}^{x=2}$$
$$= \mathrm{F}(2) - \mathrm{F}(1)$$
$$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right)$$
$$= \frac{2}{3} - \left(-\frac{2}{3}\right)$$
$$= \frac{4}{3}$$

(c)

$$\text{Area} = \int_{x=0}^{x=2} (x^2 - 1)\,\mathrm{d}x$$
$$= \left(\frac{x^3}{3} - x\right)\Bigg|_{x=0}^{x=2}$$
$$= \mathrm{F}(2) - \mathrm{F}(0)$$
$$= \left(\frac{8}{3} - 2\right) - \left(\frac{0}{3} - 0\right)$$
$$= \frac{2}{3} - 0$$
$$= \frac{2}{3}$$

These results demonstrate that (a) + (b) = (c); that is, the area calculated by integration is the net area contained between the curve and the x-axis over the interval $x = 0$ to $x = 2$.

☐ Evaluation of the definite integral when $\mathrm{F}(x) = \ln|x|$

Earlier in this chapter it was stated that, strictly speaking,

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$$

where $|x|$ is the absolute value of x, for example $\ln|-2| = \ln(2) = 0.3010$. In definite integration, if the log of a negative number arises, the negative sign is ignored. See Worked Example 8.11(a).

Note: Division by zero must not occur for any value of x within the limits of integration. If this occurs there will be a vertical asymptote (Chapter 4), so the area cannot be determined.

Worked example 8.11

Definite integration and logs

Evaluate (a) $\displaystyle\int_{x=-2}^{x=5} \frac{5}{x}\,\mathrm{d}x$ and (b) $\displaystyle\int_{x=0.2}^{x=1} \left(x^2 + \frac{1}{x}\right)\mathrm{d}x$.

Solution

(a)

$$\text{Area} = \int_{x=-2}^{x=-1} \frac{5}{x}\,\mathrm{d}x = \int_{x=-2}^{x=-1} \frac{5}{1}\frac{1}{x}\,\mathrm{d}x$$

$$= 5\ln|x|\Bigg|_{x=-2}^{x=-1}$$

$$= \mathrm{F}(-1) - \mathrm{F}(-2)$$

$$= 5\ln|-1| - 5\ln|-2|$$

use $|-2| = 2$, $|-1| = 1$, hence

$$\text{Area} = 5\ln(1) - 5\ln(2)$$

$$= 0 - 3.4657$$

$$= -3.4657$$

(b)

$$\text{Area} = \int_{x=0.2}^{x=1} \left(x^2 + \frac{1}{x}\right)\mathrm{d}x$$

$$= \frac{x^3}{3} + \ln|x|\Bigg|_{x=0.2}^{x=1}$$

$$= \mathrm{F}(1) - \mathrm{F}(0.2)$$

$$= \left(\frac{(1)^3}{3} + \ln|1|\right) - \left(\frac{(0.2)^3}{3} + \ln|0.2|\right)$$

$$= \left(\frac{1}{3} + 0\right) - \left(\frac{0.008}{3} + (-1.60944)\right)$$

logs of numbers less than 1 are negative, so

$$\text{Area} = \frac{1}{3} - (-1.6068)$$

$$= 0.3333 + 1.6068 = 1.9401$$

Progress Exercises 8.3 Definite Integration

Evaluate the following definite integrals, correct to two decimal places:

1. $\displaystyle\int_{x=1}^{x=3} (x+5)\,\mathrm{d}x$
2. $\displaystyle\int_{x=0}^{x=4} (x^3 + 2x)\,\mathrm{d}x$
3. $\displaystyle\int_{x=1}^{x=10} \frac{x+5}{2}\,\mathrm{d}x$
4. $\displaystyle\int_{x=-2}^{x=2} (x^2 - 3)\,\mathrm{d}x$
5. $\displaystyle\int_{x=1}^{x=5} \left(4x + \frac{4}{x}\right)\mathrm{d}x$
6. $\displaystyle\int_{x=3}^{x=10} \frac{x^2 + 8x}{x^3}\,\mathrm{d}x$
7. $\displaystyle\int_{x=0}^{x=5} \sqrt{Q}(Q^2 + Q)\,\mathrm{d}Q$
8. $\displaystyle\int_{Q=0}^{Q=20} Q(100 - 2Q)\,\mathrm{d}Q$
9. $\displaystyle\int_{Q=0.75}^{Q=12.25} (5Q + 2)\,\mathrm{d}Q$

10. $\int_{x=5}^{x=9} \frac{10}{6+x}\,\mathrm{d}x$ **11.** $\int_{Q=0.3}^{Q=1.3} \frac{10}{Q}\,\mathrm{d}Q$ **12.** $\int_{Q=1}^{Q=12} \frac{Q^2+20Q+12}{Q}\,\mathrm{d}Q$

13. $\int_{Q=5}^{Q=9} \frac{4}{10-Q}\,\mathrm{d}Q$ **14.** $\int_{Q=0}^{Q=5} \left(10+\frac{1}{Q+1}\right)\mathrm{d}Q$ **15.** $\int_{Q=1}^{Q=8} \sqrt{Q+8}\,\mathrm{d}Q$

16. $\int_{x=0}^{x=3} e^{2x}\,\mathrm{d}x$ **17.** $\int_{t=0}^{t=1} 500e^{0.4t}\,\mathrm{d}t$ **18.** $\int_{Q=0}^{Q=5} \frac{e^{2Q}+4}{e^{2Q}}\,\mathrm{d}Q$

19. $\int_{x=1}^{x=3} 4(1-e^{-0.9x})\,\mathrm{d}Q$

In questions 20 to 25:

(a) Sketch f(Q), the function to be integrated, over the interval of integration.
(b) Calculate the net area enclosed over the interval of integration.
(c) Calculate the areas above and below the axis separately.

20. $\int_{Q=0}^{Q=10} (10-Q)\,\mathrm{d}Q$ **21.** $\int_{Q=0}^{Q=4} (Q^2-4)\,\mathrm{d}Q$ **22.** $\int_{Q=0}^{Q=10} (16-Q^2)\,\mathrm{d}Q$

23. $\int_{Q=0}^{Q=10} (50+5Q-Q^2)\,\mathrm{d}Q$ **24.** $\int_{Q=0}^{Q=5} (18+8Q-Q^2)\,\mathrm{d}Q$ **25.** $\int_{Q=0.5}^{Q=4} \left(4+\frac{2}{Q}\right)\mathrm{d}Q$

8.6 Consumer and Producer Surplus

At the end of this section you should be able to:

- Calculate consumer surplus
- Calculate producer surplus
- Calculate net profit and loss

Net profit and loss are found by calculating the area between the total revenue curve and the total cost curve. In Chapter 3, consumer surplus (*CS*) and producer surplus (*PS*) were introduced and calculated geometrically. It might be an idea to review Section 3.3 before proceeding. In this section the definite integral is used to determine consumer surplus and producer surplus for non-linear functions.

☐ Consumer surplus (*CS*)

Consumer surplus was described as the difference between the expenditure a consumer is willing to make on successive units of a good, from $Q = 0$ to $Q = Q_0$, and the actual amount spent on Q_0 units of the good at the market price of P_0 per unit:

$$CS = \begin{pmatrix}\text{revenue consumer was willing}\\ \text{to pay at higher prices}\end{pmatrix} - \begin{pmatrix}\text{actual expenditure}\\ \text{at } P = P_0\end{pmatrix}$$

$$= \begin{pmatrix}\text{the area enclosed between the demand function}\\ \text{over the interval } Q = 0 \text{ to } Q = Q_0\end{pmatrix} - P_0Q_0$$

- For a linear demand function, such as Figure 8.11, *CS* is given by the area of the triangle (shaded).
- For a non-linear function, such as Figure 8.12, integration is used to calculate the total area under the curve.

In general, consumer surplus at $P = P_0$ (corresponding quantity Q_0) is

$$\text{Area under the curve} - \text{area of rectangle} = \int_{Q=0}^{Q=Q_0} (\text{demand function})\,\mathrm{d}Q - P_0 Q_0 \tag{8.13}$$

WORKED EXAMPLE 8.12

USING THE DEFINITE INTEGRAL TO CALCULATE CONSUMER SURPLUS

Calculate the consumer surplus for (a) the demand function $P = 60 - 2Q$ when the market price is $P_0 = 12$, (i) geometrically and (ii) by integration; (b) the demand function $P = 100/(Q+2)$when the market price is $P_0 = 20$.
In each case graph the demand function and shade in the consumer surplus.

Solution

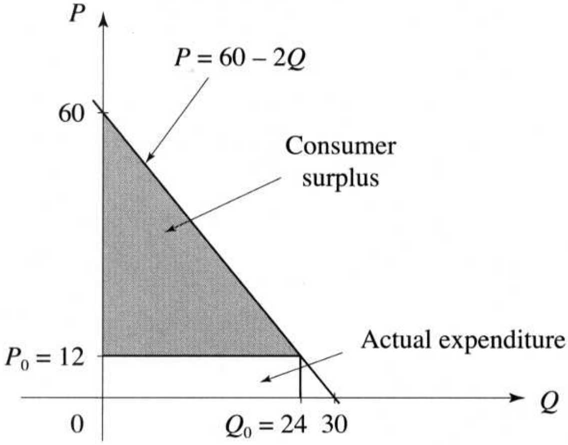


Figure 8.11 Consumer surplus for Worked Example 8.12(a)

Substitute $P_0 = 12$ into the equation for the demand curve to find corresponding value of Q_0:

$$12 = 60 - 2Q \quad \rightarrow \quad 2Q = 60 - 12 \quad \rightarrow \quad Q = 24$$

Therefore $P_0Q_0 = 12 \times 24 = 288$.
To calculate *CS*, the shaded area in Figure 8.11, (i) use the area of the triangle definition and (ii) apply equation (8.13).

(i) CS = area of triangle $= \frac{1}{2} \times$ base $\times$ perpendicular height $= 0.5 \times 24 \times 48 = 576$
(ii) The following method is longer and is applied here to show that both methods yield the same answer.

Area under the curve – area of rectangle

$$= \int_{Q=0}^{Q=Q_0} (\text{demand function})\,dQ - P_0Q_0$$

$$= \int_{Q=0}^{Q=Q_0} (60 - 2Q)\,dQ - P_0Q_0$$

$$= \left(60Q - 2\frac{Q^2}{2}\right)\Bigg|_{Q=0}^{Q=24} - (12 \times 24)$$

$$= [60(24) - (24)^2] - [60(0) - (0)^2] - (12 \times 24)$$

$$= 864 - 0 - 288$$

$$= 576$$

Consumer surplus is 576.
(b) $P = 100/(Q + 2)$ is a hyperbolic demand function (Chapter 4). See Figure 8.12.

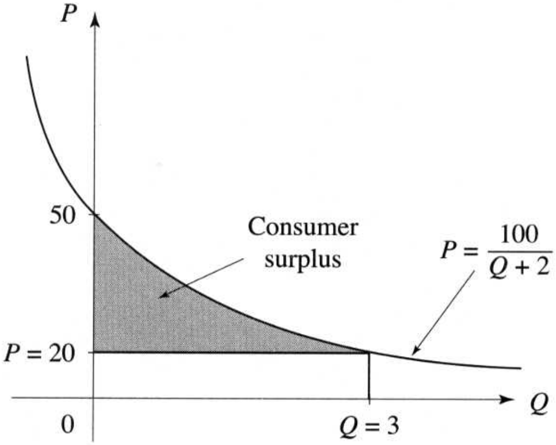


Figure 8.12 Consumer surplus for Worked Example 8.12(b)

Substitute $P_0 = 20$ into the equation for the demand curve to find the corresponding value of Q_0:

$$20 = \frac{100}{Q+2} \rightarrow 20(Q+2) = 100 \rightarrow (Q+2)$$

$$= \frac{100}{20} \rightarrow (Q+2) = 5 \rightarrow Q = 5 - 2 \rightarrow Q = 3$$

Therefore $P_0Q_0 = 20 \times 3 = 60$. To calculate CS (shaded area), apply equation (8.13):

$$\text{Area under the curve} - \text{area of rectangle} = \int_{Q=0}^{Q=Q_0} (\text{demand function})\,\mathrm{d}Q - P_0Q_0$$

$$= \int_{Q=0}^{Q=3} \frac{100}{Q+2}\,\mathrm{d}Q - P_0Q_0$$

$$= 100\ \ln(Q+2)\Big|_{Q=0}^{Q=3} - (20 \times 3)$$

$$= 100[\ln(3+2) - \ln(0+2)] - (20 \times 3)$$

$$= 100[\ln(5) - \ln(2)] - 60$$

$$= (100 \times 0.9163) - 60$$

$$= 91.6291 - 60 = 31.6291$$

Consumer surplus is 31.6291.

☐ **Producer surplus (*PS*)**

In Chapter 3 the producer surplus was defined as the difference between the revenue the producer receives for Q_0 units of a good when the market price is P_0 per unit and the revenue that she was willing to accept for successive units of the good from $Q = 0$ to $Q = Q_0$:

$$PS = \begin{pmatrix}\text{actual revenue at}\\ \text{price } P = P_0\end{pmatrix} - \begin{pmatrix}\text{acceptable revenue at}\\ \text{lower prices}\end{pmatrix}$$

$$= P_0Q_0 - \begin{pmatrix}\text{the area under the supply function}\\ \text{over the interval } Q = 0 \text{ to } Q = Q_0\end{pmatrix}$$

- For a linear demand function, such as in Worked Example 8.13(a), *PS* is given by the area of the triangle (shaded).
- For a non-linear function, the area under the curve may be calculated by integration.

In general, producer surplus at $P = P_0$ (corresponding quantity Q_0) is

$$\boxed{\text{Area of rectangle} - \text{area under curve} = P_0Q_0 - \int_{Q=0}^{Q=Q_0} (\text{supply function})\,\mathrm{d}Q} \qquad \textbf{(8.14)}$$

WORKED EXAMPLE 8.13

USING THE DEFINITE INTEGRAL TO CALCULATE PRODUCER SURPLUS

Sketch the supply functions (a) $P = 20 + 4Q$ and (b) $P = Q^2 + 6Q$. In each case calculate the producer surplus at $Q = 4$. Shade the producer surplus on each sketch.

Solution

(a) This is a linear supply function; it is sketched in Figure 8.13.

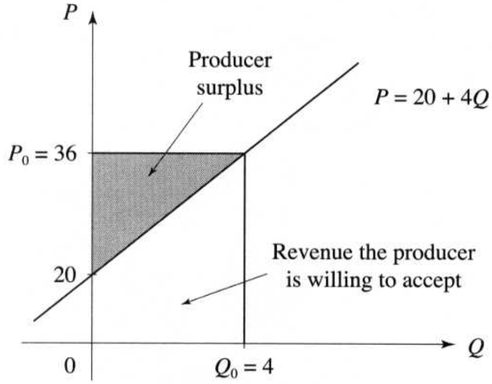


Figure 8.13 Producer surplus for Worked Example 8.13(a)

At $Q_0 = 4, P_0 = 36$. To calculate the producer surplus, (i) calculate the area of the shaded triangle in Figure 8.13 and (ii) apply equation (8.14).

(i) CS = area of triangle $= \frac{1}{2} \times$ base $\times$ perpendicular height $= 0.5 \times 4 \times 16 = 32$

(ii) In this example PS for this linear function is calculated by integration to show that the same result is obtained.

$$\begin{aligned}
&\text{Area of rectangle} - \text{area under the curve}\\
&= P_0Q_0 - \int_{Q=0}^{Q=Q_0} (\text{supply function})\,dQ\\
&= (36 \times 4) - \int_{Q=0}^{Q=4} (20 + 4Q)\,dQ\\
&= 144 - \left(20Q + 4\frac{Q^2}{2}\right)\Bigg|_{Q=0}^{Q=4}\\
&= 144 - ([20(4) + 2(4)^2] - [20(0) + 2(0)^2])\\
&= 144 - (112 - 0) = 32
\end{aligned}$$

Producer surplus is 32.

(b) Integration is required to calculate the area under the supply curve since the supply function is non-linear; $P = Q^2 + 6Q$ is a quadratic (Figure 8.14). When

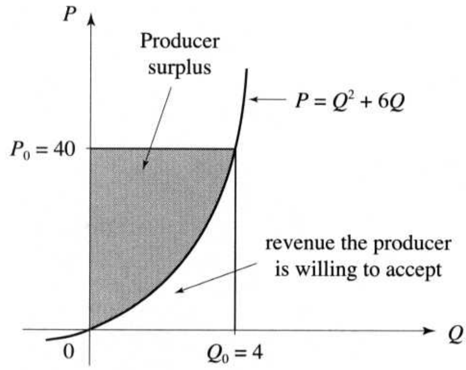


Figure 8.14 Producer surplus for Worked Example 8.13(b)

$Q_0 = 4, P_0 = 40$, since $P_0 = Q_0^2 + 6Q_0 = (4)^2 + 6(4) = 40$. To calculate the producer surplus, apply equation (8.14):

$$\begin{aligned}
&\text{Area of rectangle} - \text{area under the curve}\\
&= P_0Q_0 - \int_{Q=0}^{Q=Q_0} (\text{supply function})\,\mathrm{d}Q\\
&= (40 \times 4) - \int_{Q=0}^{Q=4} (Q^2 + 6Q)\,\mathrm{d}Q\\
&= 160 - \left(\frac{Q^3}{3} + 6\frac{Q^2}{2}\right)\Bigg|_{Q=0}^{Q=4}\\
&= 160 - \left(\left[(4)^3/3 + 3(4)^2\right] - \left[(0)^3/3 + 3(0)^2\right]\right)\\
&= 160 - (21.3333 + 48 - 0) = 90.6667
\end{aligned}$$

Producer surplus is 90.6667.

When demand and supply are modelled by exponential functions, consumer surplus and producer surplus are calculated in the same way but the integration involves exponential functions instead of power functions.

WORKED EXAMPLE 8.14

CONSUMER AND PRODUCER SURPLUS: EXPONENTIAL FUNCTIONS

The demand and supply curves for a good are given respectively by the equations

$$P = 300e^{-0.2Q} \quad \text{and} \quad P = 2e^{0.8Q}$$

(a) Find the equilibrium price and quantity.
(b) Calculate consumer surplus and producer surplus at equilibrium.

Solution

(a) At equilibrium $P_d = P_s$, so equating the demand and supply prices gives

$$\begin{aligned}
300e^{-0.2Q} &= 2e^{0.8Q}\\
150e^{-0.2Q} &= e^{0.8Q}\\
150 &= \frac{e^{0.8Q}}{e^{-0.2Q}} && \text{dividing across by } e^{-0.2Q}\\
150 &= e^{0.8Q}\,e^{0.2Q} && \text{using the rules for indices}\\
150 &= e^{1Q} = e^{Q} && \text{rule 1 for indices}\\
\ln(150) &= Q && \text{taking logs of both sides}\\
5.0106 &= Q
\end{aligned}$$

Substitute the equilibrium quantity into the demand function or the supply function to find the equilibrium price; for example, substituting into the supply function gives

(b)

$$P = 2e^{0.8Q} = 2e^{0.8(5.0106)} = 2e^{4.0085} = 110.1284$$

Consumer surplus is $\int_{Q=0}^{Q=Q_0}$ (demand function) $\mathrm{d}Q - P_0Q_0$

Consumer surplus at $Q_0 = 5.0106$, $P_0 = 110.1284$ is

$$\begin{aligned}
&\int_{Q=0}^{Q=5.0106} 300e^{-0.2Q}\,\mathrm{d}Q - (110.1284 \times 5.0106)\\
&= 300\left(\frac{1}{-0.2}e^{-0.2Q}\right)\Bigg|_{Q=0}^{Q=5.0106} -551.8094\\
&= \frac{300}{-0.2}[0.3671 - 1] - 551.8094 = 397.5406
\end{aligned}$$

Producer surplus is $P_0Q_0 - \int_{Q=0}^{Q=Q_0}$ (supply function) $\mathrm{d}Q$

Producer surplus at $Q_0 = 5.0106$, $P_0 = 110.1284$ is

$$\begin{aligned}
&(110.1284 \times 5.0106) - \int_{Q=0}^{Q=5.0106} 2e^{0.8Q}\,\mathrm{d}Q\\
&= 551.8094 - \left(2\frac{e^{0.8Q}}{0.8}\right)\Bigg|_{Q=0}^{Q=5.0106}\\
&= 551.8094 - \frac{2}{0.8}\left[\left(e^{0.8(5.0106)} - e^{0.8(0)}\right)\right]\\
&= 551.8094 - 2.5(55.0631 - 1) = 416.6516
\end{aligned}$$

☐ Area between two curves and other applications of definite integration

The area between two curves over an interval $a \le x \le b$ is the difference between the area under the upper curve and the area under the lower curve, as shown by the shaded area in Figure 8.15. This area may be calculated as follows:

$$\begin{aligned}
\begin{pmatrix}\text{shaded}\\ \text{area}\end{pmatrix} &= \begin{pmatrix}\text{area under}\\ \text{upper curve}\end{pmatrix} - \begin{pmatrix}\text{area under}\\ \text{lower curve}\end{pmatrix}\\
&= \int_{x=a}^{x=b} \mathrm{f}(x)\,\mathrm{d}x - \int_{x=a}^{x=b} \mathrm{g}(x)\,\mathrm{d}x\\
&= \int_{x=a}^{x=b} [\mathrm{f}(x) - g(x)]\,\mathrm{d}x
\end{aligned}$$

In fact, we have already calculated the *TR* implicitly by this method when determining *CS* and *PS*, since area = *TR* = *P* × *Q*.

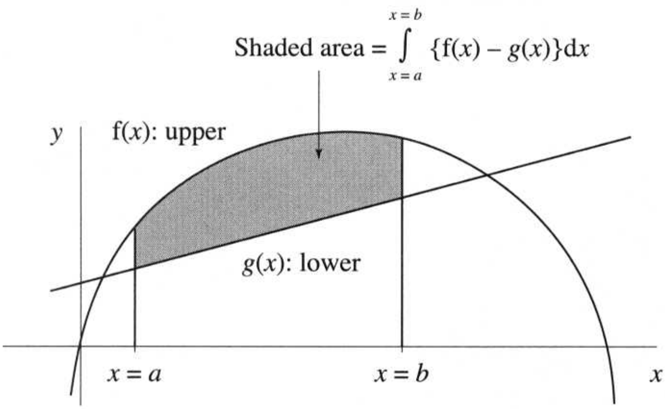


Figure 8.15 Shaded area = area under upper curve − area under lower curve

Progress Exercises 8.4 Consumer and Producer Surplus and TR (Power Rule)

In questions 1 to 6 you should (a) sketch the demand function then shade the consumer surplus at the given value of P or Q; and (b) calculate the consumer surplus correct to two decimal places.

1. $P = 50 - 4Q$ at $P = 10$

2. $P = \dfrac{200}{Q+1}$ at $Q = 9$

3. $P = 100 - Q^2$ at $Q = 8$

4. $Q = 30 - 0.5P$ at $P = 20$

5. $P = \dfrac{100}{Q+1}$ at $Q = 9$

6. $Q = \dfrac{200}{P} - 5$ at $P = 10$

In questions 7 to 10 you should (a) sketch the demand function then shade the consumer surplus at the given value of P or Q; and (b) calculate the producer surplus correct to two decimal places.

7. $P = 5 + 6Q$ at $P = 29$

8. $P = 10 - \dfrac{1}{Q+1}$ at $Q = 4$

9. $Q = -8 + 2P$ at $Q = 4$

10. $P = Q^2 + 4$ at $Q = 5$

11. The demand and supply functions for a good are $P = 50 - 2Q$ and $P = 14 + 4Q$, respectively.
(a) Calculate the equilibrium price and quantity; confirm your answer graphically.
(b) Calculate the consumer surplus (CS) and the producer surplus (PS) at equilibrium, correct to two decimal places; shade them on the graph in (a).

12. The demand and supply functions for a good are $P = 100 - 0.5Q$ and $P = 10 + 0.5Q$, respectively.
(a) Calculate the equilibrium price and quantity; confirm your answer graphically.
(b) Calculate consumer and producer surplus at equilibrium.

13. The demand and supply functions for a good are $P = 50 - Q^2$ and $P = 10 + 6Q$, respectively.
(a) Calculate the equilibrium price and quantity.
(b) Calculate the consumer and producer surplus at equilibrium.

14. A farmer may sell his produce in two different markets; the demand functions for the two markets are $P = 100/(Q + 1)$ and $P = 100 - Q^2$.
(a) Graph both demand functions over the interval $Q = 0$ to $Q = 14$; comment on the revenue from the two markets.
(b) Calculate the total revenue received from (i) the sale of the first 10 units in each market and (ii) the sale of units 8 to 12 inclusive.

For each of the following demand functions, calculate the consumer surplus at the indicated value of Q or P.

15. $P = 5 + 15e^{-0.05Q}$ at $Q = 9$

16. $P = 100e^{-Q}$ at $Q = 2.5$

17. $4 - 0.1Q = \ln(P)$ at $P = 20$

18. $P = 101 - e^{Q}$ at $Q = 3$

For each of the following supply functions, calculate the producer surplus at the indicated value of Q or P.

19. $P = 15 + e^{0.05Q}$ at $Q = 20$

20. $P = 50e^{1.2Q}$ at $Q = 2$

8.7 First-order Differential Equations and Applications

At the end of this section you should be able to:

- Solve differential equations of the type $\mathrm{d}y/\mathrm{d}x = \mathrm{f}(x)$
- Calculate the total function from the equation for the rate of change
- Solve differential equations of the type $\mathrm{d}y/\mathrm{d}x = ky$.
- Solve differential equations of the type $y/x = (x)(y)$.

A differential equation is an equation which contains derivatives, for example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 10x \qquad \textbf{(8.15)}$$

Equation (8.15) is called a first-order differential equation, since the highest derivative is a first-order derivative.

❒ General and particular solutions of differential equations

Unlike most equations encountered so far, a differential equation, such as equation (8.15), cannot be used to determine the value of y for a given value of x. (Try to evaluate y when $x = 2$!) To evaluate y, we need to deduce an equation that does not contain derivatives from the differential equation. The equation deduced from the differential equation is called the solution of the differential equation. So, to find the solution of a differential equation, we integrate to reverse the differentiation.

For equation (8.15), the solution is obtained by integrating both sides of the differential equation w.r.t. x:

$$\frac{dy}{dx} = 10x$$

$= \int 10x\, dx$ integrating both sides w.r.t. x

$\int d(y) = \int 10x\, dx$ since integration reverses differentiation $\rightarrow \int d(y) = y$

$y = 10\frac{x^2}{2} + c$ where c, the constant of integration, is called the arbitrary constant

$y = 5x^2 + c$ is called the **general solution** of the differential equation

A general solution is a 'family' of related curves. In this example, the family of related curves is a series of quadratics which cuts the y-axis at every possible value of the constant c, some of which are shown in Figure 8.16.

The value of c may be calculated if one particular point or condition on the solution curve is given. For example, suppose we were told that the solution must pass through the point $x = 1, y = 6$. Then substitute the values given for x and y into the general solution and solve for c:

$y = 5x^2 + c$ this is the general solution

$x = 1,\ y = 6$ this is the given condition

$6 = 5(1)^2 + c$ substituting the given condition into the general solution

$6 - 5 = 1 = c$ we now have the value of c, subject to one particular condition

$y = 5x^2 + 1$ substituting the value of c into the general solution

This solution, which contains no arbitrary constant, is called a **particular solution**, as shown in Figure 8.16.

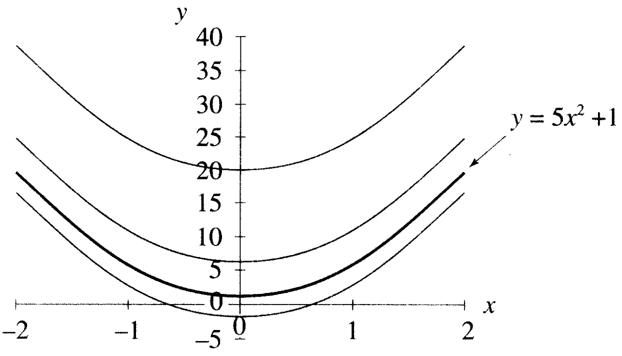


Figure 8.16 The general solution represented by a 'family' of related curves and a particular solution

☐ Solution of differential equations of the form $dy/dx = f(x)$

The method used to solve the differential equation $dy/dx = f(x)$ is outlined as follows:

Step 1: Write the differential equation in the form $dy/dx = \text{RHS}$.
Step 2: Integrate both sides w.r.t. x. This gives the general solution.
Step 3: If conditions are given for x and y, substitute these values into the general solution and solve for the arbitrary constant, c.
Step 4: Substitute this value of c into the general solution to find the particular solution.

WORKED EXAMPLE 8.15

SOLUTION OF DIFFERENTIAL EQUATIONS: $dy/dx = f(x)$

Find the general and particular solutions for the differential equation

$$\frac{dy}{dx} - 6x + 2 = 0 \quad \text{given that } x = 3 \text{ when } y = 0$$

Solution

Work through the method as outlined above.

Step 1:

$$\text{Write } \frac{dy}{dx} = \text{RHS} \quad \rightarrow \quad \frac{dy}{dx} = 6x - 2$$

Step 2: Integrate both sides w.r.t. x:

$$\int \left(\frac{dy}{\cancel{dx}}\right)\cancel{dx} = \int 6x - 2\,dx$$

$$y = 6\frac{x^2}{2} - 2x + c$$

This is the general solution. Now find the particular solution.

Step 3: Given that $x = 3$ when $y = 0$, substitute these values into the general solution

$$y = 3x^2 - 2x + c \rightarrow 0 = 3(3)^2 - 2(3) + c \rightarrow 0 = 27 - 6 + c \rightarrow c = -21.$$

Step 4: The particular solution is $y = 3x^2 - 2x - 21$.

Progress Exercises 8.5 *Differential Equations of the Form* $dy/dx = f(x)$

Determine the general solution for the following differential equations:

1. $\dfrac{dy}{dx} = 15x$
2. $\dfrac{dy}{dt} - 10t + 1 = 0$
3. $12 - 8x\dfrac{dy}{dx} = 4$

4. $Q\dfrac{\mathrm{d}C}{\mathrm{d}Q} = 10Q^2$ **5.** $\dfrac{P}{Q}\dfrac{\mathrm{d}P}{\mathrm{d}Q} - QP = 0$ **6.** $\dfrac{\mathrm{d}Y}{\mathrm{d}t} = e^{0.5t}$

Find the particular solution for each of the following differential equations:

7. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 5x$, given $x = 2$ when $y = 4$ **8.** $\dfrac{\mathrm{d}y}{\mathrm{d}t} - 10t - 2 = 0$, given $y = 15$ when $t = 0$

9. $12 - 8x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x$, given $y = 6$ when $x = 1$

10. $Q\dfrac{\mathrm{d}C}{\mathrm{d}Q} = 10 - Q^2$, given $C = 50$ when $Q = 1$

11. $\dfrac{P}{Q}\dfrac{\mathrm{d}P}{\mathrm{d}Q} - (Q + 5)P = 0$, given $P = 10$ when $Q = 0$

12. $\dfrac{\mathrm{d}Y}{\mathrm{d}t} = 12e^{0.6t}$ given $Y = 80$ when $t = 0$

☐ Differential equations in economics

In Chapter 6 the marginal function was defined as the derivative of the total function. The equation for a marginal function may be written as a differential equation, which may then be solved to obtain the equation of the total function.

WORKED EXAMPLE 8.16

FIND TOTAL COST FROM MARGINAL COST

The marginal cost for a product is given by the equation $MC = 10/Q$.

(a) Write down the differential equation for total cost in terms of Q.

(b) Find the equation of the total cost function if total costs are 500 when $Q = 10$.

Solution

(a) $MC = 10/Q$ may be written as

$$MC = \frac{\mathrm{d}(TC)}{\mathrm{d}Q} = \frac{10}{Q}$$

(b) This differential equation is solved to obtain the equation for the total cost.

Step 1: MC is

$$\frac{\mathrm{d}(TC)}{\mathrm{d}Q} = \frac{10}{Q}$$

Step 2:

$$TC = \int \frac{d(TC)}{dQ}\, dQ = \int (MC)\, dQ = \int \frac{10}{Q}\, dQ$$

$$TC = 10 \ln(Q) + c$$

where the arbitrary constant c is fixed cost. The general solution gives the general form of the total cost function.

Step 3: The condition that $TC = 500$ when $Q = 10$ is given. Substitute these conditions into the general solution to evaluate c:

$$500 = 10 \ln(10) + c \quad \rightarrow \quad 500 - 23.0259 = 476.9741 = c$$

Step 4: Therefore $TC = 10 \ln(Q) + 476.9741$

This is the particular solution. It is the equation of the total cost function which satisfies a particular condition $TC = 500$ when $Q = 10$.

❐ Differential equations and rates of change

Many practical applications are described in terms of rates of change (dy/dt) such as the rate at which the quantity of a good demanded, Q, changes over a period of time; the rate at which natural resources are depleted; the rate at which pollutants invade the environment; the rate at which the volume of sales changes; the rate at which a drug is absorbed into the bloodstream; the rate at which an epidemic spreads; the rate at which random information is absorbed, etc. These rates are described mathematically as differential equations. Frequently, the total amount accumulates over a given interval of time $t = t_0$ to $t = t_n$ is required. The total quantity accumulated is simply the definite integral of the rate of change from $t = t_0$ to $t = t_n$. The explanation is briefly outlined as follows.

The total quantity accumulated over a small interval of time Δt is approximately

$$\frac{\Delta Q}{\Delta t} \times \Delta t = \Delta Q \qquad \text{rate} \times (\text{time interval}) = (\text{change in } Q)$$

The total accumulated value of Q at t_n since t_0 is approximately equal to the sum of the quantities accumulated over each successive small interval of time from the Δt_0 to Δt_n:

$$Q = \Delta Q_1 + \Delta Q_2 + \cdots \Delta Q_n = \sum \Delta Q_i = \sum_{\Delta t_i} \frac{\Delta Q}{\Delta t_i} \times \Delta t_i$$

The exact quantity accumulated is obtained by letting the time intervals become infinitesimally small. Consequently, the summation sign is replaced by the integral sign (as in the introduction to the definite integral):

$$\text{Total } Q = \int_{t=t_1}^{t=t_n} (\text{rate of change of } Q)\, dt = \int_{t=t_1}^{t=t_n} \frac{dQ}{dt}\, dt$$

The total quantity accumulated between two points in time is calculated by integrating the equation for the rate of change between the two given time limits. This idea is illustrated graphically in Figure 8.17. The area under the curve between any two points in time $t = a$ and $t = b$ gives the total amount accumulated (or consumed) during that time interval.

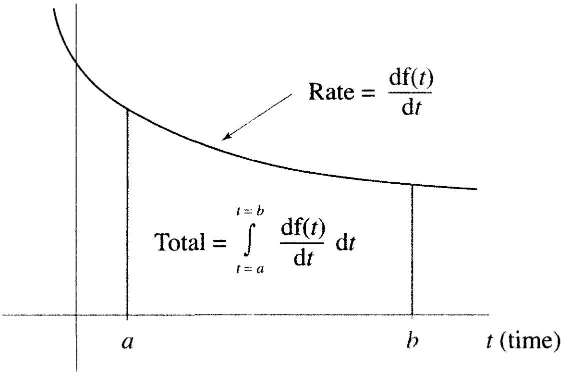

Rate = $\frac{df(t)}{dt}$
Total = $\int_{t=a}^{t=b} \frac{df(t)}{dt}\,dt$
a
b
t (time)

Figure 8.17 The definite integral of the rate = total accumulation

Note: integrating marginal functions to obtain total functions is, in effect, integrating the rate to obtain the total.

WORKED EXAMPLE 8.17

DIFFERENTIAL EQUATIONS AND RATES OF CHANGE

The present rate at which heating oil (in thousands of litres) consumed annually by a large department store is given by the equation

Rate of consumption $= \frac{dQ}{dt} = 1560e^{0.012t}$, where t is in years

(a) Plot the rate of consumption over the interval $0 \le t \le 20$. Derive the equation of the consumption function, given $Q = 0$ at $t = 0$.
Assuming that present rates of consumption continue,
(b) Calculate the total amount consumed during the first 10 years.
(c) Calculate the total amount consumed in the years, $10 \le t \le 20$

Solution

(a) The rate of consumption is an exponential, as sketched in Figure 8.18.

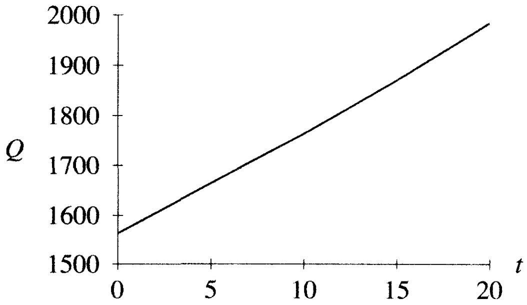


Figure 8.18 Consumption of oil for years 0 to 20

$$\frac{dQ}{dt} = 1560e^{0.012t} \rightarrow \int \frac{dQ}{dt}\,dt = \int 1560e^{0.012t}\,dt \rightarrow Q = 130\,000e^{0.012t} + C$$

Substitute $Q = 0$ at $t = 0$ into this equation to find $C \rightarrow 130\,000 + C \rightarrow c = -130\,000$. Therefore $Q = 130\,000(e^{0.012t} - 1)$.

(b) The total amount consumed during the first 10 years is given by the equation

$$\begin{aligned}\text{Total consumed} &= \int_{t=0}^{t=10} \frac{dQ}{dt}\,dt\\ &= \int_{t=0}^{t=10} 1560e^{0.012t}\,dt\\ &= 1560\left(\frac{e^{0.012t}}{0.012}\right)\Bigg|_{t=0}^{t=10}\\ &= \frac{1560}{0.012}(e^{0.012(10)} - e^{0.012(0)}) = \frac{1560}{0.012}(0.1275)\\ &= 16\,574.6 \text{ thousand litres}\end{aligned}$$

(c) The total amount consumed between the years 11 (starts at $t = 10$) to 20 inclusive, is calculated as in part (b) above, with the limits of integration changed to $t = 10$ to $t = 20$:

$$\begin{aligned}\text{Total consumed} &= \int_{t=10}^{t=20} 1560e^{0.012t}\,dt\\ &= 1560\left(\frac{e^{0.012t}}{0.012}\right)\Bigg|_{t=10}^{t=20}\\ &= \frac{1560}{0.012}(e^{0.012(20)} - e^{0.012(10)}) = \frac{1560}{0.012}(0.1438)\\ &= 18\,694 \text{ thousand litres}\end{aligned}$$

Progress Exercises 8.6 Integrating Marginals/Rates to Obtain Totals

1. The marginal cost function for a good is given by the equation $MC = -Q^2 + 80Q$.
   (a) Write down the differential equation which relates total cost to marginal costs.
   (b) Find the equation of the total cost function if fixed costs are 500.
   (c) Calculate the cost of producing successive units from $Q = 3$ to $Q = 12$.
2. The marginal propensity to consume is given by the equation $MPC = e^{-0.8Y}$.
   (a) Write the differential equation which relates MPC and total consumption.
   (b) Solve for the consumption function if it is known that consumption is 25 at $Y = 0$.
3. The marginal revenue function for a firm is given by the equation $MR = 120 - 2Q$.
   (a) Write down the differential equation relating marginal revenue and total revenue.
   (b) Find the equation of the total revenue function if total revenue is zero when $Q = 0$.

4. A firm's marginal cost is given by the equation $MC = Q^2 - 42Q + 200$.
(a) Find the equation of the total cost function if fixed costs are 900.
(b) Calculate the total cost of producing 12 units.

5. The marginal labour cost function is given by the equation $MLC = 3 + 4L$.
(a) Write the differential equation which relates MLC and total labour cost.
(b) Find the equation of the total labour cost function if total labour cost is zero when $L = 0$.
(c) Calculate the cost of employing successive labour units from $L = 1$ to $L = 7$.

6. The marginal productivity of labour is given by the equation $MP_L = 10 - 2L$.
(a) Write down the differential equation which relates output Q and MP_L.
(b) Find the equation of the production function if $Q = 0$ when $L = 0$.
(c) Calculate the total output of employing successive labour units from $L = 2$ to $L = 5$.

7. A firm expects its annual income, (Y), and expenditure to change according to the equations $Y(t) = 250 + 0.5t$ and $E(t) = t^2 - 17t + 287.5$ (t is time in years).
(a) Sketch the graphs of income and expenditure on the same diagram over the interval $0 \le t \le 20$. Confirm the break-even points algebraically.
(b) Use integration to calculate the net savings (total income – total expenditure) over the intervals

(i) $0 \le t \le 2.5$ (ii) $2.5 \le t \le 15$

8. An intensive pig-fattening unit is adding pollutant (in hundreds of gallons) to a river at the rate of $r(t) = \sqrt{t}$, where t is the number of months that the unit has been in operation.
(a) Graph the rate of pollution for $t = 0$ to 12 months. Comment.
(b) Calculate the volume of pollutant in the river during the first six months.

9. The rate at which a certain individual can memorise a sequence of items is given by the equation

$$n(t) = \frac{10}{(1+t)} \quad \text{where } t \text{ is in minutes}$$

(a) Graph and hence describe the rate of memory-retention curve.
(b) Derive an expression for the total number of items memorised over an interval of time, given that no items have been memorised at $t = 0$. Graph the number of items retained in memory for $t = 0$ to $t = 30$.
(c) Calculate the number of items memorised during
(i) the first 10 minutes, (ii) the second 10 minutes, (iii) the third ten minutes

10. A tool-hire company has estimated the rate of increase in maintenance cost for power drills which is given by the equation $C(t) = 2 + t^{1.5}$, where costs are in £, time in weeks.
(a) Plot the rate of increase in maintenance costs for $t = 0$ to 30.
(b) Calculate the total maintenance costs during (i) the first 5 weeks and (ii) the next 5 weeks.
(c) What is the average weekly cost of maintenance during
(i) The first 5 weeks?
(ii) The second 5 weeks?

❐ Solution of differential equations of the form $dy/dx = ky$

Equations of this type are solved by the same method as outlined above for $dy/dx = f(x)$, with the inclusion of one further step before step 2.

Step 1: Write the differential equation in the form $dy/dx = \text{RHS}$, i.e. $dy/dx = ky$. In this case, it is not possible to integrate both sides w.r.t. x immediately since the RHS contains y:

$$\int\left(\frac{dy}{dx}\right)dx = \int ky\,dx$$

We can only integrate functions of x w.r.t. x and functions of y w.r.t. y, etc. Therefore, separate the y terms with dy and the x terms with dx before integrating. This is called separating the variables.

Step 2: Separate the variables: separate the x terms with dx, and the y terms with dy:

$$\frac{1}{y}\frac{dy}{dx} = k \quad \rightarrow \quad \frac{1}{y}\,dy = k\,dx$$

Step 3: Integrate both sides. This gives the general solution.

Step 4: If conditions are given for x and y, substitute these values into the general solution and solve for the arbitrary constant, c.

Step 5: Substitute this value of c into the general solution to find the particular solution.

WORKED EXAMPLE 8.18

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = ky$

Solve the differential equations:

(a) $\dfrac{dy}{dx} - 0.5y = 0$, given $y = 10$ when $x = 2$

(b) $\dfrac{dI}{dt} = -0.07I$, given $I = 500$ when $t = 0$

Solution

(a) **Step 1:** $\dfrac{dy}{dx} = 0.5y$

Step 2: $\dfrac{1}{y}\dfrac{dy}{dx} = 0.5 \quad \rightarrow \quad \dfrac{1}{y}\,dy = 0.5\,dx$

Step 3: $\displaystyle\int \frac{1}{y}\,dy = \int 0.5\,dx$

$\ln(y) = 0.5x + c$ integrating

$y = e^{0.5x+c} = e^{0.5x}e^{c}$ going from log form to index form

$y = Ae^{0.5x}$ since e^{c} is a constant term, call it A

This is the general solution. The general solution is a family of growth curves which cut the vertical axis at A.

Step 4: The condition that $y = 10$ when $x = 2$ is given, so substitute these values into the general solution, and you may solve for A:

$$y = Ae^{0.5x} \qquad \text{this is the general solution}$$

Substituting $y = 10$, $x = 2$ into the general solution gives the equation

$$10 = Ae^{0.5(2)} \quad \rightarrow \quad 10 = A(2.7183) \quad \rightarrow \quad \frac{10}{2.7183} = 3.6788 = A$$

Step 5: Substitute the value calculated for A into the general solution to obtain the particular solution $y = 3.6788e^{0.5x}$.

(b) $\dfrac{dI}{dt} = -0.07I$, given $I = 500$ when $t = 0$.

Step 1: $\dfrac{dI}{dt} = -0.07I$

Step 2: $\dfrac{1}{I}\dfrac{dI}{dt} = -0.07 \quad \rightarrow \quad \dfrac{1}{I}\,dI = -0.07\,dt$

Step 3: $\displaystyle\int \frac{1}{I}\,dI = \int -0.07\,dt$

$$\ln(I) = -0.07t + c \qquad \text{integrating}$$

$$I = e^{-0.07t+c} = e^{-0.07t}e^{c} \qquad \text{going from log form to index form}$$

$$I = Ae^{-0.07t} \qquad \text{since } e^{c} \text{ is a constant term, call it } A$$

This is the general solution. The general solution is a family of growth curves which cut the vertical axis at A. See Figure 8.19.

Step 4: The condition that $I = 500$ when $t = 0$ is given, so substitute these values into the general solution, and solve for A:

$$I = Ae^{-0.07t} \qquad \text{this is the general solution}$$

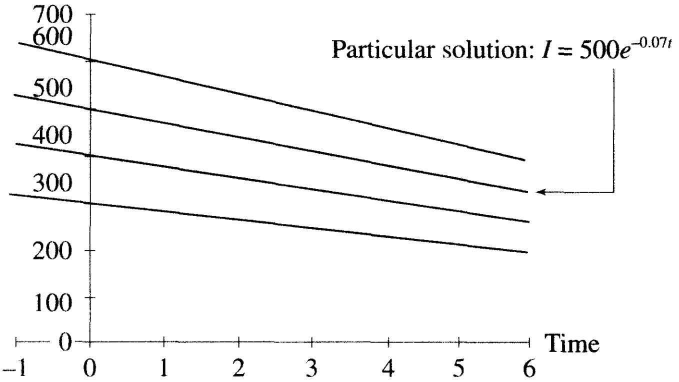


Figure 8.19 General solution, with particular solution indicated

Substituting $I = 500$, $t = 0$ into the general solution, gives the equation

$$500 = Ae^{-0.07(0)} \rightarrow 500 = A(1) \rightarrow 500 = A$$

Step 5: Substitute the value calculated for A into the general solution to obtain the particular solution $I = 500e^{-0.07t}$.

☐ Solution of differential equations of the form $dy/dx = f(x)g(y)$

In this case, the RHS of the differential equation may be factored into the product of functions of x, $f(x)$ and functions of y, $g(y)$. Such equations are solved by separating the variables as outlined above.

WORKED EXAMPLE 8.19

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = f(x)g(y)$

Solve the differential equation $dP/dQ = P(Q+1)$.

Solution

Step 1: $\dfrac{dP}{dQ} = P(Q+1)$

Step 2: $dP = P(Q+1)\,dQ \rightarrow \dfrac{1}{P}\,dP = (Q+1)\,dQ$

Step 3: $\displaystyle\int \frac{1}{P}\,dP = \int (Q+1)\,dQ$

$$\ln(P) = \frac{Q^2}{2} + Q + c$$

$P = e^{0.5Q^2+Q+c} = e^{0.5Q^2+Q}e^c$ going from log form to index form

$P = Ae^{0.5Q^2+Q}$ since e^c is a constant term, call it A

This is the general solution. No conditions were given, so a particular solution cannot be obtained.

Progress Exercises 8.7 Differential Equations of the Form $dy/dx = f(y)$; $dy/dx = f(y)g(y)$

Find the general solution and particular solutions of the following differential equations.

1. $\dfrac{dy}{dt} = 2y$, given $y = 50$ at $t = 0$

2. $\dfrac{dy}{dt} + \dfrac{y}{10} = 0$, given $y = 800$ at $t = 0$

3. $0.4N + \frac{dN}{dt} = 0$, given $N = 812$ at $t = 2$

4. $\frac{dy}{dx} = 2\frac{x}{y}$, given $y = 10$ when $x = 10$

5. $\frac{dy}{dt} = 3t^2y$, given $y = 50$ at $t = 0$

6. $3\frac{dP}{dt} + \frac{t}{2} = 60$, given $P = 21$ at $t = 0$

7. $\frac{dy}{dx} = xy$, given $y = 20$ when $x = 0$

8. $0.05I + \frac{dI}{dt} = 0$, given $I = 900$ at $t = 0$

9. $\frac{dQ}{dP}\frac{P}{Q} = -1$, given $P = 0.25$ when $Q = 4$

10. $\frac{dQ}{dP}\frac{P}{Q} = -\frac{1}{Q}$, given $P = 120$ when $Q = 0$

8.8 Differential Equations for Limited and Unlimited Growth

In this section you shall be introduced to:

- Differential equations of the form $dy/dt = ry$, which describe unlimited growth
- Differential equations of the form $y/t = r(A - y)$, which describe limited growth.

☐ Law of unlimited growth

Differential equations of the type $dy/dt = ry$ describe systems in which the proportional rate of growth, r, is constant. The solution of this type of differential equation is $y = Ae^{rt}$, where A is the arbitrary constant, describing unlimited growth (or decay, if the index is negative). See Figure 8.20. The solution to this type of equation has already been outlined in Worked Example 8.18.

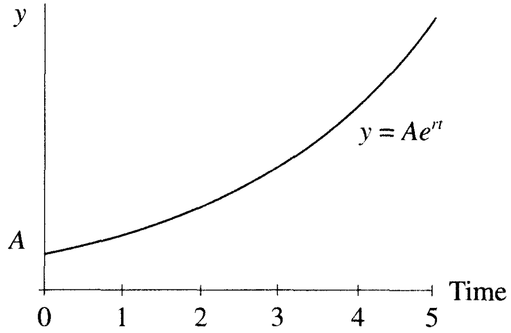


Figure 8.20 Unlimited growth $y = Ae^{rt}$

❐ Law of limited growth

Differential equations of the type $\mathrm{d}y/\mathrm{d}t = r(A - y)$ describe systems which grow fast initially but level off just short of a maximum limit, $y = A$ as illustrated in Figure 8.21.

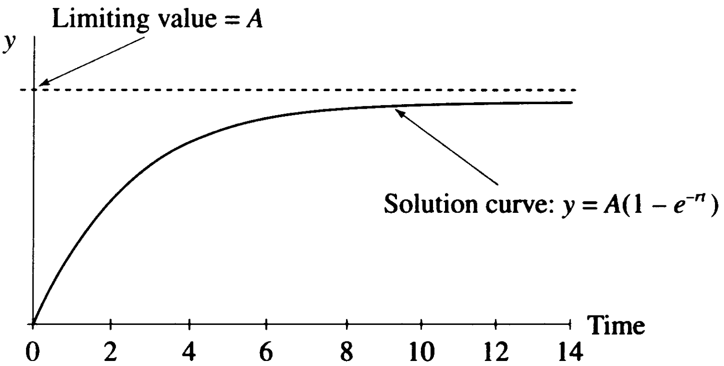


Figure 8.21 The solution of the differential equation: $\mathrm{d}y/\mathrm{d}t = r(A - y)$ models limited growth

WORKED EXAMPLE 8.20

LIMITED GROWTH

The rate at which the volume of sales (Q) for a new type of printer increase after an advertising campaign is given by the equation $\mathrm{d}Q/\mathrm{d}t = 0.04(700 - Q)$, given that $Q = 0$ at $t = 0$. Q is the number of printers sold, t is time in years.

(a) Solve the differential equation to obtain an expression for Q in terms of t.

(b) Sketch the solution curve, hence describe verbally how the volume of sales should increase in time.

Solution

(a) To solve this equation the variables must be separated:

Step 1: $\dfrac{\mathrm{d}Q}{\mathrm{d}t} = 0.04(700 - Q)$

Step 2:

$$\mathrm{d}Q = 0.04(700 - Q)\,\mathrm{d}t$$

$$\frac{1}{700 - Q}\,\mathrm{d}Q = 0.04\,\mathrm{d}t$$

Step 3:

$$\int \frac{1}{700 - Q}\,\mathrm{d}Q = \int 0.04\,\mathrm{d}t$$

$$\frac{\ln(700 - Q)}{-1} = 0.04t + c$$

$$\ln(700 - Q) = -0.04t - c$$

$$700 - Q = e^{-0.04t - c} = e^{-0.04t}e^{-c} = Be^{-0.04t}$$

$$Q = 700 - Be^{-0.04t}$$

This is the general solution. Substituting the given condition that $Q = 0$ at $t = 0$ gives an equation which may be solved for the arbitrary constant B:

$$0 = 700 - Be^{(0)} = 700 - B \quad \rightarrow \quad B = 700$$

Therefore, the particular solution is $Q = 700(1 - e^{-0.04t})$.

(b) The sketch of the solution curve is given in Figure 8.22, demonstrating that the limit or 'ceiling' towards which the volume of sales increases is 700, at a rate of 0.04 per year.

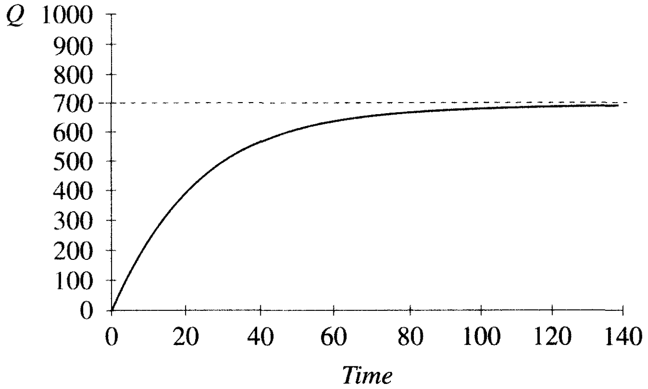


Figure 8.22 Limited growth, where the limiting value is 700

❒ Constant proportional rates of growth

In Chapter 4 the growth in populations, investment, etc., were modelled by the exponential function of the form given in equation (8.16):

$$P_t = P_0e^{rt} \tag{8.16}$$

where P_t, P_0 are the population sizes at time t and at some base time $t = 0$, respectively. The constant r is called the **constant proportional rate of growth**, since the proportional rate, or the relative rate, of growth is defined in general as

$$\frac{\text{rate of change of } P \text{ at time } t}{\text{size of } P \text{ at time } t} = \frac{\mathrm{d}P_t/\mathrm{d}t}{P_t} \tag{8.17}$$

To show that r is a constant proportional rate of growth, take equation (8.16) and calculate the rate of change:

$$P_t = P_0e^{rt}$$

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = P_0(re^{rt}) = rP_0e^{rt}$$

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = rP_t \qquad (\text{since } P_t = P_0e^{rt})$$

That is, the rate of change of the size of P at time $t = r \times$ the size of P at time t. Therefore

$$\frac{\mathrm{d}P_t/\mathrm{d}t}{P_t} = r$$

This means that the proportional rate of growth (or decay), r, for functions of the form $P_t = P_0 e^{rt}$ is the same at any point in time. In other growth (and decay) models, the proportional rate of growth may have different values at different points in time, for example

$$P_t = \frac{12}{t},\quad P_t = \frac{P_0}{500 - t},\quad P_t = \frac{P_0}{500 - t},\quad L_t = 200\ \ln(t + 1)$$

WORKED EXAMPLE 8.21

DETERMINING THE PROPORTIONAL RATES OF GROWTH

For each of the following functions:

(i) $P_t = 5312e^{0.015t}$ (ii) $P_t = 47e^{-0.02t}$ (iii) $P_t = \dfrac{12}{t}$

(a) Derive a general expression for the proportional rate of growth.
(b) Calculate the proportional rates of growth at $t = 5$.

Solution

The proportional rate of growth is given by

$$r = \frac{\mathrm{d}P_t/\mathrm{d}t}{P_t}$$

For each of the functions, the calculations are as follows.

(i)

(a) $P_t = 5312e^{0.015t}$

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = 5312(0.015e^{0.015t})$$

Therefore

$$r = \frac{\mathrm{d}P_t/\mathrm{d}t}{P_t} = \frac{5312(0.015e^{0.015t})}{5312e^{0.015t}} = 0.015$$

r does not depend on time, therefore r is constant, as

(ii)

(a) $P_t = 47e^{-0.02t}$

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = 47(-0.02e^{-0.02t})$$

Therefore

$$r = \frac{\mathrm{d}P_t/\mathrm{d}t}{P_t} = \frac{47(-0.02e^{-0.02t})}{47e^{-0.02t}} = -0.02$$

r does not depend on time, therefore r is constant, as

(iii)

(a) $P_t = \dfrac{12}{t} = 12t^{-1}$

$$\frac{\mathrm{d}P_t}{\mathrm{d}t} = 12(-1)t^{-2} = -12t^{-2}$$

Therefore

$$r = \frac{\mathrm{d}P_t/\mathrm{d}t}{P_t} = -\frac{12t^{-2}}{12t^{-1}} = -t^{-1} = -\frac{1}{t}$$

r changes as time changes, therefore r is not constant, as

(i)	(ii)	(iii)
demonstrated in part (b) below:	demonstrated in part (b) below:	shown in (b) below:
(b) At $t = 5$, $r = 0.015$ At $t = 10$, $r = 0.015$	(b) At $t = 5$, $r = -0.02$ At $t = 10$, $r = -0.02$	(b) At $t = 5$, $r = -\frac{1}{t} = -\frac{1}{5} = -0.2$ At $t = 10$, $r = -\frac{1}{t} = -\frac{1}{10} = -0.1$

Progress Exercises 8.8 Growth, Decay and Other Applications

1. The rate at which an ore is mined is given by the differential equation $\mathrm{d}A/\mathrm{d}t = 0.02A$.
   (a) Solve the differential equation and graph the general solution.
   (b) If the total reserves of ore in 1989 were 400 million tons, find the particular solution.
   (c) Calculate the total amount of ore mined from the beginning of 1995 to the beginning of 2000 if mining continues at the present rate.
2. The differential equation $\mathrm{d}P/\mathrm{d}t = 0.01P$ models the relationship between the numbers in a population at t. What is the proportional rate of population growth?
   (a) Deduce the equation which describes the total population at any time t.
   (b) Calculate the numbers in the population in 2200, given that $P = 58.6$ million in 1998 (let $t = 0$ at the start of 1998).
   (c) Calculate the number of years taken for the population to rise to 70 million.
3. The amount of sulphur dioxide (in tons) present in the atmosphere is given by the equation

$$\frac{\mathrm{d(pollutant)}}{\mathrm{d}t} = -0.2P \quad (t \text{ is in hours})$$

   (a) If 50 tons are released into the atmosphere (i.e. pollutant = 50 at $t = 0$), derive an expression for the amount of sulphur dioxide present at any time t.
   (b) Use your expression from (a) to determine the amount of sulphur dioxide (i) after 4 hours and (ii) after 8 hours.
   (b) Use your expression from (a) to calculate the time taken for sulphur dioxide to reduce to 5 tons.
4. A capital investment (I) depreciates continuously at the rate of 5% annually,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -0.05I$$

   (a) If the investment is valued at £12 000 initially (i.e. at $t = 0$), calculate the value of the investment after 5.5 years.
   (b) How many years will it take for the value of the investment to fall to £5000?

5. The price elasticity of demand for a good is constant at $\varepsilon = -1$.
   (a) Write down the differential equation that relates the price of the good to the quantity demanded. (**Hint:** Use the definition of point elasticity.)
   (b) Deduce an expression for the demand function, given that $P = 12$ when $Q = 48$.
6. The rate at which fish stocks increase in an inland lake is given by the equation $\mathrm{d}F/\mathrm{d}t = 0.12(80 - F)$, where F is in thousands, t is in years.
   (a) Derive an expression for the total number of fish present at any time t given $F = 0$ at $t = 0$.
   (b) Graph the total fish stock as a function of time from $t = 0$ to $t = 30$. Does the lake appear to have a maximum carrying capacity?
   (c) Calculate the number of years taken for the fish stocks to reach 40 000.
7. A hospital is considering the installation of two alternative heating systems. The rate of increase in costs for each system is given by the equations

$$\frac{\mathrm{d}C_1}{\mathrm{d}t} = 0.01C_1 \qquad \text{and} \qquad \frac{\mathrm{d}C_2}{\mathrm{d}t} = t^{0.25}$$

   (a) Derive expressions for the total cost of each system at any time t given that cost = £50 000 at $t = 0$.
   (b) Graph both cost functions from $t = 0$ to $t = 10$ years. Which system is more cost-effective in the long run? Give reasons.
8. The price elasticity of demand is given by $\varepsilon = -10/Q$.
   (a) Write down the differential equation in terms of P and Q.
   (b) Find the equation of the demand function if $P = 20$ when $Q = 0$.

In questions 9 to 12, determine the proportional rate of growth or decay, as defined in equation (8.18) for each of the following functions at $t = 5$ and $t = 10$. State whether the functions have a constant proportional rate of growth.

9. $P_t = 500e^{0.2t}$ 10. $P_t = 685e^{-0.1t}$ 11. $P_t = \dfrac{2}{t+2}$ 12. $I_t = 20t^{2.5}$

8.9 Summary

□ Rules for integration

Standard functions		**Functions of linear functions**	
$\mathrm{f}(x)$	$\mathrm{F}(x) = \int \mathrm{f}(x)\,\mathrm{d}x$	$\mathrm{f}(mx+c)(m \neq 0)$	$\mathrm{F}(mx+c) = \int \mathrm{f}(mx+c)\,\mathrm{d}x$
K	$Kx + c$	N/A	N/A
$x^n, n \neq -1$	$\dfrac{x^{n+1}}{n+1} + c$	$(mx+c)^n, n \neq -1$	$\dfrac{1}{m}\dfrac{(mx+c)^{n+1}}{n+1} + c$
x^{-1} or $\dfrac{1}{x}$	$\ln(x) + c$	$\dfrac{1}{mx+c}$	$\dfrac{1}{m}\ln(mx+c) + c$
e^x	$e^x + c$	$e^{(mx+c)}$	$\dfrac{1}{m}e^{(mx+c)} + c$

The definite integral

$$\int_{x=a}^{x=b} \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$$

This is the net area enclosed between the curve and the x-axis over the interval $x = a$ and $x = b$

❒ First-order differential equations

Use integration to solve first-order differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x), \qquad \text{solve by direct integration}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = Ky\left(\frac{\mathrm{d}y}{\mathrm{d}t} = ry\right), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = r(A - y) \qquad \text{solve by separating the variables}$$

(Solutions model unlimited and limited growth)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y). \qquad \text{solve by separating the variables}$$

❒ Consumer and producer surplus

At $P = P_0$, $Q = Q_0$

(a) Consumer surplus is defined as the

$$\text{Area under the curve} - \text{area of rectangle} = \int_{Q=0}^{Q=Q_0} (\text{demand function})\,\mathrm{d}Q - P_0Q_0$$

(b) Producer surplus is defined as the

$$\text{Area of rectangle} - \text{area under the curve} = P_0Q_0 - \int_{Q=0}^{Q=Q_0} (\text{supply function})\,\mathrm{d}Q$$

❒ Integrate marginal functions to obtain total functions

$$TC = \int \frac{\mathrm{d}(TC)}{\mathrm{d}Q}\,\mathrm{d}Q = \int (MC)\,\mathrm{d}Q$$

$$TR = \int \frac{\mathrm{d}(TR)}{\mathrm{d}Q}\,\mathrm{d}Q = \int (MR)\,\mathrm{d}Q$$

❒ Integrate rates (w.r.t. time) to obtain the total amount accumulated over a given time interval

Example: Given the rate at which a resource is consumed: $\mathrm{d}Q/\mathrm{d}t = Ae^{bt}$, then the total amount used between $t = t_0$ and t_1 is given by the integral

$$\int_{t=t_0}^{t=t_1} \frac{\mathrm{d}Q}{\mathrm{d}t}\,\mathrm{d}t = \int_{t=t_0}^{t=t_1} Ae^{bt}\,\mathrm{d}t$$

☐ Solution of certain first-order differential equations

Differential equations are used to model situations which involve change, for example limited and unlimited growth:

(a) $dy/dt = ry$ has a solution of the form $y = Ae^{rt}$, which describes unlimited growth.
(b) $dy/dt = r(A - y)$ has a solution of the form $Q = A - Be^{-rt}$, which describes limited growth.

Test Exercises 8

1. (a) Integrate the following:

   (i) $\int x^4 + 2x \, dx$ (ii) $\int \sqrt{x} + 2 \, dx$ (iii) $\int \frac{2}{x} + 2x \, dx$

   (b) Given the following functions,

   (i) $y = 4 - 2x$ (ii) $y = 4 - x^2$ (iii) $y = x^2 - x + 4$

   (i) Graph each function for $0 \leq x \leq 4$
   (ii) Use integration to determine the net area over the interval $0 \leq x \leq 4$.
   (iii) Use integration to determine the total area enclosed between the graph and the axis over the interval $0 \leq x \leq 4$.

2. Solve the differential equations

   (a) $\frac{dy}{dx} = 0.2y$ (b) $\frac{dy}{dx} = 120(1 - 0.2y)$

   Which solution models limited growth? What is the limit?

3. 500 tons of farm effluent is released into a river. The amount of effluent (E) present is given by the differential equation $dE/dt = -0.1E$, where t is in hours
   (a) Deduce an expression for E in terms of t.
   (b) Calculate the amount of effluent present after 5 hours.
   (c) Calculate the time taken for the amount of effluent to reduce to 10.50 tons.

4. (a) Evaluate the following:

   $$\int_{t=0}^{t=5} 40e^{-0.3t} \, dt$$

   (b) The demand and supply functions for a good are given by the equations

   $$P = 4(Q - 16)^2 \qquad P = (Q + 2)^2 \quad \text{respectively}$$

   (i) Determine the equilibrium price and quantity.
   (ii) Calculate the consumer and producer surplus at equilibrium.

5. (a) Solve the differential equation $dy/dt = (1 - y)$.
   (b) The rate at which an infection spreads in a poultry house is given by the equation $dP/dt = 0.75(2500 - P)$, where t is time in days. If $P = 0$ at $t = 0$,
   (i) Solve the differential equation to determine an expression for the number of poultry (P) infected at any time t.
   (ii) Graph the solution for $t = 0$ to 12. Hence describe the time path.
   (iii) Calculate the time taken for 1500 poultry to become infected.

CHAPTER 9

Linear Algebra and Applications

In this chapter you shall be introduced to:

9.1 Linear programming
9.2 Matrices
9.3 Solution of equations: elimination methods
9.4 Determinants
9.5 The inverse matrix and input/output analysis
9.6 Summary

9.1 Linear Programming

In Chapter 7 the optimal value(s) of non-linear functions (utility, production, cost, etc.) were found, subject to one constraint only. In this section the optimal value of linear functions (cost, profit, output, etc.) subject to several constraints will be found. If both the function to be optimised and the constraints are restricted to two variables, the optimum value may be found graphically as well as algebraically. This is the subject of this section. When the number of variables exceed two, the graphical approach is no longer possible. The optimal value is found by more general linear programming methods, such as the simplex method, which is beyond the scope of this text. The graphical and mathematical methods used in solving a typical linear programming problem are described in Worked Example 9.1.

☺ **Remember:** cost, budget and income constraints were presented in Chapter 2. These will be required in the graphical methods which follow.

WORKED EXAMPLE 9.1

FIND THE MINIMUM COST SUBJECT TO CONSTRAINTS

A daily diet requires a minimum of:

600 mg of Vitamin C, 360 mg of Vitamin D, 40 mg of Vitamin E
The content of these vitamins (per portion) in two food mixes, X and Y, are given in Table 9.1.

Table 9.1

	Vitamin C (mg)	Vitamin D (mg)	Vitamin E (mg)	Cost per 1 mg
One portion of X	20	10	4	5
One portion of Y	30	20	1	4
Minimum daily requirement	600	360	40	

(a) Express the information given on vitamins in terms of inequality constraints.
(b) Graph the inequality constraints and shade in the feasible region.
(c) Write down the equation of the cost function. Plot the cost function for costs = 200.
(d) From the graph determine the number of portions of mixes X and Y which fulfil the daily requirements at a minimum cost. Confirm your answer algebraically.

Solution

Let the optimum number of portions of mixes X and Y be x and y respectively. The problem sets out to find the values of x and y which satisfy the minimum dietary requirements at a minimum cost.

(a) The expressions for the inequality constraints
The constraints are the minimum daily requirements of each vitamin. The constraints are called **inequality constraints** since the requirements are not exact values but minimum values, so the variables may be greater than minimum. To express the inequality constraints mathematically proceed as follows:

The quantities of vitamins C, D and E provided by one portion of mix X is given in row one, Table 9.1. The quantities of vitamins C, D and E provided by x portions of mix X are calculated by multiplying the quantities provided by one portion (given in row one, Table 9.1) by x. The quantities of vitamins C, D and E provided by y portions of mix Y are calculated by multiplying the quantities provided by one portion of mix Y by y. These calculations are summarised in Table 9.2.

Table 9.2 Vitamin content of x and y portions of mix X and Y respectively

	Vitamin C	**Vitamin D**	**Vitamin E**	**Cost per portion**
x portions of mix X	$20 \times x$	$10 \times x$	$4 \times x$	$5 \times x$
y portions of mix Y	$30 \times y$	$20 \times y$	$1 \times y$	$4 \times y$
Minimum daily requirement	600	360	40	

Therefore, to satisfy the minimum daily requirements we must have

Total vitamin $C \geq 600$ that is, $20x + 30y \geq 600$ **(9.1)**

Total vitamin $D \geq 360$ that is, $10x + 20y \geq 360$ **(9.2)**

Total vitamin $E \geq 40$ that is, $4x + y \geq 40$ **(9.3)**

Since the number of portions is always greater than or equal to zero, $x \geq 0$ and $y \geq 0$ are two further constraints. These are the **inequality constraints**.

(b) To graph the inequalities, treat each inequality as an equation, then plot the equation line. The inequality is represented graphically by every point on or above the plotted line. The easiest way to plot these lines is to find the points of intersection with the axes. The points of intersection with the axes are summarised for all three lines, equations (9.1), (9.2), (9.3), as follows:

	Equation of line	Cuts x-axis when $y = 0$	Cuts y-axis when $x = 0$
Vitamin C	(1) $20x + 30y = 600$	$20x = 600 \rightarrow x = 30$	$30y = 600 \rightarrow y = 20$
Vitamin D	(2) $10x + 20y = 360$	$10x = 360 \rightarrow x = 36$	$20y = 360 \rightarrow y = 18$
Vitamin E	(3) $4x + y = 40$	$4x = 40 \rightarrow x = 10$	$y = 40$
	$x \geq 0, y \geq 0$	First quadrant	

The unknowns are x and y. Therefore, when plotting graphs either one of the variables may be plotted on the horizontal axis, the other on the vertical axis. In this example, x is plotted on the horizontal and y on the vertical. The three inequality constraints are plotted in Figure 9.1.

All three inequality constraints are satisfied simultaneously by the overlap region in the plane, as shown in Figure 9.1. This overlap region is called the **feasible region**.

(c) To find the equation of the cost function

The function to be minimised (or maximised) is called the **objective function**. In this example, the objective function is cost, where

$$\text{Cost} = (\text{units of } X \times \text{price per unit}) + (\text{units of } Y \times \text{price per unit})$$

Since the price of each unit of X and Y is given, therefore the cost of x units of mix X and y units of mix Y is: cost $= (x \times 5) + (y \times 4)$, that is,

$$\text{Cost} = 5x + 4y \qquad \textbf{(9.4)}$$

This cost function, is in fact, an **isocost** function: it represents the combinations of all quantities of products X and Y which result in the same overall cost.

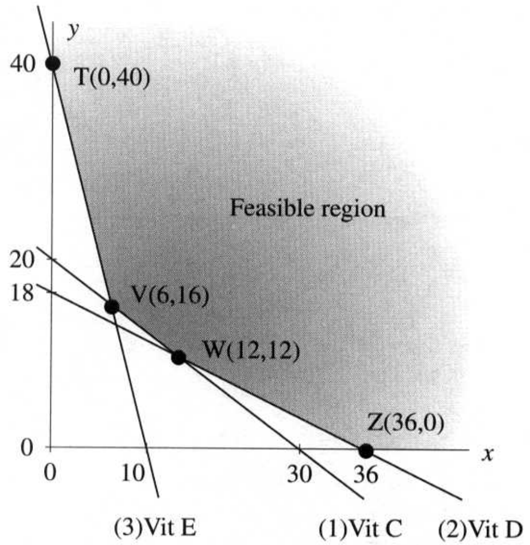


Figure 9.1 Inequality constraints and the feasible region

To plot the cost function (isocost line)

Since the value of cost is unknown, begin by assuming any value for cost, calculate the points of intersection with the axes, then plot the line. For example, when cost $= 200$, the isocost line is given by cost $= 5x + 4y \rightarrow 200 = 5x + 4y$.

This isocost line cuts the x axis when $y = 0$, that is, $200 = 5x \rightarrow x = 40$. This isocost line cuts the y axis when $x = 0$, that is, $200 = 4y \rightarrow y = 50$.

When one isocost line is drawn, any number of isocost lines may be drawn by drawing lines parallel to this isocost line (see Chapter 2). As cost decreases, the cost line moves downwards towards the origin as shown in Figure 9.2.

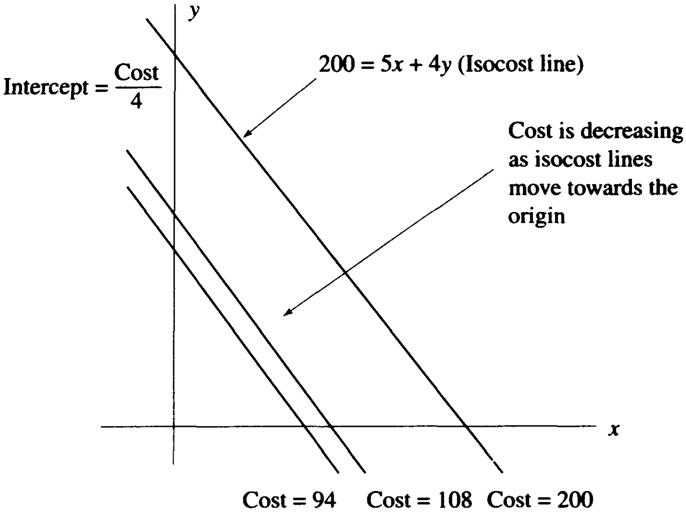

y
Intercept = $\frac{\text{Cost}}{4}$
200 = 5x + 4y (Isocost line)
Cost is decreasing as isocost lines move towards the origin
x
Cost = 94
Cost = 108
Cost = 200

Figure 9.2 Cost decreases as isocost lines move towards the origin

Alternatively, write the isocost equation in the form $y = mx + c$; read off the slope, m; plot any line with slope m:

$$y = -\frac{5}{4}x + \frac{\text{cost}}{4}: \quad m = -\frac{5}{4} = \frac{\Delta y}{\Delta x}:$$ plot a line which cuts the y-axis at 5 and the x-axis at 4

(d) To find the minimum cost graphically

The minimum cost is the point in the feasible region which is on the 'lowest' isocost line. To find this point graphically, move the isocost line downwards towards the origin until it contains just one point in the feasible region. From Figure 9.3, this point is at V, the point which has coordinates $x = 6$, $y = 16$.

Therefore, $x = 6$ portions of mix X and $y = 16$ portions of mix Y will satisfy the minimum daily vitamin requirements at the lowest possible cost. Since cost $= 5x + 4y$, the minimum cost is cost $= 5x + 4y = 5(6) + 4(16) = 94$.

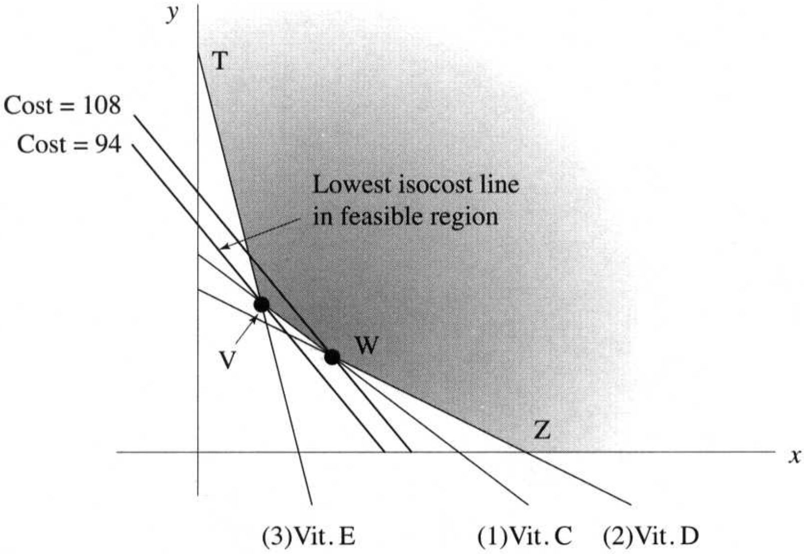


Figure 9.3 Minimum cost at point V, subject to constraints

❒ The minimum cost by mathematical methods

The extreme point theorem: This theorem states that if an optimal solution exists, it is found at one of the corner points of the feasible region. In Figure 9.3, the corner points, T, V, W and Z are found by solving each pair of equations bounding the feasible region for their point of intersection. For example, point V is the point of intersection of lines (1), vitamin C and, (3), vitamin E: $20x + 30y = 600$ and $4x + y = 40$. So solving these two equations gives point V, as follows:

(1) $20x + 30y = 600$

(4) → (3) × 5 $20x + 5y = 200$

subtract (1)–(4) $25y = 400$, therefore, $y = 16$

Substituting $y = 16$ into any of these three equations gives $x = 6$.
This answer agrees with that obtained graphically.

❒ To demonstrate the extreme point theorem

That is, show that the cost is lower at point V than at any other corner point of the feasible region. The points at the corners of the feasible region are calculated by solving the appropriate pair of simultaneous equations, as was done for point V above. The results for all corner points should be: T: (0, 40), V: (6, 16), W: (12, 12), Z: (36, 0).

The total cost is evaluated at each of these points and the results summarised as:

Point (x, y)	**Cost** $= 5x + 4y$
T(0, 40)	Cost $= 5(0) + 4(40) = 160$
V(6, 16)	Cost $= 5(6) + 4(16) = 94$
W(12, 12)	Cost $= 5(12) + 4(12) = 108$
Z(36, 0)	Cost $= 5(36) + 4(0) = 180$

Therefore the mathematical result, a minimum at V, agrees with the graphical result.

Note: If a constraint is not required to find at least one corner point in the feasible region, then this constraint is not a limitation.

❒ Maximisation

Worked Example 9.2 demonstrates how to find the maximum values of a function subject to several constraints.

WORKED EXAMPLE 9.2

PROFIT MAXIMISATION SUBJECT TO CONSTRAINTS

A company manufactures two types of wrought iron gates. The number of man-hours required to produce each type of gate along with the maximum number of hours available are given in Table 9.3.

Table 9.3 Requirements for gates type I and II

	Welding	**Finishing**	**Admin.**	**Selling price (£)**	**Profit (£)**
Type I gate	6	2	1	120	55
Type II gate	2	1	1	95	25
Max. hours available	840	300	250		

(a) Write down expressions for:
  (i) The constraints
  (ii) Total revenue
  (iii) Profit.
(b) Plot the constraints and shade in the feasible region.
(c) Plot the revenue and profit lines, indicating the direction of increasing revenue and profit.
(d) Determine, graphically, the number of gates which should be produced and sold to maximise
  (i) Revenue
  (ii) Profit.
Confirm these answers algebraically, using the extreme point theorem.
(e) Calculate the number of man-hours which are not used when:
  (i) Revenue is maximised
  (ii) Profit is maximised.

Solution

(a) For x type I gates and y type II gates, the number of man-hours, revenue and profit are calculated by multiplying the requirements for one gate of each type, as given in Table 9.3, by x and y respectively. This is given in Table 9.4.

Table 9.4 Requirements for x type I gates and y type II gates

	Welding	Finishing	Admin.	Selling price (£)	Profit (£)
x type I gates	$6x$	$2x$	x	$120x$	$55x$
y type II gates	$2y$	$1y$	y	$95y$	$25y$
Max. hours available	840	300	250		

The constraints are:

Welding $\quad 6x + 2y \leq 840 \quad$ **(9.5)**

Finishing $\quad 2x + y \leq 300 \quad$ **(9.6)**

Administration $\quad x + y \leq 250 \quad x \geq 0,\ y \geq 0 \quad$ **(9.7)**

The equations for total revenue and profit are:

Total revenue: $\quad TR = 120x + 95y \quad$ **(9.8)**

Profit: $\quad \pi = 55x + 25y \quad$ **(9.9)**

(b) Plot the constraints, by calculating the points of intersection with the axes.

Activity	Equation	Cuts x-axis when $y = 0$	Cuts y-axis when $x = 0$
Welding	(1) $6x + 2y = 840$	$6x = 840, \rightarrow x = 140$	$2y = 840, \rightarrow y = 420$
Finishing	(2) $2x + y = 300$	$2x = 300, \rightarrow x = 150$	$y = 300$
Administration	(3) $x + y = 250$	$x = 250$	$y = 250$
	$x \geq 0, y \geq 0$	First quadrant	

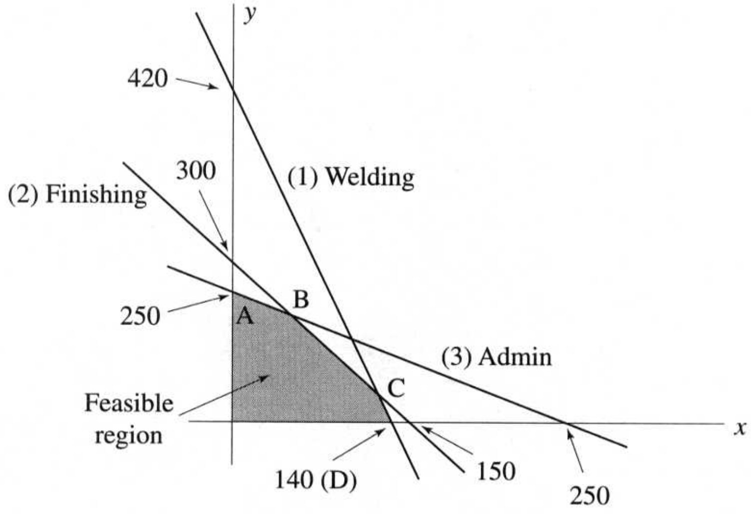


Figure 9.4 The constraints defining the feasible region for gate manufacturing

These lines are sketched in Figure 9.4. Since the inequalities are all less than or equal to a limit, the feasible region is the overlap region of all three constraints.

(c) $TR = 120x + 95y$

To plot one revenue line, choose any value of TR, for example $TR = 9500$. This line (called an isorevenue line) cuts the y-axis at $y = 100.00$ (when x is zero) and the x-axis at $x = 79.167$ (when $y = 0$). See Figure 9.5. As revenue increases, the isorevenue lines move upwards away from the origin.

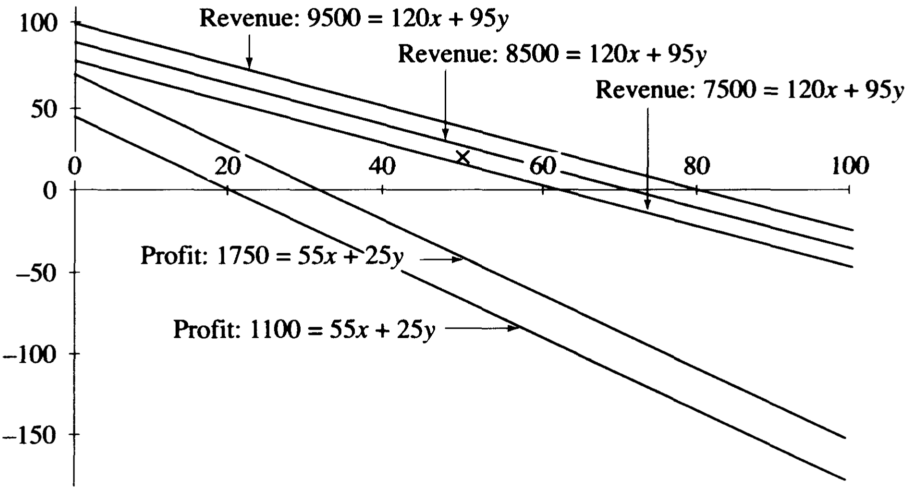


Figure 9.5 Profit and revenue increases as isoprofit and isorevenue lines move upwards from the origin

Profit: $\pi = 55x + 25y$

To plot one isoprofit line, choose any value of π, for example $\pi = 1100$. This line cuts the y-axis at $y = 44$ (when x is zero) and the x-axis at $x = 20$ (when $y = 0$). As profit increases, the isoprofit lines move upwards away from the origin. See Figure 9.5.

(d) For maximum revenue, it is required that the isorevenue line be as 'high' as possible, but contain one point in the feasible region. From Figure 9.6(a), this point is at B, where $x = 50$, $y = 200$. For maximum profit, it is required that the isoprofit line be as 'high' as possible, but still contain one point in the feasible region. From Figure 9.6(b), this point is at C, where $x = 120$, $y = 60$.

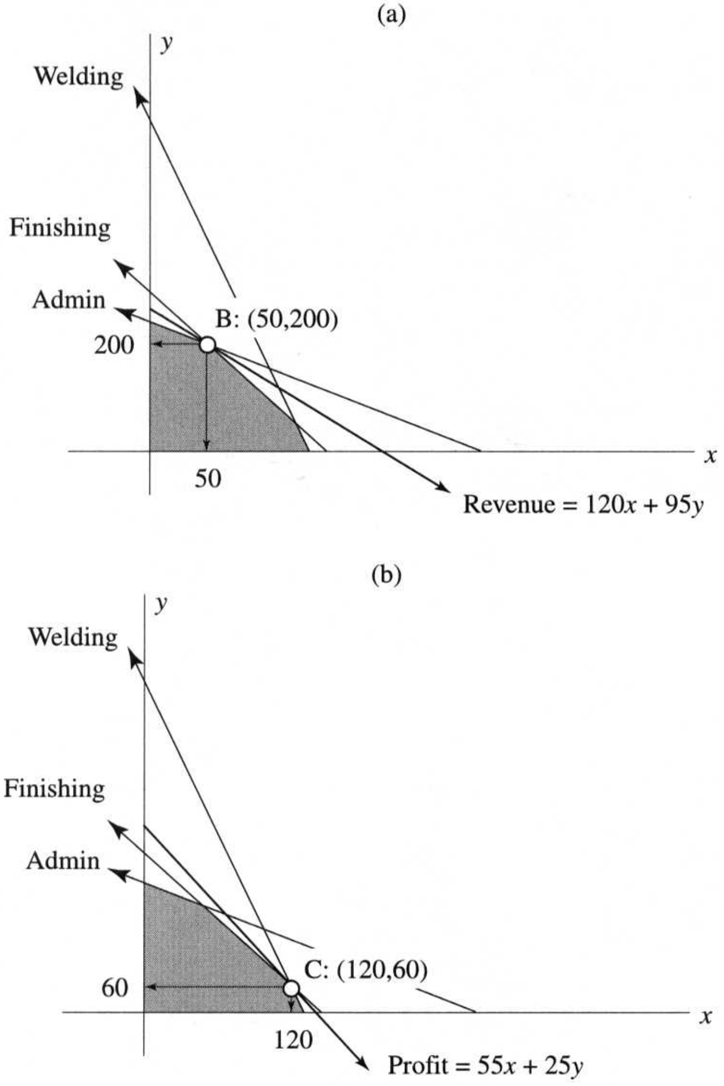


Figure 9.6 (a) Revenue is a maximum at point B; (b) profit is a maximum at point C

The answer is confirmed algebraically by solving the appropriate pairs of equations at points B and C.

Point B: Maximum revenue is at the intersection of equations (2), finishing, and (3), administration.

(2) $2x + y = 300$ | The solution of these equations is $x = 50$, $y = 200$

(3) $x + y = 250$

Point C: The maximum profit is at the intersection of equation (1), welding, and (2) finishing.

(1) $6x + 2y = 840$ | The solution of these equations is $x = 120$, $y = 60$

(2) $2x + y = 300$

To demonstrate the extreme point theorem, calculate profit and revenue at all corner points (just to check the answers above).

Point (x, y)	$TR = 120x + 95y$	**Profit** $= 55x + 25y$
A ($x = 0$, $y = 250$)	$TR = 120(0) + 95(250) = 23\,750$	Profit $= 55(0) + 25(250) = 6250$
B ($x = 50$, $y = 200$)	$TR = 120(50) + 95(200) = 25\,000$*	Profit $= 55(50) + 25(200) = 7750$
C ($x = 120$, $y = 60$)	$TR = 120(120) + 95(60) = 20\,100$	Profit $= 55(120) + 25(60) = 8100$*
D ($x = 140$, $y = 0$)	$TR = 120(140) + 95(0) = 16\,800$	Profit $= 55(140) + 25(0) = 7700$

* Maximum value, subject to contraints.

(e) When revenue is maximised ($x = 50$, $y = 200$), there are 140 unused welding hours:

	Welding	**Finishing**	**Admin.**
50 type I gates	6 (50)	2 (50)	1 (50)
200 types II gates	2 (200)	1 (200)	1 (200)
Hours used	700	300	250
Max. hours available	840	300	250
Unused hours	140	0	0

When profit is maximised ($x = 120$, $y = 60$) there are 70 unused administration hours:

	Welding	**Finishing**	**Admin.**
120 type I gates	6 (120)	2 (120)	1 (120)
60 types II gates	2 (60)	1 (60)	1 (60)
Hours used	840	300	180
Max. hours available	840	300	250
Unused hours	0	0	70

Progress Exercises 9.1 Linear Programming

Assume the non-negativity constraint ($x \geq 0, y \geq 0$) applies in each of the following problems. In questions 1 to 4,

(a) Graph the inequality constraints.
(b) Shade in the feasible region.
(c) Calculate the corner points of the feasible region.

1. (i) $\begin{array}{l} 3x + 2y \geq 15 \\ 6x + 9y \geq 36 \end{array}$ (ii) $\begin{array}{l} 6x + 2y \leq 30 \\ 2x + 6y \leq 26 \end{array}$

2. (i) $4y + 5x \leq 20$ (ii) $y + 3x \leq 6$ (iii) $3y + 7x \leq 21$

3. (i) $4y + 5x \leq 20$ (ii) $8y + 15x \leq 48$ (iii) $3y + 7x \leq 21$

4. (i) $y + 5x \leq 10$ (ii) $y + 2x \leq 6$ (iii) $2y + 7x \geq 14$

5. Draw two isofunction lines for each of the following:

(a) $TC = 12x + 24y$ (b) $TR = 9x + 3y$ (c) $\pi = 5.5x + 2.85y$

6. (See question 3)
Find the maximum value of $W = 3x + 2y$, subject to the constraints

(i) $4y + 5x \leq 20$ (ii) $8y + 15x \leq 48$ (iii) $3y + 7x \leq 21$

(a) algebraically and (b) graphically.

7. (See question 6)
Find the minimum value of $C = 3x + 2y$, subject to the constraints

(i) $4y + 5x \leq 20$ (ii) $8y + 15x \leq 48$ (iii) $3y + 7x \leq 21$

(a) algebraically and (b) graphically.

8. For the following constraints

(i) $12x + 36y \leq 720$ (ii) $60x + 70y \leq 4200$ (iii) $24x + 40y \leq 1200$

(a) Graph the inequalities.
(b) Shade in the feasible region. Hence state whether each constraint is acting as a limitation.
(c) Calculate the corner points of the feasible region.

9. For the following constraints

(i) $25x + 80y \geq 800$ (ii) $22x + 5y \leq 220$ (iii) $x + y \geq 10$

(a) Graph the inequalities.
(b) Shade in the feasible region. Hence state whether each constraint is acting as a limitation.
(c) Calculate the corner points of the feasible region.

10. A company can use two types of machine A and B, in a manufacturing plant. The number of operators required and the running cost per day are given as

	Cost per day	**Available operators**	**Floor area (m^2)**	**Profit per machine**
Machine A	6	2	2	20
Machine B	3	4	2	30
Maximum available	360	280	160	

(a) Write down the inequality constraints and the profit function.

(b) Graph the inequalities. From the graph determine the number of machines A and B which should be used to maximise profits.

(c) Confirm your answer in (b) algebraically.

11. A gardener requires fertiliser to have a minimum of 80 units of nitrogen, 15 units of potassium and 10 units of iron. Two fertiliser mixes are available.
Brand X contains 20, 2 and 1 units of the minerals while
Brand Y contains 4, 1, 2 units of the minerals respectively.
Brand X costs £18 per kg while brand Y costs £6 per kg.

(a) Write down the inequality constraints and the equation of the cost function.

(b) Graph the inequality constraints. Hence determine the number of kg of each brand which provide the minimum mineral requirements at minimum cost.

(c) Confirm your answer in (b) algebraically.

9.2 Matrices

At the end of this section you should be able to:

- Define matrices
- Add, subtract and transpose matrices
- Multiply matrices
- Apply matrix arithmetic
- Use matrix arithmetic to simplify calculations for large arrays of data.

Matrices are rectangular arrays of numbers or symbols. The usual arithmetic operations—addition, subtraction and multiplication—are performed with these arrays. However, dealing with arrays of numbers, and not just single numbers as in ordinary arithmetic, means that there will be restrictions on all these arithmetic operations. Since matrices and determinants (see next section) are both arrays of numbers, much of the terminology used to describe matrices is the same as that used in determinants such as minors, cofactor, dimension.

9.2.1 Matrices: definition

Matrices are rectangular arrays of numbers or symbols.

The dimension of a matrix is stated as the number of rows by number of columns. The following are examples of matrices, with their dimensions:

$$A=\begin{pmatrix}1 & 2\\ -2 & 4\end{pmatrix} \qquad B=\begin{pmatrix}0 & 2\\ 1 & 0\\ 2 & 5\end{pmatrix} \qquad D=(-2 \quad 1 \quad 4)$$

Dimension 2×2 3×2 1×3

❐ **Special matrices**

The null matrix is a matrix of any dimension in which every element is zero, such as

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \qquad \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}, \qquad (0 \quad 0 \quad 0 \quad 0)$$

Dimension 2×2 3×1 1×4

The unit matrix or identity matrix is any square matrix in which every element is zero, except the elements on the main diagonal, each of which has the value 1, such as

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad I=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

The unit or identity matrix is represented by the symbol I.

Matrices are equal if they are of the same dimension and the corresponding elements are identical:

$$\begin{pmatrix}1 & -1\\ 0 & 3\end{pmatrix}, \qquad \begin{pmatrix}1 & -1\\ 0 & 3\end{pmatrix}$$ are equal, but the matrices

$$\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}, \qquad \begin{pmatrix}1 & 1\\ 1 & 1\\ 1 & 1\end{pmatrix}$$ are not equal. The elements are identical, but the dimensions are not.

The transpose of a matrix is the matrix obtained by writing the rows of any matrix as columns or vice versa as follows:

$$\begin{array}{r} \\ \\ \text{row 1} \rightarrow \\ \text{row 2} \rightarrow \end{array} \begin{pmatrix}a & b\\ c & d\end{pmatrix}^{\mathrm{T}} = \begin{array}{c} \begin{array}{cc}\text{row 1} & \text{row 2}\\ \downarrow & \downarrow\end{array} \\ \begin{pmatrix}a & c\\ b & d\end{pmatrix}\end{array}$$

where the superscript T indicates that the matrix is to be transposed.

9.2.2 Matrix addition and subtraction

To add matrices, add the corresponding elements, for example,

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}+\begin{pmatrix}e & f\\ g & h\end{pmatrix}=\begin{pmatrix}a+e & b+f\\ c+g & d+h\end{pmatrix} \tag{9.10}$$

To subtract matrices, subtract the corresponding elements, for example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix} \tag{9.11}$$

WORKED EXAMPLE 9.3

ADDING AND SUBTRACTING MATRICES

Given the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} \quad C = \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} \tag{9.12}$$

(a) Calculate
  (i) $A + C$
  (ii) $A - C$

(b) Why is it not possible to calculate $A + B$ or $A - B$?
Hence, state the restrictions on matrix addition and subtraction.

Solution

(a) (i) $$\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 1+3 & 2+(-2) \\ -2+5 & 4+0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & 4 \end{pmatrix}$$

(ii) $$\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 1-3 & 2-(-2) \\ -2-5 & 4-0 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -7 & 4 \end{pmatrix}$$

(b) When attempting to add the corresponding elements of the matrices A and B it is found that there is no third column in matrix A, therefore it is not possible to add pairs of corresponding elements in the two matrices. Matrix addition is not possible.

$$\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1+0 & 2+2 & ?+2 \\ -2+1 & 4+0 & ?+5 \end{pmatrix}$$

No corresponding elements to add!

The same problem arises when attempting to subtract matrices A and B: there is no third column in matrix A from which to subtract the elements in column three of matrix B.

To add or subtract two or more matrices, all matrices must have exactly the same dimensions

❒ Scalar multiplication

A scalar is an ordinary number: 2, 5, −8, etc. When a matrix is multiplied by a scalar, each element in the matrix is multiplied by the scalar:

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \tag{9.13}$$

WORKED EXAMPLE 9.4

MULTIPLICATION OF A MATRIX BY A SCALAR

(a) Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix}$$

calculate $5A$.

(b) Calculate $3I$, where I is the 3×3 unit matrix.

Solution

(a) $$5A = \begin{pmatrix} 5 \times 1 & 5 \times 2 \\ 5 \times (-2) & 5 \times 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ -10 & 20 \end{pmatrix}$$

(b) $$3I = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

9.2.3 Matrix multiplication

Two matrices A and B are multiplied as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} \end{pmatrix} \tag{9.14}$$

where row 1, row 2 are from matrix A, col. 1, col. 2 from matrix B.

To multiply a row by a column, for example (row 1 × col. 1), proceed as follows:

$$\begin{pmatrix} a & b \\ - & - \end{pmatrix} \times \begin{pmatrix} e & - \\ g & - \end{pmatrix} = \begin{pmatrix} (a \times e) + (b \times g) & - \\ - & - \end{pmatrix} \tag{9.15}$$

that is, from row 1 and column 1, multiply the first pair of elements in each, the second pair of elements in each, then add the products of these pairs.

The product of the two matrices written out in full is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \tag{9.16}$$

$$\text{dimension:} \quad (2 \times 2) \times (2 \times 2) \quad = (2 \times 2)$$

WORKED EXAMPLE 9.5

MATRIX MULTIPLICATION

Given the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} \quad C = \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix}$$

evaluate: (a) AC (b) CA (c) AB (d) BA

Compare your answers for (a) and (b).

Solution

(a) The product of the matrices A and C is given in general as:

$$AC = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \times \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} \end{pmatrix}$$

$$= \begin{pmatrix} (1)(3)+(2)(5) & (1)(-2)+(2)(0) \\ (-2)(3)+(4)(5) & (-2)(-2)+(4)(0) \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -2 \\ 14 & 4 \end{pmatrix}$$

(b) Similarly, CA is calculated as follows:

$$CA = \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} \end{pmatrix}$$

$$= \begin{pmatrix} (3)(1)+(-2)(-2) & (3)(2)+(-2)(4) \\ (5)(1)+(0)(-2) & (5)(2)+(0)(4) \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -2 \\ 5 & 10 \end{pmatrix}$$

In the above example

$$AC = \begin{pmatrix} 13 & -2 \\ 14 & 4 \end{pmatrix}$$

and

$$CA = \begin{pmatrix} 7 & -2 \\ 5 & 10 \end{pmatrix}$$

therefore $AC \neq CA$. (Since two matrices are equal only if all corresponding elements are identical.)

So in matrix multiplication, the order of multiplication is important. In general, in matrix multiplication, $AC \neq CA$.

(c) The product of the matrices A and B is given in general as

$$AB = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \times \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} & \text{row 1} \times \text{col. 3} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} & \text{row 2} \times \text{col. 3} \end{pmatrix}$$

$$\text{dimension:} \quad (2 \times 2) \times (2 \times 3) = (2 \times 3)$$

Multiplying the rows by the columns,

$$AB = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \times \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(0) + (2)(1) & (1)(2) + (2)(0) & (1)(2) + (2)(5) \\ (-2)(0) + (4)(1) & (-2)(2) + (4)(0) & (-2)(2) + (4)(5) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 12 \\ 4 & -4 & 16 \end{pmatrix}$$

(d) To evaluate BA, proceed as usual:

$$BA = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} \end{pmatrix}$$

$$\text{dimension} \quad (2 \times 3) \times (2 \times 2)$$

Then multiplying the rows in the first matrix by the columns in the second matrix, it is found that:

$$BA = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix}$$

$$\boxed{\text{no third element in the columns of } A \text{ to multiply the third elements in the rows of } B}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$= \begin{pmatrix} (0)(1) + (2)(-2) + (2)(?) & (0)(2) + (2)(4) + (2)(?) \\ (1)(1) + (0)(-2) + (5)(?) & (1)(2) + (0)(4) + (5)(?) \end{pmatrix}$$

So the rows by the columns cannot be multiplied, since the rows in B contain three elements and the columns in A only two elements. Therefore, matrix multiplication is not possible. In general,

> Matrix multiplication BA is possible if the number of elements in the rows of the first matrix (B) is the same as the number of elements in the columns of the second matrix (A).

This condition for matrix multiplication can be established quickly by writing down the dimensions of the matrices to be multiplied, in order:

$A \times B =$ product

dimension of product: 2×3

dimensions: $(2 \times 2) \times (2 \times 3) = (2 \times 3)$

the same, so multiplication is possible

The 'inside' numbers are the same, therefore multiplication is possible. The 'outside' numbers give us the dimension of the product.

9.2.4 Applications of matrix arithmetic

WORKED EXAMPLE 9.6

APPLICATIONS OF MATRIX ARITHMETIC

A distributor records the weekly sales of personal computers (PCs) in three retail outlets in different parts of the country (see Table 9.5). The cost price of each model is:

Table 9.5 Number of computers sold in each shop

	Pentium (basic)	Pentium (extra)	Pentium (latest)
Shop A	150	320	180
Shop B	170	420	190
Shop C	201	63	58

Pentium (basic) £480, Pentium (extra) £600, Pentium (latest) £1020

The retail price of each model in each of the three shops is given in Table 9.6.

Table 9.6 Selling price of computers in each shop

	Pentium (basic)	Pentium (extra)	Pentium (latest)
Shop A	560	750	1580
Shop B	520	690	1390
Shop C	590	720	1780

Use matrix multiplication to calculate

(a) The total weekly cost of computers to each shop.
(b) The total weekly revenue for each model for each shop.
(c) The total weekly profit for each shop.

Which shop makes the greatest overall profit?

Solution

(a) The numbers sold from Table 9.5 may be written as a matrix, Q:

$$Q = \begin{array}{l} \text{PC} \rightarrow \\ \text{Shop A} \\ \text{Shop B} \\ \text{Shop C} \end{array} \begin{array}{c} \text{basic extra latest} \\ \begin{pmatrix} 150 & 320 & 180 \\ 170 & 420 & 190 \\ 201 & 63 & 58 \end{pmatrix} \end{array}$$

Write the cost of each type of computer as a column matrix:

$$C = \begin{pmatrix} 480 \\ 600 \\ 1020 \end{pmatrix}$$

If this cost matrix C is premultiplied by the numbers sold matrix, Q, the product will be a 3×1 matrix, in which the elements in each row gives the total cost of computers to each shop:

$$Q \cdot C = \text{Total cost}$$

$$\text{dimension:} \quad (3 \times 3)(3 \times 1) = (3 \times 1)$$

$$\begin{pmatrix} 150 & 320 & 180 \\ 170 & 420 & 190 \\ 201 & 63 & 58 \end{pmatrix} \begin{pmatrix} 480 \\ 600 \\ 1020 \end{pmatrix} = \begin{pmatrix} 150(480) + 320(600) + 180(1020) \\ 170(480) + 420(600) + 190(1020) \\ 201(480) + 63(600) + 58(1020) \end{pmatrix}$$

$$= \begin{pmatrix} 447\,600 \\ 527\,400 \\ 193\,440 \end{pmatrix}$$

Cost to: shop A = £447 600, shop B = £527 400, shop C = £193 440.

(b) The total revenue = price × quantity. The quantities are given by the matrix, Q, for the data in Table 9.5. The prices are obtained from the data in Table 9.6. Matrix multiplication, however, is carried out by multiplying rows by columns; therefore in order to multiply quantity × price for each PC, the rows in Table 9.6 must be written as columns in the prices matrix. That is, the prices matrix must be transposed before premultiplying by the quantities matrix, Q.

$$P = \begin{pmatrix} 560 & 750 & 1580 \\ 520 & 690 & 1390 \\ 590 & 720 & 1780 \end{pmatrix}^{\mathrm{T}} = \begin{array}{r} \text{Price of basic PC} \rightarrow \\ \text{Price of extra PC} \rightarrow \\ \text{Price of latest PC} \rightarrow \end{array} \begin{pmatrix} 560 & 520 & 590 \\ 750 & 690 & 720 \\ 1580 & 1390 & 1780 \end{pmatrix}$$

Now multiply Q (quantities matrix) by P (prices matrix from data in Table 9.6):

$$\begin{array}{lc} \text{PC type} & \begin{array}{ccc}\text{basic} & \text{extra} & \text{latest}\\ \downarrow & \downarrow & \downarrow\end{array} \\ Q \times P = \begin{array}{r}\text{shop A}\\ \text{shop B}\\ \text{shop C}\end{array} & \begin{pmatrix}150 & 320 & 180\\ 170 & 420 & 190\\ 201 & 63 & 58\end{pmatrix}\end{array}$$

$$\times \begin{array}{r}\text{Price of basic} \rightarrow\\ \text{Price of extra} \rightarrow\\ \text{Price of latest} \rightarrow\end{array} \begin{pmatrix}560 & 520 & 590\\ 750 & 690 & 720\\ 1580 & 1390 & 1780\end{pmatrix}$$

$$\begin{array}{lcc} Q \times P & \text{shop A} & \text{shop B} \\ & \downarrow & \downarrow \\ = & \left(\begin{array}{c} 150(560) + 320(750) + 180(1580) \\ \text{N/A} \\ \text{N/A} \end{array}\right. & \begin{array}{c} \text{N/A} \\ 170(520) + 420(690) + 190(1390) \\ \text{N/A} \end{array} \end{array}$$

$$\begin{array}{c} \text{shop C} \\ \downarrow \\ \times \left.\begin{array}{c} \text{N/A} \\ \text{N/A} \\ 201(590) + 63(720) + 58(1780) \end{array}\right) \end{array}$$

The total revenue for each shop is given by the elements on the main diagonal of the product matrix. Total revenue for shops A, B and C are summarised in a column matrix, TR;

$$\begin{array}{rcl} & \begin{array}{ccc}\text{Basic PC} & \text{Extra PC} & \text{Latest PC}\end{array} & \\ \begin{array}{r}\text{Revenue matrix,}\\ TR =\end{array} & \begin{pmatrix} 84\,000 + 240\,000 + 284\,400 \\ 88\,400 + 289\,800 + 264\,100 \\ 118\,590 + 45\,360 + 103\,240 \end{pmatrix} & \begin{array}{l} \leftarrow \text{shop A} \\ \leftarrow \text{shop B} \\ \leftarrow \text{shop C} \end{array} = \begin{pmatrix} 604\,400 \\ 642\,300 \\ 267\,190 \end{pmatrix} \end{array}$$

(c) Profit $= TR - TC$

$$= \begin{pmatrix} 604\,400 \\ 642\,300 \\ 267\,190 \end{pmatrix} - \begin{pmatrix} 447\,600 \\ 527\,400 \\ 193\,440 \end{pmatrix} = \begin{pmatrix} 156\,800 \\ 114\,900 \\ 73\,750 \end{pmatrix} \begin{array}{l} \leftarrow \text{shop A} \\ \leftarrow \text{shop B} \\ \leftarrow \text{shop C} \end{array}$$

So shop A makes the highest profit.

Progress Exercises 9.2 Introductory Matrix Algebra and Applications

1. Given the matrices: $A = \begin{pmatrix} 1 & -4 \\ 0 & -9 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ -7 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 5 & -1 & -1 \\ 12 & 0 & 2 \end{pmatrix}$

   calculate, if possible:

   (a) $A + B$ (b) $A - B$ (c) $A + 4B$ (d) $A + I$ (e) AI (f) $A + C$
   (g) $A + B^{\mathrm{T}}$ (h) BC (i) CB (j) CB^{T} (k) $(AB)^{\mathrm{T}}$ (l) $C + 5I$
   (m) $C^{\mathrm{T}}A$ (n) $(BC)^{\mathrm{T}}$ (o) $AC + B$

2. Given the matrices: $A = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 2 & -5 \\ -1 & 1 \end{pmatrix}$

   Determine each of the following, if possible:

   (a) $A + C$ (b) $B^{\mathrm{T}} + C$ (c) BC (d) $A^{\mathrm{T}}B^{\mathrm{T}}$ (e) $B^{\mathrm{T}}A^{\mathrm{T}}$

3. A fast-food chain has three shops, A, B and C. The average daily sales and profit in each shop is given in the following table:

	Units sold			**Units profit**		
	Shop A	Shop B	Shop C	Shop A	Shop B	Shop C
Burgers	800	400	500	20p	40p	33p
Chips	950	600	700	50p	45p	60p
Drinks	500	1200	900	30p	35p	20p

   Use matrix multiplication to determine

   (a) The profit for each product
   (b) The profit for each shop

4. The percentage of voters who will vote for party candidates A, B, and C is given in the following table:

	A	B	C	No. of voters
Area 1	60%	20%	20%	25 000
Area 2	45%	30%	25%	60 000
Area 3	38%	30%	32%	98 000

   Use matrix multiplication to calculate the total number of votes for each candidate.

5. Given the following matrices:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

(a) Show that $AB \neq BA$.
(b) Determine the following if possible:

(i) AC (ii) AD (iii) DC (iv) DCC^T

9.3 Solution of Equations: Elimination Methods

In Chapter 3, simultaneous equations were solved by adding multiples of equations to other equations until one equation in one variable was obtained. With one variable known, the remaining variables were easily found. See Worked Examples 3.1, 3.2, 3.3, 3.6. Worked Example 3.6 will be solved here by the method of Gaussian elimination.

9.3.1 Gaussian elimination

In its simplest form, Gaussian elimination is a technique for solving a system of n linear equations in n unknowns by systematically adding multiples of equations to other equations in such a way that we end up with a series of n equations, each containing one less unknown than the previous equation, with the last equation containing just one unknown. In other words, a set of equations such as

$$\begin{aligned} x + y - z &= 3 \\ 2x + y - z &= 4 \\ 2x + 2y + z &= 12 \end{aligned} \tag{9.17}$$

may be reduced systematically to

$$\begin{aligned} x + y - z &= 3 \\ -y + z &= 2 \\ 2z &= 4 \end{aligned} \tag{9.18}$$

The solutions to both sets of equations (9.17) and (9.18) are identical, but the reduced set of equations is easily solvable. The last equation contains just one unknown, z, so solve this first. Substitute this value of z into the middle equation to solve for y. Finally, with y and z known, solve the first equation for x. This is called solving by back substitution. The following worked example will demonstrate the method of Gaussian elimination and solving by back substitution.

WORKED EXAMPLE 9.7

SOLUTION OF A SYSTEM OF EQUATIONS: GAUSSIAN ELIMINATION

Solve the following equations by Gaussian elimination:

$$x + y - z = 3 \quad (1)$$

$$2x + y - z = 4 \quad (2)$$

$$2x + 2y + z = 12 \quad (3)$$

Solution

All three equations must be written in the same format: variables x, y, z (in the same order) on the LHS and constants on the RHS. With the equations written in order, the variables may be dropped in the calculations which follow.

$$\begin{array}{rl} x + y - z = 3 & (1) \\ 2x + y - z = 4 & (2) \\ 2x + 2y + z = 12 & (3) \end{array} \quad \longrightarrow \quad \begin{array}{c} \begin{array}{cccc} x & y & z & \text{RHS} \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & 1 & -1 & 4 \\ 2 & 2 & 1 & 12 \end{pmatrix} \end{array} \quad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

The 3×4 matrix is called the augmented matrix, A. The aim of Gaussian elimination is to systematically reduce this array of numbers to upper triangular form, similar in form to the reduced set of equations (9.18). A matrix of upper triangular form has zeros beneath the main diagonal (tinted background):

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & a_{3,3} & a_{3,4} \end{pmatrix}$$

Therefore, in column 1, we require zeros beneath the first element, and in column 2 we require zeros beneath the second element.

Action	**Augmented matrix**	**Calculations**
To get the required 0s, you must add (−2 × row 1) to row 2 and (−2 × row 1) to row 3	$\begin{array}{ccccc} x & y & z & \text{RHS} & \\ \downarrow & \downarrow & \downarrow & \downarrow & \end{array}$ $\begin{pmatrix} 1 & 1 & -1 & 3 \\ \textcircled{2} & 1 & -1 & 4 \\ \textcircled{2} & 2 & 1 & 12 \end{pmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$	^{1}Updated once ^{2}Updated twice
row 2 + (−2 × row 1) ⟶	$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 1 & -2 \\ 2 & 2 & 1 & 12 \end{pmatrix} \begin{array}{c} (1) \\ (2)^1 \\ (3) \end{array}$	$\left\{ \begin{array}{rrrrl} -2 & -2 & 2 & -6 & (1) \times -2 \\ 2 & 1 & -1 & 4 & (2) \\ \hline 0 & -1 & 1 & 2 & \text{adding} \end{array} \right.$
row 3 + (−2 × row 1) ⟶	$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 3 & 6 \end{pmatrix} \begin{array}{c} (1) \\ (2)^1 \\ (3)^1 \end{array}$	$\left\{ \begin{array}{rrrrl} -2 & -2 & 2 & -6 & (1) \times -2 \\ 2 & 2 & 1 & 12 & (3) \\ \hline 0 & 0 & 3 & 6 & \text{adding} \end{array} \right.$

No further elimination is necessary, since there is a 0 beneath the second element in column 2. However, divide row 3 by 3 to simplify (optional)

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{array}{c} (1) \\ (2)^1 \\ (3)^2 \end{array} \longleftarrow \boxed{\text{divide row } 3^1 \text{ by } 3}$$

Rewrite the equations from the augmented matrix:

$$\begin{aligned} x+y-z&=3 && (1)\\ -y+z&=-2 && (2)^1\\ z&=2 && (3)^2 \end{aligned} \qquad \textbf{(9.19)}$$

Solve by back substitution

Start with equation $(3)^2$

$(3)^2 \rightarrow z=2$

Substitute $z=2$ into equation $(2)^1$

$(2)^1 \rightarrow -y+z=-2$ hence $-y+2=-2 \rightarrow y=4$

Substitute $z=2$ and $y=4$ into equation $(3)^2$

$(3)^2 \rightarrow x+y-z=3$ hence $x+4-2=3 \rightarrow x=1$

Solution: $x=1$, $y=4$, $z=2$

In Worked Example 9.7 the elimination worked out quickly and easily; the numbers in the augmented matrix made the arithmetic easy. However, when the numbers are not so convenient, divide the row that is being used in the elimination by the value of the eliminating element, as illustrated in Worked Example 9.8.

WORKED EXAMPLE 9.8

MORE GAUSSIAN ELIMINATION

Solve the following equations by Gaussian elimination:

$$\begin{aligned} 2x+y+z&=12\\ 6x+5y-3z&=6\\ 4x-y+3z&=5 \end{aligned}$$

Solution

The equations are already written in the required form: variables in order on the LHS and constants on the RHS. Therefore, write down the augmented matrix and proceed with the Gaussian elimination:

$$\begin{aligned} 2+y+z&=12\\ 6x+5y-3z&=6\\ 4x-y+3z&=5 \end{aligned} \quad \rightarrow \quad \begin{array}{c} \begin{array}{cccc} x & y & z & \text{RHS} \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 12\\ 6 & 5 & -3 & 6\\ 4 & -1 & 3 & 5 \end{pmatrix} \end{array}$$

Now proceed with the elimination on the augmented matrix.

Action	Augmented matrix	Calculations
Make the tinted 2 into a 1 by dividing row 1 by 2	$\begin{array}{cc} \begin{array}{cccc} x & y & z & \text{RHS} \end{array} & \\ \begin{pmatrix} 2 & 1 & 1 & 12 \\ 6 & 5 & -3 & 6 \\ 4 & -1 & 3 & 5 \end{pmatrix} & \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \end{array}$	[1] Updated once [2] Updated twice
	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 6 & 5 & -3 & 6 \\ 4 & -1 & 3 & 5 \end{pmatrix} \begin{matrix} (1)^1 \\ (2) \\ (3) \end{matrix}$	
row 2 + (−6 × row 1^1)	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 0 & 2 & -6 & -30 \\ 4 & -1 & 3 & 5 \end{pmatrix} \begin{matrix} (1)^1 \\ (2)^1 \\ (3) \end{matrix}$	$\left\{ \begin{array}{rrrrl} -6 & -3 & -3 & -36 & (1)^1 \times -6 \\ 6 & 5 & -3 & 6 & (2) \\ \hline 0 & 2 & -6 & 30 & \text{adding} \end{array} \right.$
row 3 + (−4 × row 1^1)	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 0 & 2 & -6 & -30 \\ 0 & -3 & 1 & -19 \end{pmatrix} \begin{matrix} (1)^1 \\ (2)^1 \\ (3)^1 \end{matrix}$	$\left\{ \begin{array}{rrrrl} -4 & -2 & -2 & -24 & (1)^1 \times -4 \\ 4 & -1 & 3 & 5 & (3) \\ \hline 0 & -3 & 1 & -19 & \text{adding} \end{array} \right.$
Make the tinted 2 into a 1 by dividing row 2^1 by 2	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 0 & 2 & -6 & -30 \\ 0 & -3 & 1 & -19 \end{pmatrix} \begin{matrix} (1)^1 \\ (2)^1 \\ (3)^1 \end{matrix}$	
	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 0 & 1 & -3 & -15 \\ 0 & -3 & 1 & -19 \end{pmatrix} \begin{matrix} (1)^1 \\ (2)^2 \\ (3)^1 \end{matrix}$	
row 3^1 + (3 × row 2^2)	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & -8 & -64 \end{pmatrix} \begin{matrix} (1)^1 \\ (2)^2 \\ (3)^2 \end{matrix}$	$\left\{ \begin{array}{rrrrl} 0 & 3 & -9 & -45 & (2)^2 \times 3 \\ 0 & -3 & 1 & -19 & (3)^1 \\ \hline 0 & 0 & -8 & -64 & \text{adding} \end{array} \right.$

The elimination is now complete; solve by back substitution.
From equation $(3)^2$

$$-8z = -64 \quad \text{hence} \quad z = \frac{-64}{-8} = 8$$

Substitute $z = 8$ into equation $(2)^2$

$$y - (3 \times 8) = -15 \quad \text{hence} \quad y = -15 + (3 \times 8) = 9$$

Substitute $z = 8$ and $y = 9$ into equation $(1)^1$

$$x + (0.5 \times 9) + (0.5 \times 8) = 6 \quad \text{hence} \quad x = 6 - 4.5 - 4 = -2.5$$

Solution: $x = -2.5$, $y = 9$, $z = 8$

9.3.2 Gauss–Jordan elimination

Gauss–Jordan elimination goes further than Gaussian elimination, producing an augmented matrix with a main diagonal of ones:

$$\begin{pmatrix} 1 & 0 & 0 & b_{1.4} \\ 0 & 1 & 0 & b_{2.4} \\ 0 & 0 & 1 & b_{3.4} \end{pmatrix}$$

In this form the solutions may be read off immediately, as in Worked Example 9.9.

WORKED EXAMPLE 9.9

GAUSS–JORDAN ELIMINATION

Solve the following equations by Gauss–Jordan elimination:

$$2x + y + z = 12$$

$$6x + 5y - 3z = 6$$

$$4x - y + 3z = 5$$

Solution

Rearrange the equations to have variables on the LHS and constants on the RHS, as for Gaussian elimination. Then write them as an augmented matrix. Start by carrying out the Gaussian elimination, i.e. reducing the augmented matrix to upper triangular form. Since this is the same set of equations as those in Worked Example 9.8, we will continue from this point.

Action	Augmented matrix	Calculations
Make the tinted -8 into a 1 by dividing row 3^2 by -8	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & -8 & -64 \end{pmatrix} \begin{array}{l} (1)^1 \\ (2)^2 \\ (3)^2 \end{array}$	
Add multiples of row 3 to rows 1 and 2 to generate 0s in column 3	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & 1 & 8 \end{pmatrix} \begin{array}{l} (1)^1 \\ (2)^2 \\ (3)^3 \end{array}$	
$(-0.5 \times \text{row } 3^3) + \text{row } 1^1$ $(3 \times \text{row } 3^3) + \text{row } 2^2$	$\begin{pmatrix} 1 & 0.5 & 0 & 2 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 8 \end{pmatrix} \begin{array}{l} (1)^2 \\ (2)^3 \\ (3)^3 \end{array}$	Calculate $(-0.5 \times \text{row } 3^3)$ $\rightarrow (0\ \ 0\ \ -0.5\ \ -4)$ Calculate $(3 \times \text{row } 3^3)$ $\rightarrow (0\ \ 0\ \ 3\ \ 24)$
Add multiples of row 2 to row 1 to generate 0s in column 2	$\begin{pmatrix} 1 & 0 & 0 & -2.5 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 8 \end{pmatrix} \begin{array}{l} (1)^3 \\ (2)^3 \\ (3)^3 \end{array}$	Calculate $(-0.5 \times \text{row } 2^3)$ $\rightarrow (0\ \ -0.5\ \ 0\ \ -4.5)$

Write down the equations from the augmented matrix:

$$x + 0y + 0z = -2.5$$
$$0x + y + 0z = 9$$
$$0x + 0y + z = 8$$

or read off the solution directly: $x = -2.5$, $y = 9$, $z = 8$

Note: If the elimination process produces fewer equations than unknowns then there is no unique solution; see Chapter 8 of Hartley and Wynn-Evans.

Gauss–Jordan elimination will be used in Section 9.5 to calculate the inverse of a matrix.

Progress Exercises 9.3 Gaussian and Gauss–Jordan Elimination

1. Write the following system of equations as an augmented matrix:

$$\begin{aligned} 3x + 3y + 6z &= 12 \\ x - 3y + 5z &= 5 \\ 2x + 10y - 3z &= 0 \end{aligned}$$

(a) Reduce the augmented matrix to upper triangular form, then solve by back substitution.
(b) Solve by Gauss–Jordan elimination.

2. Solve the following systems of equations by (i) Gaussian elimination and (ii) Gauss–Jordan elimination:

(a) $$\begin{aligned} x + y &= 12 \\ 2x + 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned}$$

(b) $$\begin{aligned} x + y &= 12 \\ 2x - 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned}$$

(c) $$\begin{aligned} x + y &= 12 \\ 2x + 2y &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned}$$

(d) $$\begin{aligned} x + y - 2z &= 12 \\ x - 5y + 4z &= 20 \\ -6x + 3y - 15z &= 0 \end{aligned}$$

Solve the equations in questions 3 to 7 and leave your answers as fractions.

3. $$\begin{aligned} 3x + 4y - 9z &= -2 \\ 6x + 15z - 21 &= 0 \\ 5x - 4y - 9 &= 0 \end{aligned}$$

4. $$\begin{aligned} P_1 + 4P_2 + 8P_3 &= 26 \\ 5P_1 + 7P_2 &= 38 \\ 8P_1 + 12P_2 + 2P_3 &= 66 \end{aligned}$$

5. $$\begin{aligned} 2Y - 5C + 0.8T &= 580 \\ -Y + C + 0.6T + 340 &= 0 \\ 0.4Y - T &= 100 \end{aligned}$$

8. Use an elimination method to find the equilibrium prices and quantities where the supply and demand functions for each good are as follows:

$$\begin{aligned} Q_{d1} &= 50 - 2P_1 + 5P_2 - 3P_3 & Q_{s1} &= 8P_1 - 5 \\ Q_{d2} &= 22 + 7P_1 - 2P_2 + 5P_3 & Q_{s2} &= 12P_2 - 5 \\ Q_{d3} &= 17 + P_1 + 5P_2 - 3P_3 & Q_{s3} &= 4P_3 - 1 \end{aligned}$$

9.4 Determinants

At the end of this section you should be able to:

- Evaluate 2 × 2 and 3 × 3 determinants
- State and use Cramer's rule to solve two and three simultaneous equations in the same number of unknowns.
- Find equilibrium values for the national income model and other applications.

9.4.1 Evaluate 2 × 2 determinants

❑ Determinants: definitions

- A determinant is a square array of numbers or symbols, for example,

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad B = \begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} \qquad D = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

- The dimensions of a determinant are stated as (number of rows, r) × (number of columns, c). Therefore, the dimensions of the determinants A, B and D are

$$A\colon\ 2 \times 2, \qquad B\colon\ 2 \times 2, \qquad D\colon\ 3 \times 3.$$

The elements within a determinant (or matrix) are referred to by the row and column in which the element occurs, for example,

$$B = \begin{vmatrix} \overset{\substack{\text{row 1, col. 1, etc.} \\ \downarrow}}{b_{1,1}} = 2 & b_{1,2} = 5 \\ b_{2,1} = 3 & b_{2,2} = -4 \end{vmatrix}$$

❑ How to evaluate a 2 × 2 determinant

A 2 × 2 determinant is evaluated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a)(d) - (c)(b) \tag{9.20}$$

Hence the value of determinant B is:

$$\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = (2)(-4) - (3)(5) = -8 - (15) = -23$$

Warning: Most mistakes made in evaluating determinants arise from signs. So use brackets.

9.4.2 Use determinants to solve equations: Cramer's rule

You might wonder 'how can determinants be used to solve equations?' The use of determinants are demonstrated through Worked Example 9.10.

WORKED EXAMPLE 9.10

USING DETERMINANTS TO SOLVE SIMULTANEOUS EQUATIONS

Solve the simultaneous equations by eliminating one variable, then solve for the other variable.

$$\text{(a)}\quad \begin{aligned} 2x + 5y &= 10 \\ 3x + 4y &= 5.8 \end{aligned} \qquad (b)\quad \begin{aligned} a_1x + b_1y &= d_1 \\ a_2x + b_2y &= d_2 \end{aligned}$$

Describe the method of solution in general terms using the equations given in (b). Hence, deduce a method of solution which uses determinants.

Solution

To describe the method of solution in general we shall solve (a) and (b) in parallel.

(a)	**(b)**
(1) $2x + 5y = 10$	(1) $a_1x + b_1y = d_1$
(2) $3x + 4y = 5.8$	(2) $a_2x + b_2y = d_2$
To eliminate y, multiply equation (1) by 4 and equation (2) by 5.	To eliminate y, multiply equation (1) by b_2 and equation (2) by b_1
(1) × 4 $8x + 20y = 40$	(1) × b_2 $a_1b_2x + b_1b_2y = d_1b_2$
(2) × 5 $15x + 20y = 29$	(2) × b_1 $a_2b_1x + b_1b_2y = d_2b_1$
Subtract	Subtract
$8x - 15x + 0y = 40 - 29$	$a_1b_2x - a_2b_1x = d_1b_2 - d_2b_1$
Gathering the x terms	Gathering the x terms
$-7x = 11$	$(a_1b_2 - a_2b_1)x = d_1b_2 - d_2b_1$
Divide across by -7	Divide across by $(a_1b_2 - a_2b_1)$
$x = \frac{11}{-7} = -1.57$	$x = \frac{d_1b_2 - d_2b_1}{(a_1b_2 - a_2b_1)}$ **(9.21)**

Looking carefully at the numerator and the denominator of equation (9.21), notice that in each case there is (product of two values) – (product of two values). This is exactly the same format that was used when evaluating a 2 × 2 determinant. Therefore, the numerator and denominator in equation (9.21) may be written as determinants:

$$x = \frac{d_1b_2 - d_2b_1}{(a_1b_2 - a_2b_1)} = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{9.22}$$

Similarly, if x was eliminated in equations (b) and solved for y, it would be found that:

$$y = \frac{a_1d_2 - a_2d_1}{(a_1b_2 - a_2b_1)} = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{9.23}$$

The formulae for x and y are now examined in detail. A general rule is deduced, called Cramer's rule, which uses determinants to solve simultaneous equations.

Cramer's rule

The solution of the simultaneous equations

$$a_1x + b_1y = d_1$$
$$a_2x + b_2y = d_2$$

is given by the formulae

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

When these formulae are examined in detail, it is noted that:

- The denominator is the same in each:

$$\underbrace{x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}}_{\boxed{\text{same}}} \tag{9.24}$$

the denominator in each case is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The columns in this determinant are related to the original equations as follows:

$$\begin{array}{l} a_1x + b_1y = d_1 \\ a_2x + b_2y = d_2 \\ \downarrow \quad\ \ \downarrow \\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{array}$$

Column 1 consists of the coefficients of the x variables from the original set of equations. Column 2 consists of the coefficients of the y variables from the original set of equations. The determinant of the coefficients shall be referred to as Δ, that is,

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Looking again at equation (9.24), a rule for writing down the numerators may be established:

$$\boxed{x \text{ col. in } \Delta \text{ replaced by col. } (d_1 \ \ d_2)} \qquad \boxed{y \text{ col. in } \Delta \text{ replaced by col. } (d_1 \ \ d_2)}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$\boxed{\qquad\qquad \Delta \qquad\qquad}$$

The determinant in which the column of x-coefficients are replaced by the column constants is referred to as Δ_x:

$$\Delta_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$$

The determinant in which the column of y-coefficients are replaced by the column constants is referred to as Δ_y:

$$\Delta_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$$

These observations are summarised:

The solution of two simultaneous equations in two unknowns is given by

$$x = \frac{\Delta_x}{\Delta} \qquad y = \frac{\Delta_y}{\Delta} \tag{9.25}$$

In fact, this general formula may be extended to any number of equations in the same number of unknowns. This general rule is called Cramer's rule. For example,

Cramer's rule

The solution of three linear equations in three unknowns is given by

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta}, \qquad z = \frac{\Delta_z}{\Delta} \tag{9.26}$$

Note: if $\Delta = 0$, there is division by zero when applying Cramer's rule formulae. $\Delta = 0$ means that the set of equations has no unique solution (see Chapter 3).

WORKED EXAMPLE 9.11

USING CRAMER'S RULE TO SOLVE SIMULTANEOUS EQUATIONS

Use Cramer's rule to solve the equations:

$$\text{(a)}\ \begin{aligned} y &= 10x + 12 \\ 4x + 2y &= 36 \end{aligned} \qquad \text{(b)}\ \begin{aligned} P &= 50 - 2Q \\ P &= 5 + 3Q \end{aligned}$$

Solution

In each case, Cramer's rule is used:

The solution of two simultaneous equations in two unknowns is

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta}$$

The procedure is as follows:

Step 1. Write the equations in order: LHS (variable terms in the same order) = RHS (constants).

Step 2. Write down the determinants Δ, Δ_x, Δ_y. Evaluate these determinants.

Step 3. Use Cramer's rule to solve for the unknowns.

$$\text{(a)}\ \begin{aligned} y &= 10x + 12 \\ 4x + 2y &= 36 \end{aligned}$$

Step 1

$$\begin{aligned} -10x + y &= 12 \\ 4x + 2y &= 36 \end{aligned}$$

Step 2

$$\Delta = \begin{vmatrix} -10 & 1 \\ 4 & 2 \end{vmatrix} = (-10)(2) - (4)(1)$$

$$= -20 - 4 = -24$$

$$\Delta_x = \begin{vmatrix} 12 & 1 \\ 36 & 2 \end{vmatrix} = (12)(2) - (36)(1)$$

$$= 24 - 36 = -12$$

$$\text{(b)}\ \begin{aligned} P &= 50 - 2Q \\ P &= 5 + 3Q \end{aligned}$$

Step 1

$$\begin{aligned} P + 2Q &= 50 \\ P - 3Q &= 5 \end{aligned}$$

Step 2

$$\Delta = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} = (1)(-3) - (1)(2)$$

$$= -3 - 2 = -5$$

$$\Delta_P = \begin{vmatrix} 50 & 2 \\ 5 & -3 \end{vmatrix} = (50)(-3)$$

$$- (5)(2) = -160$$

$$\Delta_y = \begin{vmatrix} -10 & 12 \\ 4 & 36 \end{vmatrix} = (-10)(36)$$

$$-(4)(12) = -408$$

Step 3. Use Cramer's rule.

$$x = \frac{\Delta_x}{\Delta} = \frac{-12}{-24} = 0.5$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-408}{-24} = 17$$

$$\Delta_Q = \begin{vmatrix} 1 & 50 \\ 1 & 5 \end{vmatrix} = (1)(5) - (1)(50)$$

$$= -45$$

Step 3. Use Cramer's rule.

$$P = \frac{\Delta_P}{\Delta} = \frac{-160}{-5} = 32$$

$$Q = \frac{\Delta_Q}{\Delta} = \frac{-45}{-5} = 9$$

WORKED EXAMPLE 9.12

FIND THE MARKET EQUILIBRIUM USING CRAMER'S RULE

Given the supply and demand functions for two related goods, A and B,

$$\text{Good A:}\quad \begin{cases} Q_{\mathrm{d}a} = 30 - 8P_a + 2P_b \\ Q_{\mathrm{s}a} = -15 + 7P_a \end{cases} \qquad \text{Good B:}\quad \begin{cases} Q_{\mathrm{d}b} = 28 + 4P_a - 6P_b \\ Q_{\mathrm{s}b} = 12 + 2P_b \end{cases}$$

(a) Write down the equilibrium condition for each good. Hence, deduce two equations in P_a and P_b.
(b) Use Cramer's rule to find the equilibrium prices and quantities for goods A and B.

Solution

(a) The equilibrium condition for each good is that $Q_{\mathrm{s}} = Q_{\mathrm{d}}$.

For good A:

$$-15 + 7P_a = 30 - 8P_a + 2P_b$$
$$15P_a - 2P_b = 45$$

For good B:

$$12 + 2P_b = 28 + 4P_a - 6P_b$$
$$-4P_a + 8P_b = 16$$

Therefore, the simultaneous equations are:

(1) $15P_a - 2P_b = 45$

(2) $-4P_a + 8P_b = 16$

(b) Applying Cramer's rule:

$$P_a = \frac{\Delta_{P_a}}{\Delta} = \frac{\begin{vmatrix} 45 & -2 \\ 16 & 8 \end{vmatrix}}{\begin{vmatrix} 15 & -2 \\ -4 & 8 \end{vmatrix}} = \frac{(45)(8) - (16)(-2)}{(15)(8) - (-4)(-2)} = \frac{360 - (-32)}{120 - (8)} = \frac{392}{112} = 3.5$$

$$P_b = \frac{\Delta_{P_b}}{\Delta} = \frac{\begin{vmatrix} 15 & 45 \\ -4 & 16 \end{vmatrix}}{\begin{vmatrix} 15 & -2 \\ -4 & 8 \end{vmatrix}} = \frac{(15)(16)-(-4)(45)}{(15)(8)-(-4)(-2)} = \frac{240-(-180)}{120-(8)} = \frac{420}{112} = 3.75$$

Substituting these values of P_a and P_b into any of the original equations, solve for Q_a and Q_b. Solution: $Q_a = 9.5$, $Q_b = 19.5$.

☐ General expressions for equilibrium in the income-determination model

Cramer's rule may be used to find general expressions for the equilibrium level of income, consumption, investment, etc. in the income determination model.

WORKED EXAMPLE 9.13

USE CRAMER'S RULE FOR THE INCOME-DETERMINATION MODEL

Given the general income-determination model (no government sector):

$$Y = C + I \qquad \textbf{(9.27)}$$

$$C = C_0 + bY \qquad \textbf{(9.28)}$$

where $I = I_0$, $0 < b < 1$ (b, I_0 and C_0 are constants).

(a) Write equations (9.24) and (9.25) in the form:

$$a_1 Y + a_2 C = a_3 \qquad a_1, a_2, a_3 \text{ are constants}$$

(b) Hence, use Cramer's rule to express the equilibrium levels of income (Y_e) and consumption (C_e) in terms of the constants b, I_0 and C_0.

Solution

(a) Rearranging equations (9.27) and (9.28) in the required form, variable terms on LHS and constant terms on RHS, and substituting I_0 for I:

$$Y - C = I_0 \qquad \textbf{(9.29)}$$

$$-bY + C = C_0 \qquad \textbf{(9.30)}$$

(b) Solve equations (9.29) and (9.30) for Y and C by Cramer's rule:

$$Y_e = \frac{\Delta_Y}{\Delta} = \frac{I_0 + C_0}{1-b} \quad \text{and} \quad C_e = \frac{\Delta_C}{\Delta} = \frac{C_0 + bI_0}{1-b}$$

Since:

$$\Delta = \begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix} = 1 - (-b)(-1) = 1 - b$$

$$\Delta_Y = \begin{vmatrix} I_0 & -1 \\ C_0 & 1 \end{vmatrix} = (I_0) - (-C_0) = I_0 + C_0$$

$$\Delta_C = \begin{vmatrix} 1 & I_0 \\ -b & C_0 \end{vmatrix} = C_0 - (-b)(I_0) = C_0 + bI_0$$

Progress Exercises 9.4 2 × 2 Determinants, with Applications

1. Evaluate the following determinants:

(a) $\begin{vmatrix} 4 & 1 \\ 9 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -2 & 0 \\ 2 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 4 & -3 \\ 1 & 8 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix}$ (e) $\begin{vmatrix} 6 & -1 \\ -6 & 2 \end{vmatrix}$

(f) $\begin{vmatrix} 6 & -1 \\ -6 & 1 \end{vmatrix}$ (g) $\begin{vmatrix} c & 2c \\ 1 & c \end{vmatrix}$ (h) $\begin{vmatrix} a & (1-a) \\ -a & a \end{vmatrix}$ (i) $\begin{vmatrix} \frac{1}{a} & -\frac{1}{1-a} \\ -a & a \end{vmatrix}$

Use Cramer's rule to solve the simultaneous equations (correct to two decimal places) in questions 2 to 7.

2. $x + 2y = 10, \quad 5x + 8y = 40$

3. $2x - y = 12, \quad 21x + 12y = 63$

4. $5P_1 + 9P_2 = 36, \quad 12P_1 + 2P_2 = 16$

5. $P = 100 - 8Q, \quad P = 30 + 5Q$

6. $50P + 8Q - 190 = 0, \quad P = 25 + 7Q$

7. $Y = 0.2r + 20, \quad Y = -0.05r + 42$

8. The demand and supply functions for two related products (1: pens, 2: paper) are given by the equations:

$$Q_{d1} = 30 - P_1 + 4P_2, \quad Q_{s1} = 3P_1 - 6$$
$$Q_{d2} = 36 + 3P_1 - 2P_2, \quad Q_{s2} = 12P_2 - 4$$

Use Cramer's rule to find the equilibrium prices and quantities.

9. Given the general income determination model:

(1) $Y = C + I_0$ and (2) $C = C_0 + bY$ where $0 < b < 1$, I_0 and C_0 are constants,

(a) Write equations (1) and (2) in the form $a_1Y + a_2C = a_3$ where a_1, a_2, a_3 are constants.

(b) Hence use Cramer's rule to express the equilibrium condition for income (Y) and consumption (C) in terms of the constants b, I_0 and C_0.

10. If the equilibrium condition in

(i) the goods market is given by the equation

$Y = C + I$ where $C = 237.8 + 0.2Y$ and $I = 10 - 0.4r$

(ii) the money market is given by the equation

$M_d = M_s$ where $M_d = 100 + 0.1Y - 0.3r$ and $M_s = 129.225$

(a) Write the equilibrium equations for each market in the form $aY + cr = c$ where a, b and c are constants

(b) Use Cramer's rule to solve for the equilibrium levels of income (Y) and interest rate (r), for which the product and money markets are simultaneously in equilibrium.

11. Given the demand function for two goods,

$$Q_1 = 5 - P_1 + 4P_2$$
$$Q_2 = -3P_2 + 2P_1$$

use Cramer's rule to derive expressions for P_1 and P_2 in terms of Q_1 and Q_2.

9.4.3 Evaluate 3 × 3 determinants

Using Cramer's rule to solve three equations in three unknowns involves the evaluation of 3 × 3 determinants. This is quite a different procedure from evaluating 2 × 2 determinants. To evaluate the general 3 × 3 determinant A, proceed as follows:

$$A = \begin{vmatrix} a_{1.1} & a_{1.2} & a_{1.3} \\ a_{2.1} & a_{2.2} & a_{2.3} \\ a_{3.1} & a_{3.2} & a_{3.3} \end{vmatrix} = (a_{1.1})\begin{vmatrix} a_{2.2} & a_{2.3} \\ a_{3.2} & a_{3.3} \end{vmatrix} - (a_{1.2})\begin{vmatrix} a_{2.1} & a_{2.3} \\ a_{3.1} & a_{3.3} \end{vmatrix} + (a_{1.3})\begin{vmatrix} a_{2.1} & a_{2.2} \\ a_{3.1} & a_{3.2} \end{vmatrix} \tag{9.31}$$

Note: brackets are used to avoid errors when substituting negative numbers.

WORKED EXAMPLE 9.14

EVALUATION OF A 3 × 3 DETERMINANT

Evaluate the 3 × 3 determinant, D, where

$$D = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

Solution

Applying the method given in equation (9.31), the value of determinant D is calculated as:

$$\begin{aligned} D &= 1\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} - (0)\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 1[(2)(2) - (3)(3)] - 0[(2)(2) - (1)(3)] + (-2)[(2)(3) - (1)(2)] \\ &= 1(-5) + (0)(1) + (-2)(4) = -5 + 0 - 8 = -13 \end{aligned}$$

WORKED EXAMPLE 9.15

SOLVE THREE SIMULTANEOUS EQUATIONS BY CRAMER'S RULE

The equilibrium condition for three related products simplifies to the following equations:

$$\begin{aligned} 15P_1 - 4P_2 - 7P_3 &= 14 \\ -4P_1 + 6P_2 - 2P_3 &= 34 \\ -3P_1 - 2P_2 + 12P_2 &= 1 \end{aligned}$$

Use Cramer's rule to solve for the equilibrium prices P_1, P_2 and P_3.

Solution

Go through the usual steps for Cramer's rule:

Step 1. The equations are already arranged in the required form

LHS (variable terms, in order) = RHS (constants).

Step 2. Write out and evaluate the four determinants: Δ, Δ_{P_1}, Δ_{P_2}, Δ_{P_3}.

The determinants are:

$$\Delta = \begin{vmatrix} 15 & -4 & -7 \\ -4 & 6 & -2 \\ -3 & -2 & 12 \end{vmatrix} \qquad \Delta_{P_1} = \begin{vmatrix} 14 & -4 & -7 \\ 34 & 6 & -2 \\ 1 & -2 & 12 \end{vmatrix} \qquad \Delta_{P_2} = \begin{vmatrix} 15 & 14 & -7 \\ -4 & 34 & -2 \\ -3 & 1 & 12 \end{vmatrix}$$

$$\Delta_{P_3} = \begin{vmatrix} 15 & -4 & 14 \\ -4 & 6 & 34 \\ -3 & -2 & 1 \end{vmatrix}$$

Evaluate each determinant as follows:

$$\Delta = \begin{vmatrix} 15 & -4 & -7 \\ -4 & 6 & -2 \\ -3 & -2 & 12 \end{vmatrix} = 15\begin{vmatrix} 6 & -2 \\ -2 & 12 \end{vmatrix} - (-4)\begin{vmatrix} -4 & -2 \\ -3 & 12 \end{vmatrix} + (-7)\begin{vmatrix} -4 & 6 \\ -3 & -2 \end{vmatrix}$$

$$\begin{aligned} &= 15[(6)(12) - (-2)(-2)] + 4[(-4)(12) - (-3)(-2)] \\ &\quad - (7)[(-4)(-2) - (-3)(6)] \\ &= (15)(68) + (4)(-54) + (-7)(26) \\ &= 1020 - 216 - 182 = 622 \end{aligned}$$

$$\Delta_{P_1} = \begin{vmatrix} 14 & -4 & -7 \\ 34 & 6 & -2 \\ 1 & -2 & 12 \end{vmatrix} = 14\begin{vmatrix} 6 & -2 \\ -2 & 12 \end{vmatrix} - (-4)\begin{vmatrix} 34 & -2 \\ 1 & 12 \end{vmatrix} + (-7)\begin{vmatrix} 34 & 6 \\ 1 & -2 \end{vmatrix}$$

$$\begin{aligned} &= 14[(6)(12) - (-2)(-2)] + 4[(34)(12) - (1)(-2)] - 7[(34)(-2) - (1)(6)] \\ &= 14[68] + 4[410] - 7[-74] \\ &= 952 + 1640 + 518 = 3110 \end{aligned}$$

Similarly, Δ_{P_2} is evaluated as follows:

$$\Delta_{P_2} = \begin{vmatrix} 15 & 14 & -7 \\ -4 & 34 & -2 \\ -3 & 1 & 12 \end{vmatrix} = 15\begin{vmatrix} 34 & -2 \\ 1 & 12 \end{vmatrix} - 14\begin{vmatrix} -4 & -2 \\ -3 & 12 \end{vmatrix} + (-7)\begin{vmatrix} -4 & 34 \\ -3 & 1 \end{vmatrix}$$

Now evaluate each 2×2 determinant:

$$\begin{aligned} \Delta_{P_2} &= 15[(34)(12) - (1)(-2)] - 14[(-4)(12) - (-3)(-2)] \\ &\quad - 7[(-4)(1) - (-3)(34)] \\ &= 15[410] - 14[-54] - 7[98] \\ &= 6150 + 756 - 686 = 6220 \end{aligned}$$

Similarly, Δ_{P_3} is evaluated as

$$\Delta_{P_3} = \begin{vmatrix} 15 & -4 & 14 \\ -4 & 6 & 34 \\ -3 & -2 & 1 \end{vmatrix} = 15\begin{vmatrix} 6 & 34 \\ -2 & 1 \end{vmatrix} - (-4)\begin{vmatrix} -4 & 34 \\ -3 & 1 \end{vmatrix} + (14)\begin{vmatrix} -4 & 6 \\ -3 & -2 \end{vmatrix}$$

Evaluate the 2 × 2 determinants,

$$\begin{aligned}\Delta_{P_3} &= 15[(6)(1) - (-2)(34)] + 4[(-4)(1) - (-3)(34)] \\ &\quad + 14[(-4)(-2) - (-3)(6)] \\ &= 15[74] + 4[98] + 14[26] \\ &= 1110 + 392 + 364 = 1866\end{aligned}$$

Step 3. Use Cramer's rule to solve for P_1, P_2 and P_3:

$$P_1 = \frac{\Delta_{P_1}}{\Delta} = \frac{3110}{622} = 5, \quad P_2 = \frac{\Delta_{P_2}}{\Delta} = \frac{6220}{622} = 10, \quad P_3 = \frac{\Delta_{P_3}}{\Delta} = \frac{1866}{622} = 3$$

❐ Applications

Cramer's rule may be used in any application which involves the solution of three or more simultaneous linear equations, such as in equilibrium, break-even, etc. In Worked Example 9.12 the equilibrium for a two-product market was calculated. The equilibrium for markets with three products or more may now be calculated. Worked Example 9.13, the income-determination model, may also be extended to three or more variables.

WORKED EXAMPLE 9.16

EQUILIBRIUM LEVELS IN THE NATIONAL INCOME MODEL

Given the following national income model:

$$Y = C + I_0 + G_0$$

$$C = C_0 + b(Y - T), \quad \text{and} \quad T = T_0 + tY$$

where Y = income, C = consumption, T = taxation, C_0, b, t, I_0, G_0 and T_0 (autonomous taxation) are constants and $C_0 > 0$; $0 < b < 1$; $0 < t < 1$:

(a) Write this model as three equations in terms of the variables Y, C and T.
(b) Use Cramer's rule to derive expressions for the equilibrium level of income, consumption and taxation.

Solution

(a) The variable terms containing Y, C and T are arranged on the LHS of each equation, the constants on the RHS:

$$\begin{aligned} Y - C &= I_0 + G_0 \\ -bY + C + bT &= C_0 \\ -tY + T &= T_0 \end{aligned}$$

(b) Using Cramer's rule, solve for Y, C, T:

$$Y = \frac{\Delta_Y}{\Delta} = \frac{I_0 + G_0 + C_0 - bT_0}{1 - b + bt} = \text{the equilibrium level of income}$$

$$C = \frac{\Delta_C}{\Delta} = \frac{C_0 - bT_0 + b(1-t)(I_0 + G_0)}{1 - b + bt} = \text{equilibrium level of consumption}$$

$$T = \frac{\Delta_T}{\Delta} = \frac{T_0(1-b) + t(C_0 + I_0 + G_0)}{1 - b + bt} = \text{equilibrium level of taxation}$$

Since:

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix} = (1)\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + (0) = 1 + (-b - (-bt))$$

$$= 1 - b + bt$$

$$\Delta_Y = \begin{vmatrix} I_0 + G_0 & -1 & 0 \\ C_0 & 1 & b \\ T_0 & 0 & 1 \end{vmatrix} = (I_0 + G_0)\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} C_0 & b \\ T_0 & 1 \end{vmatrix} + (0)$$

$$= (I_0 + G_0) + (C_0 - bT_0)$$

$$\Delta_C = \begin{vmatrix} 1 & I_0 + G_0 & 0 \\ -b & C_0 & b \\ -t & T_0 & 1 \end{vmatrix} = (1)\begin{vmatrix} C_0 & b \\ T_0 & 1 \end{vmatrix} - (I_0 + G_0)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + (0)$$

$$= (C_0 - bT_0) - (I_0 + G_0)(-b - (-bt))$$

$$= C_0 - bT_0 + b(I_0 + G_0)(1 - t)$$

$$\Delta_T = \begin{vmatrix} 1 & -1 & I_0 + G_0 \\ -b & 1 & C_0 \\ -t & 0 & T_0 \end{vmatrix} = (1)\begin{vmatrix} 1 & C_0 \\ 0 & T_0 \end{vmatrix} - (-1)\begin{vmatrix} -b & C_0 \\ -t & T_0 \end{vmatrix}$$

$$+ (I_0 + G_0)\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}$$

$$= T_0 - bT_0 + tC_0 + (I_0 + G_0)(0 - (-t))$$

$$= T_0(1 - b) + t(C_0 + I_0 + G_0)$$

Progress Exercises 9.5 3 × 3 Determinants, with Applications

1. Evaluate each of the following determinants:

$$\text{(a)}\ \begin{vmatrix} 1 & 2 & -5 \\ 0 & 6 & 5 \\ -1 & 2 & 7 \end{vmatrix} \qquad \text{(b)}\ \begin{vmatrix} -3 & 0 & 3 \\ 3 & 2 & 6 \\ 4 & 0 & 9 \end{vmatrix} \qquad \text{(c)}\ \begin{vmatrix} -2 & 3 & 2 \\ 5 & 12 & 2 \\ 3 & -4.5 & 3 \end{vmatrix}$$

2. Solve the following system of equations by Cramer's rule

$$\text{(a)}\ \begin{aligned} x + y &= 12 \\ 2x + 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned} \qquad \text{(b)}\ \begin{aligned} x + y &= 12 \\ 2x - 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned}$$

$$\text{(c)}\ \begin{aligned} x + y &= 12 \\ 2x + 2y &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned} \qquad \text{(d)}\ \begin{aligned} x + y - 2z &= 12 \\ x - 5y + 4z &= 20 \\ -6x + 3y - 15z &= 0 \end{aligned}$$

Use Cramer's rule to solve the following equations in questions 3 to 7.

$$\text{3.}\ \begin{aligned} 3x + 4y - 9z &= -2 \\ 6x + 15z - 21 &= 0 \\ 5x + 4y - 9 &= 0 \end{aligned} \qquad \text{4.}\ \begin{aligned} P_1 + 4P_2 + 8P_3 &= 26 \\ 5P_1 + 7P_2 &= 38 \\ 8P_1 + 12P_2 + 2P_3 &= 66 \end{aligned} \qquad \text{5.}\ \begin{aligned} 2Y - 5C + 0.8T &= 580 \\ -Y + C + 0.6T + 340 &= 0 \\ 0.4Y - T &= 100 \end{aligned}$$

$$\text{6.}\ \begin{aligned} 5x + 5y - 4z &= 15 \\ 7x - 3y + 4z &= 15 \\ z &= 2x - 8y \end{aligned} \qquad \text{7.}\ \begin{aligned} 5x + 1.7z &= 21.6 \\ y - z &= 0 \\ 2.5y - 2.5x &= 1.35x - 10.8 \end{aligned}$$

8. Use Cramer's rule to find the equilibrium prices and quantities where the supply and demand functions for each good are

$$\begin{aligned} Q_{d1} &= 50 - 2P_1 + 5P_2 - 3P_3, & Q_{s1} &= 8P_1 - 5 \\ Q_{d2} &= 22 + 7P_1 - 2P_2 + 5P_3, & Q_{s2} &= 12P_2 - 5 \\ Q_{d3} &= 17 + P_1 + 5P_2 - 3P_3, & Q_{s3} &= 4P_3 - 1 \end{aligned}$$

9. (a) State Cramer's rule.
Show graphically how a system of two equations in two unknowns has:

(i) no solution, (ii) a unique solution, (iii) infinitely many solutions.

Is it possible to solve all three systems above by Cramer's rule?
(b) Use Cramer's rule to solve for P_1, P_2 and P_3.

$$\begin{aligned} 0.5P_1 + 11.6P_2 - 8P_3 &= -47.39 \\ 1.9P_2 + 4.5P_3 &= 59.94 \\ 0.8P_1 + 3.5P_3 &= 49.7 \end{aligned}$$

9.5 The Inverse Matrix and Input/Output Analysis

At the end of this section you should be able to:

- Calculate the inverse of a matrix
- Use the inverse matrix to solve equations
- Solve problems related to input/output analysis

Section 9.4 used determinants to solve three equations in three unknowns. This section uses the inverse matrix for the same purpose. It also introduces one other application of the inverse matrix, input/output analysis.

☐ The inverse matrix

Section 9.2 introduced matrix arithmetic: addition, subtraction and multiplication. For each arithmetic operation, the restrictions imposed were noted because matrices were used instead of single numbers. In matrix multiplication the order of multiplication generally produced different answers. When a matrix B is divided by a matrix A, B is multiplied by A^{-1}, the inverse of A, but there are two possibilities:

$$\frac{B}{A} = \begin{cases} BA^{-1} \\ A^{-1}B \end{cases}$$

each of which will generally give a different answer. Therefore matrix division is not used directly; instead we multiply by the inverse of the matrix.

9.5.1 To find the inverse of a matrix: elimination method

To find the inverse of square matrix, set up the augmented matrix, consisting of the matrix whose inverse is required, and a unit matrix of the same dimension.

WORKED EXAMPLE 9.17

THE INVERSE OF A MATRIX: ELIMINATION METHOD

(a) Find the inverse of the matrix

$$D = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

by Gauss–Jordan elimination.
(b) Show that $DD^{-1} = I$.

Solution

Write out the augmented matrix consisting of the matrix D and the unit matrix of the same dimension:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$

Carry out Gauss–Jordan elimination on the matrix D by the method given in Worked Example 9.9.

<table>
<tr><th>Action</th><th>Augmented matrix</th><th>Calculations</th></tr>
<tr><td></td><td>$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right)\begin{matrix}(1)\\(2)\\(3)\end{matrix}$</td><td></td></tr>
<tr><td>row 2 + (−2 × row 1)
row 3 + (−1 × row 1)</td><td>$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 7 & -2 & 1 & 0 \\ 0 & 3 & 4 & -1 & 0 & 1 \end{array}\right)\begin{matrix}(1)\\(2)^1\\(3)^1\end{matrix}$</td><td>Calculate (−2 × row 1)
$\rightarrow (-2 \quad 0 \quad 4 \quad -2 \quad 0 \quad 0)$
Calculate (−1 × row 1)
$\rightarrow (-1 \quad 0 \quad 2 \quad -1 \quad 0 \quad 0)$</td></tr>
<tr><td>row $2^1 \times \frac{1}{2}$</td><td>$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 3 & 4 & 0 & 0 & 1 \end{array}\right)\begin{matrix}(1)\\(2)^2\\(3)^1\end{matrix}$</td><td></td></tr>
<tr><td>row $3^1 + (-3 \times \text{row } 2^2)$</td><td>$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{13}{2} & 2 & -\frac{3}{2} & 1 \end{array}\right)\begin{matrix}(1)\\(2)^2\\(3)^2\end{matrix}$</td><td>Calculate $(-3 \times \text{row } 2^2)$
$\rightarrow (0 \quad -3 \quad -\frac{21}{2} \quad 3 \quad -\frac{3}{2} \quad 0)$</td></tr>
<tr><td>row $3^2 \times -\frac{2}{13}$</td><td>$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{4}{13} & \frac{3}{13} & -\frac{2}{13} \end{array}\right)\begin{matrix}(1)\\(2)^2\\(3)^3\end{matrix}$</td><td></td></tr>
<tr><td>row $1 + (2 \times \text{row } 3^3)$
row $2^2 + (-\frac{7}{2} \times \text{row } 3^3)$</td><td>$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{13} & \frac{6}{13} & \frac{4}{13} \\ 0 & 1 & 0 & \frac{1}{13} & -\frac{4}{13} & \frac{7}{13} \\ 0 & 0 & 1 & -\frac{4}{13} & \frac{3}{13} & -\frac{2}{13} \end{array}\right)\begin{matrix}(1)^1\\(2)^3\\(3)^3\end{matrix}$</td><td>Calculate $(2 \times \text{row } 3^3)$
$\rightarrow (0 \quad 0 \quad 2 \quad -\frac{8}{13} \quad \frac{6}{13} \quad -\frac{4}{13})$
Calculate $(-\frac{7}{2} \times \text{row } 3^3)$
$\rightarrow (0 \quad 0 \quad -\frac{7}{2} \quad \frac{14}{13} \quad -\frac{10.5}{13} \quad \frac{7}{13})$</td></tr>
</table>

The original matrix, D, is now reduced to the unit matrix. The inverse of D is given by the transformed unit matrix: the 4th, 5th and 6th columns of the augmented matrix.

$$D^{-1} = \begin{pmatrix} \frac{5}{13} & \frac{6}{13} & -\frac{4}{13} \\ \frac{1}{13} & -\frac{4}{13} & \frac{7}{13} \\ -\frac{4}{13} & \frac{3}{13} & -\frac{2}{13} \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 6 & -4 \\ 1 & -4 & 7 \\ -4 & 3 & -2 \end{pmatrix}$$

Note: If division by zero arises at any stage, then D does not have an inverse.
(b) Every element in D^{-1} is multiplied by $1/13$. Before multiplying D^{-1} by D, factor out this scalar to simplify the arithmetic involved:

$$DD^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \frac{1}{13} \begin{pmatrix} 5 & 6 & -4 \\ 1 & -4 & 7 \\ -4 & 3 & -2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9.5.2 To find the inverse of a matrix: cofactor method

Initially, some of the terminology associated with determinants and matrices is explained. The **minor** of an element is the determinant of what is left, when the row and column containing that element are crossed out. For example, in determinant

$$D = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

the minor of the first element in the first row, $d_{1,1}$, is

$$D = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = (2)(2) - (3)(3) = -5$$

The **cofactor** of a given element is the minor of that element multiplied by either $+1$ or -1. If the given element is in the same position as $+1$ in the determinant of ± 1, given below, multiply the minor by $+1$, otherwise multiply the minor by -1. Here is the determinant of ± 1:

$$\begin{vmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{vmatrix}$$

The cofactor of an element is given the symbol C subscripted with the location (row, column) of that element. For example, $C_{1,1}$ is the cofactor of $d_{1,1}$ in determinant D above; it is calculated as follows:

$$C_{1,1} = (\text{minor of } d_{1,1}) \times (+1) = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \times (1) = (-5)(1) = -5$$

The minor, -5, was evaluated above.

The value of a 3×3 determinant is calculated as follows:

$$|D| = (d_{1,1} \times C_{1,1}) + (d_{1,2} \times C_{1,2}) + (d_{1,3} \times C_{1,3}) \tag{9.32}$$

that is, the value of $|D|$ is the sum of the products of each element in row 1 and its **cofactor**. (In fact, $|D|$ may also be evaluated by summing the products of each element × cofactor from **any** one row or column. This is particularly useful if a row or column contains several zeros.) This method of evaluation is called Laplace expansion or the cofactor method.

Given a matrix A, the inverse of A is defined as follows:

$$A^{-1} = \frac{(C)^{\mathrm{T}}}{|A|}$$

where C^{T} is the matrix in which every element is replaced by its cofactor:

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}^{\mathrm{T}} \tag{9.33}$$

Note: If

$$|A| = 0, \quad \frac{1}{|A|} \to \frac{1}{0},$$

A **has no inverse**, since division by zero is not defined. Therefore, the inverse of A is determined by:

Step 1. Evaluating $|A|$. If $|A| = 0$, there is no inverse.
Step 2. Calculating the cofactor of each element.
Step 3. Replacing each element by its cofactor, then transposing the matrix of cofactors (this matrix is called the adjoint of A).
Step 4. Multiplying the transposed matrix of cofactors by $1/|A|$.

☐ The inverse of a 2 × 2 matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (ad - bc \neq 0)$

The proof is left as an exercise.

WORKED EXAMPLE 9.18

THE INVERSE OF A 3 × 3 MATRIX

(a) Find the inverse of the matrix

$$D = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

(b) Show that $DD^{-1} = I$.

Solution

(a) Use the definition of the inverse given in equation (9.30) to determine the inverse:

Step 1. Evaluate $|D|$. This is calculated by multiplying each element in row 1 by its cofactor. If the table for calculating cofactors is set up, the required cofactors are the first three cofactors in the table. So step 1 is deferred to step 2.

Step 2. Set up the table to calculate all the cofactors.

Element (by rows)	Minor × (sign) = cofactor
1	$\begin{vmatrix}1&0&-2\\2&2&3\\1&3&2\end{vmatrix} \rightarrow \begin{vmatrix}2&3\\3&2\end{vmatrix}(1) = 4-(9) = -5$
0	$\begin{vmatrix}1&0&-2\\2&2&3\\1&3&2\end{vmatrix} \rightarrow \begin{vmatrix}2&3\\1&2\end{vmatrix}(-1) = [4-(3)](-1) = -1$
−2	$\begin{vmatrix}1&0&-2\\2&2&3\\1&3&2\end{vmatrix} \rightarrow \begin{vmatrix}2&2\\1&3\end{vmatrix}(1) = [6-(2)](1) = 4$
2	$\begin{vmatrix}1&0&-2\\2&2&3\\1&3&2\end{vmatrix} \rightarrow \begin{vmatrix}0&-2\\3&2\end{vmatrix}(-1) = [0-(-6)](-1) = -6$
2	$\begin{vmatrix}1&0&-2\\2&2&3\\1&3&2\end{vmatrix} \rightarrow \begin{vmatrix}1&-2\\1&2\end{vmatrix}(1) = [2-(-2)](1) = 4$
3	$\begin{vmatrix}1&0&-2\\2&2&3\\1&3&2\end{vmatrix} \rightarrow \begin{vmatrix}1&0\\1&3\end{vmatrix}(-1) = [3-(0)](-1) = -3$
1	$\begin{vmatrix}1&0&-2\\2&2&3\\1&3&2\end{vmatrix} \rightarrow \begin{vmatrix}0&-2\\2&3\end{vmatrix}(1) = [0-(-4)](1) = 4$

$|D|$ by row 1

$= (1)(-5) + 0(-1) + (-2)(4)$

$= -13$

3 $$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}(-1) = [3-(-4)](-1) = -7$$

2 $$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}(1) = [2-(0)](1) = 2$$

Step 3. Replace each element in D by its cofactor. Then transpose.

$$C^{\mathrm{T}} = \begin{pmatrix} -5 & -1 & 4 \\ -6 & 4 & -3 \\ 4 & -7 & 2 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix} = (\text{adjoint of } D)$$

Step 4. Multiply the adjoint matrix by

$$\frac{1}{|D|} = \frac{1}{-13}$$

$$D^{-1} = \frac{1}{-13}\begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix}$$

Every element in this matrix could be multiplied by $-(1/13)$, but this will introduce awkward fractions, so the scalar multiplication is usually left until a final single matrix is required.

(b) $$DD^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \times \frac{1}{-13}\begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix}$$

$$= -\frac{1}{13}\begin{pmatrix} -13 & 0 & 0 \\ 0 & -13 & 0 \\ 0 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

❐ To write a system of equations in matrix form

In general, a system of equations, all written in the same format, may be expressed concisely in terms of matrices as follows:

$$\begin{aligned} &(1)\quad a_1x + b_1y + c_1z = d_1 \\ &(2)\quad a_2x + b_2y + c_2z = d_2 \\ &(3)\quad a_3x + b_3y + c_3z = d_3 \end{aligned} \quad \rightarrow \quad \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \qquad \textbf{(9.34)}$$

This statement is written concisely as $AX = B$ **(9.35)**

Dimension $(3 \times 3)(3 \times 1) = (3 \times 1)$

If the matrix A is multiplied by the column matrix X, the result is a 3×1 matrix, in which each element is the LHS of equations (1), (2) and (3) above:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Equating the corresponding elements of these 3×1 matrices, AX and B, reproduces equations (1), (2) and (3).

Note: the columns of matrix A consist of the coefficients of x, y, z from equations (1), (2) and (3).

☐ To solve a set of equations using the inverse matrix

Premultiply both sides of equation (9.35), $AX = B$, by the inverse of matrix A:

$A^{-1}AX = A^{-1}B$ but $AA^{-1} = I$, the unit matrix

$IX = A^{-1}B$ but $IX = X$, since I in matrix multiplication behaves like a 1 in ordinary multiplication, therefore

$$X = A^{-1}B \tag{9.36}$$

where X is the column of unknowns, A is the matrix of coefficients, B is the column of constants from the RHS of the equations, taken in order. So, if the column matrix B is premultiplied by the inverse of the matrix A, the resulting column matrix is the solution for x, y, z.

WORKED EXAMPLE 9.19

SOLVE A SYSTEM OF EQUATIONS BY THE INVERSE MATRIX

Use the inverse matrix to solve the following simultaneous equations:

$$x - 2z = 4$$
$$2x + 2y + 3z = 15$$
$$x + 3y + 2z = 12$$

Solution

Step 1. Write all the equations in the same order: variables (in order) = RHS. Write down the matrices A and B. The matrix A is the matrix of coefficients of the three equations, all arranged in the same format; B is the column matrix consisting of the constants from the RHS of the equations.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 \\ 15 \\ 12 \end{pmatrix}$$

Step 2. Since $X = A^{-1}B$, determine the inverse of A. However, if you look carefully at A, you will see that this matrix is identical to the matrix D in Worked

Examples 9.17 and 9.18 in which the inverse of D was determined; therefore

$$A^{-1} = \frac{1}{-13}\begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix}$$

So proceed straight away to the next step.

Step 3. Premultiply the matrix B by the inverse of A:

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-13}\begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 15 \\ 12 \end{pmatrix} = -\frac{1}{13}\begin{pmatrix} -62 \\ -28 \\ -5 \end{pmatrix} = \begin{pmatrix} 4.77 \\ 2.15 \\ 0.38 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4.77 \\ 2.15 \\ 0.38 \end{pmatrix}$$

The solutions may simply be read off (correct to two decimal places), $x = 4.77$, $y = 2.15$, $z = 0.38$.

☐ Input/output analysis

Consider a three-sector economy, such as industry, agriculture and financial services. The output from any sector, such as agriculture, may be required by:

(i) The other sectors
(ii) The same sector
(iii) External demands, such as sales, exports, etc.

For example, the output from all three sectors (£million) in this three-sector economy could be distributed as follows:

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150	225	125	100	600
	Industry →	210	250	140	300	900
	Services →	170	0	30	100	300

So the total output required from each sector must satisfy the final demand (sales, exports, etc.), as well as the demands from other sectors which require this as basic raw material or input. For example, the output from the agriculture sector is required as raw material by agriculture itself (calves, foodstuffs, etc.); industry requires agricultural output for processing; government and other agencies provide financial and other services. Finally, consumers and other agencies want to buy and export agricultural products.

The table above is perfectly balanced, but suppose the 'other demands' from each sector change. How is the total output required recalculated? The answer is to use matrices. One other fundamental assumption is needed before setting up our method, and that is:

> In input/output analysis, **total input = total output** for each sector.

With this assumption, the input/output table can be completed, the total input to each sector is written in the last row. You will notice that the sum of individual inputs do not add up to the total input. Therefore a further row is added to the table to allow for other inputs.

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150	225	125	100	600
	Industry →	210	250	140	300	900
	Services →	170	0	30	100	300
	Other inputs	70	425	5		
	Total input	600	900	300		

Divide the input to each sector by the total input to calculate the fraction (or, if you prefer, multiply the fraction by 100 and quote this as the percentage) of the total input which comes from all sectors:

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150/600	225/900	125/300	100	600
	Industry →	210/600	250/900	140/300	300	900
	Services →	170/600	0/900	30/300	100	300
	Total input	600/600	900/900	300/300		

The relationship between the total output from all three sectors to the input requirements from other sectors and final other demand may now be described by the matrix equation:

$$\begin{pmatrix} \frac{150}{600} & \frac{225}{900} & \frac{125}{300} \\ \frac{210}{600} & \frac{250}{900} & \frac{140}{300} \\ \frac{170}{600} & \frac{0}{900} & \frac{30}{300} \end{pmatrix} \begin{pmatrix} 600 \\ 900 \\ 300 \end{pmatrix} + \begin{pmatrix} 100 \\ 300 \\ 100 \end{pmatrix} = \begin{pmatrix} 600 \\ 900 \\ 300 \end{pmatrix}$$

This equation may be stated in general as:

$$AX + d = X$$

where X is the column of total outputs, d is the column of final (other) demands from outside the three sectors. The matrix A is called the matrix of technical coefficients: each column of A

gives the fraction of inputs to that sector which comes from each of the three sectors. This equation may be used to solve for the total output (X) required from each sector if the final demands (d) are changed:

$d = X - AX = (I - A)X$ ☺ **remember:** $IX = X$

$(I - A)^{-1}d = (I - A)^{-1}(I - A)X$ premultiply both sides by $(I - A)^{-1}$

$(I - A)^{-1}d = X$ since $(I - A)^{-1}(I - A) = I$

For the input/output model $AX + d = X$, the total output required from each sector when final demands d are changed is given by the equation:

$$X = (I - A)^{-1}d \qquad \textbf{(9.37)}$$

WORKED EXAMPLE 9.20

INPUT/OUTPUT ANALYSIS

Given the input/output table for the three-sector economy:

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150	225	125	100	600
	Industry →	210	250	140	300	900
	Services →	170	0	30	100	300

If the final demands from each sector are changed to 500 from agriculture, 550 from industry, 300 from financial services, calculate the total output from each sector.

Solution

Step 1. Use the underlying assumption total input = total output to complete the input/output table.

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150	225	125	100	600
	Industry →	210	250	140	300	900
	Services →	170	0	30	100	300
	Other inputs	70	425	5		
	Total input	600	900	300		

Step 2. Calculate the matrix of technical coefficients, A, by dividing each column of inputs by total input.

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150/600	225/900	125/300	100	600
	Industry →	210/600	250/900	140/300	300	900
	Services →	170/600	0/900	30/300	100	300
	Total input	600/600	900/900	300/300		

Step 3. Get the inverse of the matrix $(I - A)$, since this inverse matrix is required in the equation: $X = (I - A)^{-1}d$.

$$(I - A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{150}{600} & \frac{225}{900} & \frac{125}{300} \\ \frac{210}{600} & \frac{250}{900} & \frac{140}{300} \\ \frac{170}{600} & \frac{0}{900} & \frac{30}{300} \end{pmatrix} = \begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

To calculate the inverse of $(I - A)$, (i) use the elimination method, (ii) use the cofactor method. Set out the table of cofactors:

Elements of $(I - A)$	Cofactor = minor by (± 1)
0.75	$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix} \rightarrow \begin{vmatrix} 0.72 & -0.47 \\ 0.00 & 0.90 \end{vmatrix}(1) = 0.648$
−0.25	$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix} \rightarrow \begin{vmatrix} -0.35 & -0.47 \\ -0.28 & 0.90 \end{vmatrix}(-1) = 0.4466$
−0.42	$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix} \rightarrow \begin{vmatrix} -0.35 & 0.72 \\ -0.28 & 0.00 \end{vmatrix}(1) = 0.2016$

-0.35

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$|I - A| = 0.75(0.648) + (-0.25)(0.4466) + (-0.42)(0.2016) = 0.289678$$

$$\rightarrow \begin{vmatrix} -0.25 & -0.42 \\ 0.00 & 0.90 \end{vmatrix}(-1) = 0.225$$

0.72

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$\rightarrow \begin{vmatrix} 0.75 & -0.42 \\ -0.28 & 0.90 \end{vmatrix}(1) = 0.5574$$

-0.47

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$\rightarrow \begin{vmatrix} 0.75 & -0.25 \\ -0.28 & 0.00 \end{vmatrix}(-1) = 0.070$$

-0.28

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$= \begin{vmatrix} -0.25 & -0.42 \\ 0.72 & -0.47 \end{vmatrix}(1) = 0.4199$$

0.00

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$= \begin{vmatrix} 0.75 & -0.42 \\ -0.35 & -0.47 \end{vmatrix}(-1) = 0.4995$$

0.90

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ 0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$= \begin{vmatrix} 0.75 & -0.25 \\ -0.35 & 0.72 \end{vmatrix}(1) = 0.4525$$

The inverse of $(I - A) = C^{\mathrm{T}}/|I - A|$

$$= \frac{1}{0.289678}\begin{pmatrix} 0.648 & 0.4466 & 0.2016 \\ 0.225 & 0.5574 & 0.070 \\ 0.4199 & 0.4995 & 0.4525 \end{pmatrix}^{\mathrm{T}}$$

$$= \frac{1}{0.289678}\begin{pmatrix} 0.648 & 0.225 & 0.4199 \\ 0.4466 & 0.5574 & 0.4995 \\ 0.2016 & 0.070 & 0.4525 \end{pmatrix}$$

Step 3. Finally, state the column of new external demands, d, and solve for X, by equation (9.34):

$$X = (I - A)^{-1}d$$

$$\begin{pmatrix} T_{\text{agi.}} \\ T_{\text{ind.}} \\ T_{\text{serv.}} \end{pmatrix} = \frac{1}{0.289678}\begin{pmatrix} 0.648 & 0.225 & 0.4199 \\ 0.4466 & 0.5574 & 0.4995 \\ 0.2016 & 0.070 & 0.4525 \end{pmatrix}\begin{pmatrix} 500 \\ 550 \\ 300 \end{pmatrix} = \begin{pmatrix} 1980.5 \\ 2346.5 \\ 949.5 \end{pmatrix}$$

Progress Exercises 9.6 The Inverse Matrix and Input/Output

In the following questions, determine the inverse matrices using (i) the elimination method and (ii) the cofactor method.

1. Determine the inverse of the following matrices:

(a) $\begin{pmatrix} 2 & 5 \\ 2 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 4 \\ 20 & 10 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 4 \\ 12 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 4 \\ -2 & -8 \end{pmatrix}$

In questions 2, 3, 4 and 5,

(a) Determine the inverse of the matrix.
(b) Use the inverse matrix method to solve the given system of equations.

2. (a) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 2 \\ 6 & 3 & 6 \end{pmatrix}$ (b) $\begin{aligned} x + y &= 12 \\ 2x + 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned}$

3. (a) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & -5 & 2 \\ 6 & 3 & 6 \end{pmatrix}$ (b) $\begin{aligned} x + y &= 12 \\ 2x - 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned}$

4. (a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$ (b) $\begin{aligned} x + y &= 2 \\ x - y + 2z &= 20 \\ 2x + 3y + z &= 0 \end{aligned}$

5. (a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -5 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ (b) $\begin{aligned} x + y + z &= 12 \\ 2x - 5y + 2z &= 20 \\ x + z &= 0 \end{aligned}$

6. Given the matrix of technical coefficients and the matrix of final demand (other demand),

   (i) $A = \begin{pmatrix} 0.5 & 0.4 \\ 0.25 & 0.4 \end{pmatrix}$, $B = \begin{pmatrix} 40 \\ 100 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 0.5 & 0.25 \\ 0.2 & 0.5 \end{pmatrix}$, $B = \begin{pmatrix} 120 \\ 400 \end{pmatrix}$

   (a) Determine $(I - A)^{-1}$.
   (b) Calculate the total output required from each sector.

7. Given the matrix of technical coefficients, A, and final demand (other demand) B for industries 1, 2, and 3,

$$A = \begin{pmatrix} 0.5 & 0.1 & 0 \\ 0.2 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.6 \end{pmatrix}, \quad B = \begin{pmatrix} 120 \\ 200 \\ 700 \end{pmatrix}$$

   (a) Determine $(I - A)^{-1}$.
   (b) Calculate the total output required from each sector.

8. Repeat question 7 for the following matrices:

$$A = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.25 & 0.1 \\ 0.4 & 0.2 & 0.2 \end{pmatrix}, \quad B = \begin{pmatrix} 1020 \\ 1200 \\ 700 \end{pmatrix}$$

9. Given the inter-industrial transaction table for two industries,

	Input to X	Input to Y	Final demand
Output from X	600	400	200
Output from Y	600	200	0

   (a) Write down the matrix of technical coefficients.
   (b) Calculate the total output required from each industry if the final demands from X and Y change to 500 and 1000 units respectively.

10. The input/output table for an organic farm is given below:

Input to

		Hortic.	**Dairy**	**Poultry**	**Other demand**	**Total output**
Output from	Hortic →	50	40	200	210	500
	Dairy →	100	160	0	140	400
	Poultry →	200	0	80	520	800

(a) Write down the matrix of technical coefficients.

(b) Calculate the total output required from each section when other demands change to 200, 800, 1000 from horticulture, dairy and poultry sections respectively.

11. The input/output table for a three-sector industry is given below:

		Input to				
		A	**B**	**C**	**Other demand**	**Total output**
Output from	A →	200	100	200	500	1000
	B →	100	0	500	200	800
	C →	100	80	0	620	800

(a) Write down the matrix of technical coefficients.

(b) Calculate the total output required from each industry when other demands change to 400, 400, 1000 from sectors A, B and C respectively.

12. The input/output table for three inter-dependent industries is given below:

		Input to				
		Dairy	**Beef**	**Leather**	**Other demand**	**Total output**
Output from	Dairy	400	200	0	400	1000
	Beef	100	400	200	200	1000
	Leather	200	200	100	600	1000

(a) Write down the matrix of technical coefficients.

(b) Calculate the total output required from each industry when other demands change to 750, 200, 1500 from dairy, beef and leather industries respectively.

9.6 Excel for Linear Algebra

The elimination methods involve the use of elementary row operations. Examples of elementary row operations include (i) multiplying rows by non-zero constants and (ii) adding multiples of rows to other rows. Excel is ideal for these calculations. The following worked example should demonstrate the usefulness of Excel in carrying out the tedious, error-prone calculations required for elimination methods. Excel is also useful for the evaluation of determinants, hence in solving equations by Cramer's rule.

WORKED EXAMPLE 9.21

USE EXCEL TO SOLVE SYSTEMS OF LINEAR EQUATIONS

(a) Find the inverse of

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 5 & -3 \\ 4 & -1 & 3 \end{pmatrix}$$

by the elimination method, hence solve the equations

$$2x + y + z = 12$$

$$6x + 5y - 3z = 6$$

$$4x - y + 3z = 5$$

(b) Solve the equations in (a) by Gauss–Jordan elimination.

Solution

Set up the augmented matrix, consisting of the matrix A and the unit matrix of the same dimension. Add multiples of rows to other rows to reduce A to upper triangular form. Enter the Excel formula for each row operation in column 1, then copy the formula across the remaining five columns. See Chapter 1, and remember that formulae start with =.

For example, to divide row 1 by 2, place the cursor in cell B8 and type = B4/2. To copy this formula across, click on the corner of cell B8 until a black cross appears, then drag this across the remaining five columns. This updated row is labelled row 1^1. Similarly, to add row $1^1 \times (-6)$ to row 2 in order to generate the required 0, place the cursor in cell B9 and type in the formula = B5 + B8 ∗ (−6). Copy the formula across the row. In this way carry out all the row operations as indicated.

1	A	B	C	D	E	F	G	
2	Augmented matrix							
3		col 1	col 2	col 3	col 4	col 5	col 6	
4	row 1 →	2	1	1	1	0	0	
5	row 2 →	6	5	-3	0	1	0	
6	row 3 →	4	-1	3	0	0	1	
7								Operation **Formula**
8	row 1^1 →	1	0.5	0.5	0.5	0	0	row 1 divided by 2 = **B4/2**
9	row 2^1 →	0	2	-6	-3	1	0	row 2 + row $1^1 \times (-6)$ = **B5 + B8 ∗ (−6)**
10	row 3^1 →	0	-3	1	-2	0	1	row 3 + row $1^1 \times (-4)$ = **B6 + B8 ∗ (−4)**
11								
12	row 1^1 →	1	0.5	0.5	0.5	0	0	
13	row 2^2 →	0	1	-3	-1.5	0.5	0	row 2^1 divided by 2 = **B12/2**
14	row 3^2 →	0	0	-8	-6.5	1.5	1	row 3^1 + row $2^2 \times (3)$ = **B10 + B13 ∗ (3)**

A is reduced to upper triangular form: row 1^1, row 2^2 and row 3^2. Continue with the row operations to further reduce A to the unit matrix.

15	row 3^3 →	0	0	1	0.81	-0.2	-0.1	row 3^2 divided by -8 $=$ **B14/(−8)**
16	row 1^2 →	1	0.5	0	0.09	0.09	0.06	row 1^1 + row $3^3 \times (-0.5)$ $=$ **B12 + B15 * (−0.5)**
17	row 2^3 →	0	1	0	0.94	-0.1	-0.4	row 2^2 + row $3^3 \times (3)$ $=$ **B13 + B15 * (3)**
18	row 1^3 →	1	0	0	-0.4	0.13	0.25	row 1^2 + row $2^3 \times (-0.5)$ $=$ **B16 + B17 * (−0.5)**

The reduction is now complete. Write out the most recently updated rows in order.

	col 1	col 2	col 3	col 4	col 5	col 6
row 1^2 →	1	0	0	-0.375	0.0125	0.025
row 2^3 →	0	1	0	0.9375	-0.0625	-0.375
row 3^3 →	0	0	1	0.8125	-0.8125	-0.125

Read off the inverse matrix: columns 4, 5 and 6. The inverse of A is

-0.375	0.125	0.25
0.9375	-0.0625	-0.375
0.8125	-0.1875	-0.125

Finally, to multiply A^{-1} by the column of constants from the RHS of the equations, arrange these as shown.

	B	C	D	E
19	Inverse of A			RHS
20	-0.375	0.125	0.25	12
21	0.9375	-0.0625	-0.375	6
22	0.8125	-0.1875	-0.125	5
23				
24	x = -2.5	**= B20 * E20 + C20 * E21 + D20 * E22**		
25	y = 9	**= B21 * E20 + C21 * E21 + D21 * E22**		
26	z = 8	**= B22 * E20 + C22 * E21 + D22 * E22**		

Formulae used to calculate x, y and z

(b) To solve by Gauss–Jordan elimination, go back to the augmented matrix in (a) and replace col 5 (first column of the unit matrix) by the column of constants from the RHS of each equation. Delete col 5 and col 6 (see below). To carry out the Gauss–Jordan elimination, copy across the formulae that were entered earlier to find the inverse of A. Read off the solution.

	Augmented matrix				
row 1	2	1	1	12	
row 2	6	5	-3	6	
row 3	4	-1	3	5	
row 1^1	1	0.5	0.5	6	row 1 divided by 2
row 2^1	0	2	-6	-30	row 2 + row $1^1 \times (-6)$
row 3^1	0	-3	1	-19	row 3 + row $1^1 \times (-4)$
row 2^2	0	1	-3	-15	row 2^1 divided by 2
row 3^2	0	0	-8	-64	row 3^1 + row $2^2 \times (3)$

At this stage it is possible to read off the upper triangular matrix: row 1^1, row 2^2 and row 3^2. Solve by back substitution. For Gauss–Jordan elimination continue the row operations.

row 3^3	0	0	1	8	row 3^2 divided by -8
row 1^2	1	0.5	0	2	row 1^1 + row $3^3 \times (-0.5)$
row 2^3	0	1	0	9	row 2^2 + row $3^3 \times (3)$
row 1^3	1	0	0	-2.5	row 1^2 + row $2^3 \times (-0.5)$

Read off the solution from the reduced augmented matrix: $x = -2.5$, $y = 9$, $z = 8$.

	col 1	col 2	col 3	col 4
row 1^2 →	1	0	0	-2.5
row 2^3 →	0	1	0	9
row 3^3 →	0	0	1	8

9.7 Summary

☐ Linear programming

Find the optimum value of a linear function (objective function) subject to two or more linear constraints.

Method 1: Graph the inequality constraints and one level of the objective function. Determine graphically the largest value (or smallest value for minimisation) of the objective function which contains a corner point, X, in the feasible region. The coordinates of the point X are the values of the variables for which the objective function is optimised.

Method 2: Calculate the corner points of the feasible region. Evaluate the objective function at each corner point, to obtain the point at which it is optimised.

❐ Matrix algebra

1. A matrix is a rectangular array of numbers (elements).
2. A matrix of order $m \times n$ contains m rows and n columns.
3. Matrices added by adding corresponding elements and subtracted by subtracting corresponding elements, therefore the matrices must be of the same order.
4. A matrix is transposed by writing each row as the corresponding column.
5. Scalar multiply matrices by multiplying each element in the matrix by the scalar (the constant).
6. The unit matrix is a square matrix with zeros everywhere except on the diagonal; the diagonal elements are usually 1.
7. The product AB of matrices A and B is possible only when the number of columns in A is equal to the number of rows in B. In general $AB \neq BA$.
8. Gauss–Jordan elimination is a method for solving a system of equations.

❐ Determinants

1. A determinant is a square array of numbers (elements).
2. The value of a 2 × 2 determinant is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3. The value of a 3 × 3 determinant is given by

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

4. Cramer's rule: to solve two simultaneous equations

$$a_1x + b_1y = d_1$$
$$a_2x + b_2y = d_2$$

use determinants

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\Delta_x}{\Delta} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\Delta_y}{\Delta}$$

Cramer's rule can be extended to the solution of three linear equations in three unknowns (x, y, z):

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}$$

❐ The inverse of a square matrix

1. The minor of an element is the determinant of what remains when the row and column containing that element is crossed out.
2. The cofactor of an element is the value of the minor multiplied by ±1. The ±1 is given by the determinant of signs.
3. Given a matrix A, the inverse of A is defined as

$$A^{-1} = \frac{C^{\mathrm{T}}}{|A|}$$

where C^T is the matrix in which every element is replaced by its cofactor.

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{1.1} & C_{1.2} & C_{1.3} \\ C_{2.1} & C_{2.2} & C_{2.3} \\ C_{3.1} & C_{3.2} & C_{3.3} \end{pmatrix}^{\mathrm{T}}$$

To find the inverse by Gauss–Jordan elimination, set up the augmented matrix as $(A|I)$, where A is the square matrix to be inverted and I is the unit matrix of the same dimension. Use Gauss–Jordan elimination to reduce A to the unit matrix, hence $(A|I) \rightarrow (I|A^{-1})$. Read off the inverse of A.

❐ Applications of inverse matrices

1. Simplifying calculations and other operations on large arrays of data.
2. Solving simultaneous equations.

❐ Input/output analysis

Input/output analysis is based on the condition

total input = total output

For the input/output model $AX + d = X$, the total output required from each sector when final demands d are changed to d_{new} is given by the equation

$$X = (I - A)^{-1}\, d_{\text{new}}$$

where

A = the matrix of technical coefficients (the elements in each column are the proportion of that sector's input derived from each other sector)

d = the column matrix of final demands from each sector

X = the column matrix of total output from each sector

Test Exercises 9

1. Graph the inequality constraints and shade in the feasible region:

(a) $10x + 2.5y \leq 50$: $2x + 12y \leq 24$: $x \geq 0$: $y \geq 0$

(b) $4x + 5y \geq 50$: $2x + 12y \geq 24$: $x \geq 0$: $y \geq 0$

2. A small iron works manufactures two types of gate. The requirements for each stage of production, along with the limitations on the available labour-hours are given in the following table:

Stages of production

Gate type	**Welding**	**Finishing**	**Sales**	**Selling price**	**Unit profit**
Security	4.5	1.0	1.0	£7200	£180
Ornamental	2.0	2.0	1.0	£665	£315
Max. hours	900	400	250		

(a) Write down the equations for (i) profit, (ii) total revenue.

(b) Write and graph the inequality constraints. Hence determine graphically the number of each type of gate which should be produced and sold to maximise
(i) profit and (ii) total revenue.

(c) Confirm the answers in (b) algebraically.

3. Given the matrices

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ -2 & 2 \end{pmatrix}; \quad D = \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix}$$

Calculate the following, if possible (if not possible, give reasons):

(a) $A + D$ (b) CD (c) AB (d) $A + C$ (e) AC^{T}

4. State Cramer's rule. Hence

(a) Solve the following simultaneous equations:
(i) $5x - 8y = 55$: $10x + 2y = 20$ (ii) $3Q + 12P + 60 = 0$: $10Q = 4P + 200$

(b) The equilibrium condition in
(i) The goods market is given as $Y = C + I$, where $C = 330 + 0.2Y$: $I = 150 - 0.4r$
(ii) The money market is $M_d = M_s$, where $M_d = 100 + 0.1Y - 0.2r$: $M_s = 156$

Write each equilibrium condition in the form $aY + br = c$. Hence use Cramer's rule to solve for the equilibrium levels of Y and r.

5. What is the basic assumption underlying the input–output model?

The input–output table for a three-sector economy is given in the following table:

		Sector A	**Input to Sector B**	**Sector C**	**Final demand**	**Total output**
Output from	Sector A	15	5	0	30	50
	Sector B	5	15	10	10	40
	Sector C	0	0	10	70	80

(a) Write out the matrix of technical coefficients.

(b) Calculate the total output required from each sector if the final demand changes to 10 from A, 20 from B and 100 from C.

6. The input–output table for a two-sector economy is given as follows:

		Input to sector A	**Input to sector B**	**Final demand**
Output from	Sector A	200	50	50
Output from	Sector B	100	400	100

(a) Write out the matrix of technical coefficients.

(b) Calculate the total output required from each sector when final demand increases to 200 from sector A and 250 from sector B.

(c) Confirm your answer by writing out the full input–output table.

CHAPTER

10

Difference Equations

In this chapter, the methods for solving difference equations are introduced. The chapter is divided into the following sections:

10.1 Introduction to difference equations
10.2 Solution of difference equations (first-order)
10.3 Applications of difference equations (first order)
10.4 Summary

10.1 Introduction to Difference Equations

Difference equations model dynamic systems in which changes occur at fixed, equally spaced intervals only. There are endless situations in everyday life which are modelled by difference equations. For example,

(a) If income increases by 20% each year, then, for any year, t, income may be written as 120% of the previous year's income,

$$Y_{t+1} = \frac{120}{100} Y_t$$

$$Y_{t+1} - 1.2Y_t = 0 \qquad \textbf{(10.1)}$$

Note: A difference equation describes the relationship between the value of a variable, such as income, in **any** year to its value in the previous year(s). Therefore, an integer constant may be added (or subtracted) to **each occurrence of *t*** without affecting the relationship expressed by the equation. For example,

$$Y_t - 1.2Y_{t-1} = 0; \quad Y_{t+2} - 1.2Y_{t+1} = 0; \quad \ldots \quad Y_{t+n} - 1.2Y_{t+n-1} = 0$$

and equation (10.1) all express the same relationship.

(b) If income decreases by 5% each year, but a lump sum of 1000 is given each year, then

$$Y_{t+1} = \frac{95}{100}Y_t + 1000$$

$$Y_{t+1} - 0.95Y_t = 1000 \tag{10.2}$$

(c) The size of the wild duck population in any generation is estimated to be 80% of the population size in the previous generation, plus 20% of the population size two generations before, minus a cull of 2400 per generation:

$$P_n = 0.8P_{n-1} + 0.2P_{n-2} - 2400$$

Note: This equation could also be written as

$$P_{n+2} = 0.8P_{n+1} + 0.2P_n - 2400 \tag{10.3}$$

(d) If I_t = amount invested in year t, then the difference equation,

$$I_{t+2} = 0.6I_{t+1} + 0.4I_t \tag{10.4}$$

states that investment in any year is 60% of the previous year's investment plus 40% of the investment two years ago.

Some terminology:

- A difference equation describes the relationship between an **independent variable**, such as time, t (in this chapter we shall deal mostly with time, though occasionally with n, as in example (c)) and a **dependent variable**, such as income, which changes at fixed, equally spaced intervals in time. See Table 10.1.
  In (a), the dependent variable is income, Y.
  In (b), the dependent variable is income, Y.
  In (c), the dependent variable is population, P.
  In (d), the dependent variable is investment, I.
- Y_t: a discrete function which changes at fixed points in time only, remaining constant in between, while $y(t)$: a continuous function which changes continuously in time.
- The independent variable, t, is used in functional notation for discrete functions in a similar manner to that for continuous functions. For example,

  discrete functions:

  $$Y_t = 5(t^2) + 10; \; Y_2 = 5(2^2) + 10; \; Y_{t+1} = 5(t+1)^2 + 10.$$

  continuous functions:

  $$\mathrm{f}(t) = 5(t^2) + 10; \; \mathrm{f}(2) = 5(2^2) + 10; \; \mathrm{f}(t+1) = 5(t+1)^2 + 10$$

Table 10.1 Terminology associated with difference equations, (a), (b), (c) and (d)

	Dependent variable	Independent variable	Order of difference equation	Type
(a) $Y_{t+1} - 1.2Y_t = 0$	Y_t	t	1	Homogeneous
(b) $Y_{t+1} - 0.95Y_t = 1000$	Y_t	t	1	Non-homogeneous
(c) $P_{n+2} - 0.8P_{n+1} - 0.2P_n = -2400$	P_n	n	2	Non-homogeneous
(d) $I_{t+2} - 0.6I_{t+1} - 0.4I_t = 0$	I_t	t	2	Homogeneous

- The **order** of a difference equation is the number of time intervals spanned by the difference equation. See Table 10.1.

  Example (b): $(t+1) - t = 1$: order 1

- When the difference equation is written in the form

  dependent variables = RHS

  the difference equation is classified as

  Homogeneous when the RHS = 0 as in (a) and (d) and
  Non-homogeneous when the RHS $\neq 0$ as in (b) and (c).

In this text we shall deal with linear difference equations with constant coefficients, that is, difference equations of the form,

$$aY_{t+1} + bY_t = \text{RHS}$$

10.2 Solution of Difference Equations (First-order)

If any value of the dependent variable is given, then the difference equation may be solved by progressively working through from year to year. This method is called iteration.

WORKED EXAMPLE 10.1

SOLVING DIFFERENCE EQUATIONS BY ITERATION

Solve the difference equation (10.1):

$$Y_{t+1} - 1.2Y_t = 0$$

by iteration for years, 2, 3, 4 and 5, given that income in year 1 is £18 000.

Solution

Since, $Y_{t+1} = 1.2Y_t$ and $Y_1 = 18\,000$, then, similarly,

$$Y_2 = 1.2Y_1$$
$$Y_2 = 1.2(18\,000) = 21\,600$$
$$Y_3 = 1.2(Y_2) = 1.2(21\,600) = 25\,920$$
$$Y_4 = 1.2(Y_3) = 1.2(25\,920) = 31\,104$$
$$Y_5 = 1.2(Y_4) = 1.2(31\,104) = 37\,324.8$$

A general expression for the value of Y_t may be deduced by carefully observing the pattern which evolves, as in Worked Example 10.2.

WORKED EXAMPLE 10.2

GENERAL SOLUTION OF A HOMOGENEOUS FIRST-ORDER DIFFERENCE EQUATION

(a) Write out the solution of the difference equation (10.1):

$$Y_{t+1} - 1.2Y_t = 0$$

for $t = 1, 2, 3, 4$ and 5 in terms of Y_1.
(b) Hence, deduce a general expression for Y_t in terms of t.
(c) Evaluate Y_{40}, given $Y_1 = £18\,000$.
Comment on the solution.

Solution

(a) Writing out Y_t for $t = 1, 2, 3, 4$ and 5,

$$Y_2 = 1.2Y_1$$
$$Y_3 = 1.2Y_2 = 1.2(1.2Y_1) = (1.2)^2Y_1$$
$$Y_4 = 1.2Y_3 = 1.2(1.2)^2Y_1 = (1.2)^3Y_1$$
$$Y_5 = 1.2Y_4 = 1.2(1.2)^3Y_1 = (1.2)^4Y_1$$

(b) In general,

$$Y_t = (1.2)^{t-1}Y_1$$

(c) Substitute $t = 40$ and $Y_1 = £18\,000$ into the expression in (b),

$$Y_{40} = (1.2)^{39}(18\,000)$$
$$= 22\,046\,574$$

Comment: A very desirable income 40 years on and a good example of unlimited growth. **Excel** is particularly useful for repetitive calculations such as those in Worked Examples 10.1 and 10.2.

❒ General and particular solutions of difference equations

From Worked Example 10.2(b) the general expression $Y_t = Y_1(1.2)^{t-1}$ or $Y_t = [(1.2)^{-1}Y_1] \times (1.2)^t = \text{constant} \times (1.2)^t$ is called the **general solution** of the homogeneous difference equation.

To generalise: the general solution of an homogeneous first-order difference equation is of the form

$$Y_t = A(a)^t \tag{10.5}$$

where the base, a, is determined from the difference equation (see Worked Example 10.3). The constant, A, may be determined from the general solution, equation (10.5), if conditions are given. When the constant, A, is evaluated, we then have a particular solution of the given difference equation. (Similar terminology is used in differential equations, Chapter 8.)

WORKED EXAMPLE 10.3

GENERAL AND PARTICULAR SOLUTIONS OF FIRST-ORDER HOMOGENEOUS DIFFERENCE EQUATIONS

(a) Find the general solution of the difference equation, $Y_{t+1} - 0.8Y_t = 0$.
(b) If $Y_2 = 80$, find the particular solution.
(c) Evaluate Y_1, Y_{20} and Y_{50}.

Solution

(a) The general form of the solution is

$$Y_t = A(a)^t$$

$$\text{therefore} \quad Y_{t+1} = A(a)^{t+1}$$

Substituting these into the given equation, solve for a:

$$Y_{t+1} - 0.8Y_t = 0$$

$$A(a)^{t+1} - 0.8A(a)^t = 0$$

$$Aa^ta^1 - 0.8Aa^t = 0 \quad \text{since } a^{t+1} = a^ta^1$$

$$\underbrace{Aa^t}_{\substack{A\neq 0\\ a^t\neq 0}}\underbrace{(a-0.8)}_{\substack{a-0.8=0\\ a=0.8}} = 0 \quad \text{factoring out } Aa^t$$

Note: If $A = 0$, then $Y_t = A(a)^t = 0(a)^t = 0$. If $a = 0$, then $Y_t = A(a)^t = A(0)^t = 0$. The solution $Y_t = A(a)^t = 0$ is called a trivial solution. This solution is of no interest, therefore, we proceed to look for a non-trivial solution, if it exists. Therefore, the general solution is

$$Y_t = A(0.8)^t$$

(b) Given that $Y_2 = 80$, substitute $t = 2$ and $Y_t = 80$ into the general solution to get the particular solution:

$$\begin{aligned} Y_t &= A(0.8)^t \\ Y_2 &= A(0.8)^2 \\ 80 &= A(0.8)^2 \\ \frac{80}{(0.8)^2} &= A \\ 125 &= A \end{aligned}$$

Therefore, the particular solution is

$$Y_t = 125(0.8)^t$$

(c) $Y_t = 125(0.8)^t$,

when $t = 1$, $\quad Y_1 = 125(0.8)^1 = 100$

$t = 20$, $\quad Y_{20} = 125(0.8)^{20} = 1.44$

$t = 50$, $\quad Y_{50} = 125(0.8)^{50} = 0.00178$

❒ Stability and the time path to stability

From Worked Examples 10.2 and 10.3 the general solution of a difference equation is of the form

$$Y_t = A(a)^t$$

This expression for the dependent variable, Y_t, may then be used to

- Forecast income in future years
- Determine whether income will stabilise (approach some fixed value) in time
- Trace how income changes each year until stability is reached, that is, trace the time path to stability.

The stability of the solution as t increases, and the path to stability are readily deduced from the exponential term $(a)^t$:

$(a)^t$ as t increases for $t > 0$

$-\infty < a < -1$	$-1 < a < 0$	$0 < a < 1$	$1 < a < \infty$
$(a)^t \rightarrow \pm\infty$	$(a)^t \rightarrow = 0$	$(a)^t \rightarrow 0$	$(a)^t \rightarrow \infty$
Solution unstable: time path alternates	Solution stable: time path alternates	Solution stable: time path tends to zero	Solution unstable: time path tends to infinity

For example, in Worked Example 10.3(c), $Y_t = 125(0.8)^t$, as t increased to 20, 50, Y_t decreased to 1.44, 0.00178, and eventually to zero.

For example, in Worked Example 10.2, $Y_t = 18\,000(1.2)^{t-1}$, Y_t increased from £18 000 to £22 046 574 for $t = 1$ to $t = 40$, eventually going to ∞.

Worked Example 10.4 illustrates the use of these rules.

WORKED EXAMPLE 10.4

STABILITY OF SOLUTIONS OF FIRST-ORDER DIFFERENCE EQUATIONS

(a) Find the general and particular solutions of the difference equations:

(i) $Y_{t+1} - 2Y_t = 0$ given $Y_1 = 900$ (ii) $Y_{t+1} + 0.7Y_t = 0$ given $Y_1 = 49$ (iii) $Y_{t+1} + Y_t = 0$ given $Y_1 = 10$

In each case, state whether the solution is stable, and if so, state the stable value.

(b) In each case, write out and plot the time path to stability for $t = 0$ to $t = 4$ in steps of one.

Solution

(a) In all three problems the general form of the solution is

$$Y_t = A(a)^t$$

$$\text{therefore} \quad Y_{t+1} = A(a)^{t+1} = A.a^t.a$$

Substituting the general form for Y_t and Y_{t+1}, solve for a, the non-trivial solution.

(i)	**(ii)**	**(iii)**
$Y_{t+1} - 2Y_t = 0$	$Y_{t+1} + 0.7Y_t = 0$	$Y_{t+1} + Y_t = 0$
$Aa^{t+1} - 2Aa^t = 0$	$Aa^{t+1} + 0.7Aa^t = 0$	$Aa^{t+1} + Aa^t = 0$
$Aa^t(a-2) = 0$	$Aa^t(a+0.7) = 0$	$Aa^t(a+1) = 0$
$a = 2$	$a = -0.7$	$a = -1$
General solution:	General solution:	General solution:
$Y_t = A(2)^t$	$Y_t = A(-0.7)^t$	$Y_t = A(-1)^t$
Not stable	Stable	Not stable, but finite

To find the particular solutions, substitute the given conditions:

$Y_t = A(2)^t$	$Y_t = A(-0.7)^t$	$Y_t = A(-1)^t$
$Y_1 = A(2)^1$	$Y_1 = A(-0.7)^1$	$Y_1 = A(-1)^1$
$900 = A2$	$49 = A(-0.7)$	$10 = A(-1)$
$A = 450$	$A = -\frac{49}{0.7} = -70$	$A = -10$
Particular solution:	Particular solution:	Particular solution:
$Y_t = 450(2)^t$	$Y_t = -70(-0.7)^t$	$Y_t = -10(-1)^t$

(b) Time path to stability: In each case, set up a table of points, then plot the graph, remembering that within each time interval, Y_t is constant, so the graph is a horizontal line segment.

(i) Table 10.2 gives the values for $Y_t = 450(2)^t$ over $t = 0$ to $t = 4$. These points are plotted in Figure 10.1.

(ii) Table 10.3 gives the values for $Y_t = -70(-0.7)^t$ over $t = 0$ to $t = 4$. These points are plotted in Figure 10.2.

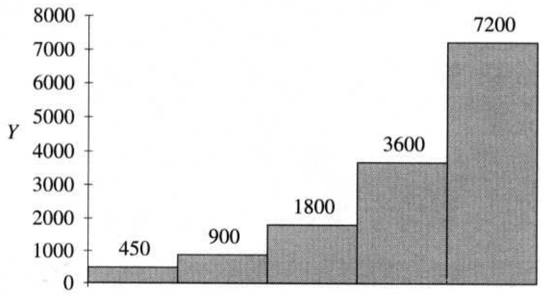


Figure 10.1 $Y_t = 450(2)^t$: Y_t increases without bound

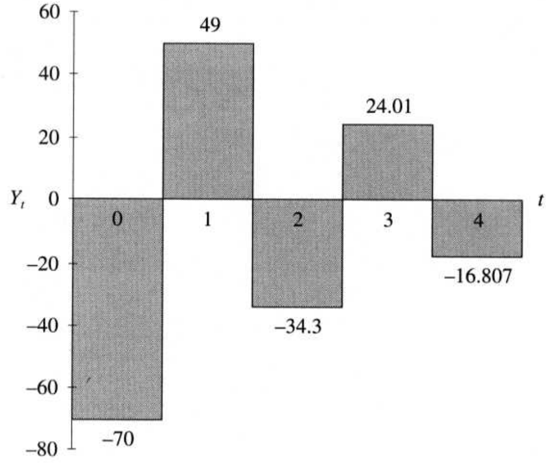


Figure 10.2 Solution oscillates to stability at $Y = 0$

(iii) Table 10.4 gives the values for $Y_t = -10(-1)^t$ over $t = 0$ to $t = 4$. These points are plotted in Figure 10.3.

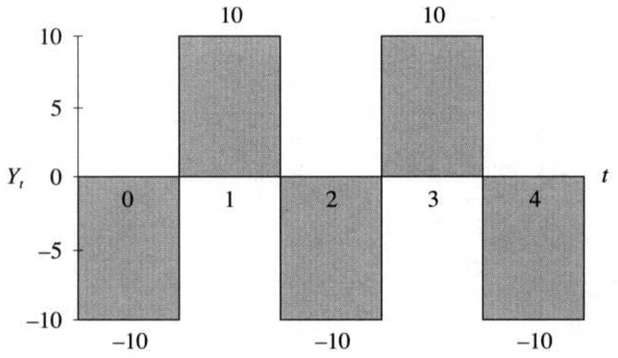


Figure 10.3 Unstable time path: solution oscillates between +10 and −10

❐ Non-homogeneous difference equations (RHS $\neq 0$)

The solution of a non-homogeneous difference equation is the sum of two terms: a complementary function and a particular integral: $Y_t = \text{CF} + \text{PI}$.

- The complementary function (CF): this is the solution of the homogeneous part of the difference equation.
  For example, equation (10.2), $Y_{t+1} - 0.95Y_t = 1000$. The complementary function is the solution of $Y_{t+1} - 0.95Y_t = 0$.
- The particular integral (PI): this is a function which satisfies the full difference equation. The general form of the PI is deduced from the RHS of the difference equation. In this chapter we shall cover only two types of functions: constants and $(b)^t$:

RHS	**PI (general form)**
Constant	$Y_{t,p} = k$ (another constant, k)
Constant × b^t	$Y_{t,p} = k(b)^t$

Again, the method shall be demonstrated through a Worked Example.

WORKED EXAMPLE 10.5

SOLVE NON-HOMOGENEOUS FIRST-ORDER DIFFERENCE EQUATIONS 1

(a) Find the general solution of the difference equation $Y_{t+1} - 0.95Y_t = 1000$.
(b) Find the particular solution, given $Y_5 = 20\,950$.
(c) Determine whether the system will stabilise and if so, what the stable value is. Plot the time path to stability for $t = 0$ to 10 in steps of one.

Solution

(a) The general solution is

$$Y_t = \text{CF} + \text{PI}$$

$$\text{or } Y_t = Y_{t,c} + Y_{t,p}$$

First, find the complementary function by solving the homogeneous equation.

Complementary function (CF): $Y_{t+1} - 0.95Y_t = 0$

General form: $Y_{t,c} = A(a)^t$, therefore $Y_{t+1,c} = A(a)^{t+1} = A.a^t.a$

Substitute the general form for Y_t and Y_{t+1} into the given equation and solve for a,

$$Aa^{t+1} - 0.95Aa^t = 0$$

$$Aa^t(a - 0.95) = 0$$

$$a = 0.95$$

Therefore, the complementary function, $Y_{t,c}$ is,

$$Y_{t,c} = A(0.95)^t$$

Particular integral (PI): Next, find the particular integral, $Y_{t,p}$. Since the RHS of the full equation is a constant, the general form of the PI is

$$Y_{t,p} = k, \qquad \text{therefore} \qquad Y_{t+1,p} = k$$

(a constant, k, has the same value at all times; t, $t + 1$, etc.) Substitute the general form into the full difference equation and solve for k:

$$Y_{t+1} - 0.95Y_t = 1000$$

$$k - 0.95k = 1000$$

$$0.05k = 1000$$

$$k = \frac{1000}{0.05} = 20\,000$$

Therefore $Y_{t,p} = 20\,000$.

So the general solution is

$$Y_t = Y_{t,c} + Y_{t,p}$$

$$Y_t = A(0.95)^t + 20\,000$$

(b) Since a condition is given, $Y_5 = 20\,950$, we may substitute this condition into the general equation to find a value for the constant A:

$$\begin{aligned} Y_5 &= A(0.95)^5 + 20\,000 \\ 20\,950 &= A(0.95)^5 + 20\,000 \\ 950 &= A(0.773781) \\ 1228 &= A \end{aligned}$$

Therefore the particular solution is

$$Y_t = 1228(0.95)^t + 20\,000$$

(c) Since $a = 0.95 < 1$, the solution will stabilise, that is, $(0.95)^t$ will approach 0 as t increases. Therefore, the stable income is

$$\begin{aligned} Y_t &= 1228(0) + 20\,000 \\ &= 20\,000 \end{aligned}$$

To plot the time path, evaluate Y_t for several values of t, as in Table 10.5. These points are plotted in Figure 10.4.

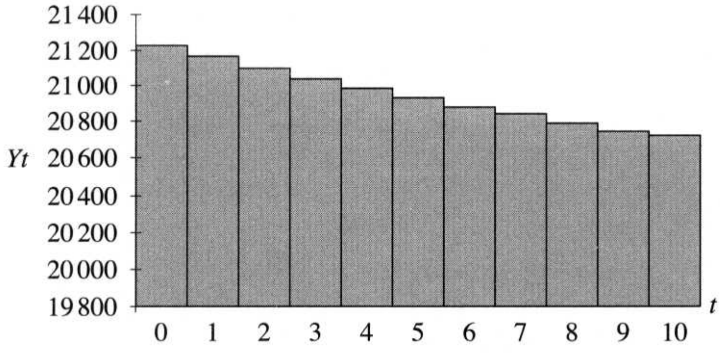


Figure 10.4 Stable time path: solution decreases steadily to 20 000

WORKED EXAMPLE 10.6

SOLVE NON-HOMOGENEOUS FIRST-ORDER DIFFERENCE EQUATIONS 2

(a) Solve the difference equation $3Y_{t+1} + 2Y_t = 44(0.8)^t$ given $Y_0 = 900$.
(b) Show that the solution stabilises and plot the time path to stability.

Solution

(a) The general solution has the form

$$Y_t = \text{CF} + \text{PI}$$

$$\text{or } Y_t = Y_{t.c} + Y_{t.p}$$

Complementary function (CF):

General form: $Y_{t.c} = A(a)^t$, therefore $Y_{t+1.c} = A(a)^{t+1} = A.a^t.a$

Substitute into the homogeneous equation:

$$3Y_{t+1} + 2Y_t = 0$$
$$3Aa^ta + 2Aa^t = 0$$
$$Aa^t(3a + 2) = 0$$
$$Aa^t \neq 0 \qquad 3a + 2 = 0$$
$$a = -\frac{2}{3}$$

Therefore, $Y_{t.c} = A\left(-\frac{2}{3}\right)^t$.

Particular integral (PI):

Since the RHS $= 44(0.8)^t$, the general form is

$$Y_{t.p} = k(0.8)^t, \text{ therefore, } Y_{t+1.p} = k(0.8)^{t+1} = k(0.8)^t(0.8)^1$$

Substitute into the full difference equation:

$$3k(0.8)^{t+1} + 2k(0.8)^t = 44(0.8)^t$$
$$k(0.8)^t[3(0.8)^1 + 2] = 44(0.8)^t$$
$$k[2.4 + 2] = 44 \qquad \text{divide both sides by } (0.8)^t$$
$$k = \frac{44}{4.4} = 10$$

Therefore $Y_{t.p} = k(0.8)^t = 10(0.8)^t$.

The general solution is

$$Y_t = Y_{t.c} + Y_{t.p}$$
$$= A\left(-\frac{2}{3}\right)^t + 10(0.8)^t$$

Since a condition, $Y_0 = 900$, is given, find the particular solution:

$$Y_0 = A\left(-\frac{2}{3}\right)^0 + 10(0.8)^0$$
$$900 = A + 10$$
$$A = 890$$

Therefore, the particular solution is

$$Y_t = 890\left(-\frac{2}{3}\right)^t + 10(0.8)^t$$

(b) **Stability:** Since the base, a, in each of the exponential terms is less than 1, these terms will tend to zero as t increases. So,

$$Y_t = 890(0) + 10(0) = 0$$

Y_t will stabilise at $Y_t = 0$. The time path will fluctuate to stability since the base of one of the exponential terms, $(-2/3)^t$, is negative. To plot the time path, evaluate Y_t for several values of t, as in Table 10.6. These points are plotted in Figure 10.5.

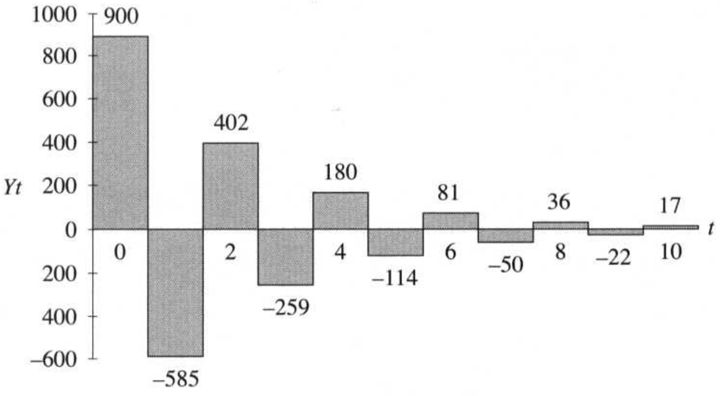


Figure 10.5 Solution oscillates to stability

Progress Exercises 10.1 Introduction to Difference Equations

1. For each of the following difference equations, state
   (i) The order of the equation.
   (ii) Whether the equation is homogeneous or not.

(a) $P_{t+1} - 0.8P_t = 0$ (b) $Y_{t+2} = 8 - Y_{t+1}$ (c) $Y_{t+2} = 80 + Y_t$

2. Solve each of the following difference equations for the indicated variable by the iteration method.
   (a) $Y_{t+1} - 0.8Y_t = 10$, given $Y_1 = 5$. Find Y_5.
   (b) $P_{t+2} = 4P_{t+1} - 8P_t$, given $P_1 = 20$, $P_2 = 18$. Find P_5.
   (c) $P_t = 0.6P_{t-1} + 80$, given $P_1 = 100$. Find P_5.
3. Find the general and particular solutions of the following. Hence evaluate Y_{20} and Y_{50}.
   (a) $Y_{t+1} - 0.8Y_t = 0$, given $Y_1 = 5$.
   (b) $Y_{t+1} + 0.8Y_t = 0$, given $Y_1 = 5$.
   (c) $Y_{t+1} - 0.8Y_t = 10$, given $Y_1 = 5$.
4. For each part in question 3, state
   (a) Whether the solution is stable, and if so, state the value of the stable solution.
   (b) Plot the solution for $t = 0, 1, 2, 3, 4$ and 5. Describe the time path to stability.
5. Find the general and particular solutions of the following:
   (a) $Y_{t+1} = 10Y_t + 900$, given $Y_0 = 20$.
   (b) $P_{t+1} = 0.9P_t + 80$, given $P_1 = 170$.
6. The relationship between P_t, the price of a good in season t, to its price during the previous season is given by the equation $P_t = 1.05P_{t-1}$.
   (a) Outline verbally the relationship defined by the difference equations.
   (b) Solve the difference equation, given that $P_1 = 6300$. Use the solution to describe how the price will change in future years.
7. (a) Which of the following difference equations are different from $P_{t+1} = 1.05P_t$? Explain.
   (i) $P_{t+5} - 1.05P_{t+4} = 0$ (ii) $P_t = 1.05P_{t+1}$ (iii) $1.05P_{t+1} - P_{t+2} = 0$

   (b) Which of the following difference equations are different from $P_{n+2} = P_n + (2)^n$? Explain.
   (i) $P_n = P_{n-2} + (2)^{n-2}$ (ii) $P_n = P_{n-2} + (2)^n$
   (iii) $P_{n+1} - P_{n-1} = (2)^n$ (iv) $P_{n+1} = P_{n-1} + (2)^{n-1}$
8. I_t is the amount invested in period t, where,

   $$I_t = 0.9I_{t-1} + 60$$

   (a) Describe investment, I_t, in terms of the amount invested in previous periods.
   (b) For each of the initial investments (at $t = 0$), £15m; £180m; £900m,
      (i) Solve the difference equation.
      (ii) Calculate the number of years taken for investment to reach £400m.
9. (a) Solve the difference equation $2Y_t + 10 = Y_{t-1}$.
   (b) If $Y_2 = 80$, find the particular solution.
   (c) Plot and describe the time path for $t = 1, 2, 3, 4$ and 5. Will the system stabilise?
10. A fish population increases by 5% each generation, but a catch of 8000 is lost from each generation. If P_n represents the size of the population for generation n,
    (a) Write the difference equation which expresses P_n in terms of P_{n-1}.
    (b) Solve the difference equation, given $P_0 = 100\,000$. Hence, describe the time path.

11. Solve the difference equations

(a) $Y_t = 0.6Y_{t-1} + 21(2)^t$ (b) $0.8Y_t = 0.56Y_{t-1} + 12(0.8)^t$, given $Y_4 = 0$

12. Find the general and particular solutions of the following:

(a) $P_{t+1} = 0.75P_t + 9(3)^t$, given $P_0 = 190$.
(b) $Y_t = 0.84Y_{t-1} + 5$, given $Y_0 = 40$.

State whether each solution is stable. Describe and plot the time path for $t = 0$ to $t = 10$.

10.3 Applications of Difference Equations (First-order)

As already mentioned, difference equations may be used to model many situations in everyday life. In this section, difference equations are used in some standard well-known models in both microeconomics and macroeconomics.

☐ The lagged income model

This is a macroeconomic application of difference equations using the standard national income model, in which equilibrium is expressed by the equation $Y = C + I$ with all variables referring to the same time period, that is,

$$Y_t = C_t + I_t$$

(There exists no government or foreign sectors.) However, if consumption depends on the income during the previous period, then

$$C_t = C_0 + bY_{t-1}$$

and

$$I_t = I_0, \text{ a constant}$$

Note: The consumption function $C_t = C_0 + bY_{t-1}$, is different from the consumption functions that have been used so far. It assumes that planned consumption is dependent on past income, not present income. This is quite logical since present income may not yet be known.

Substitution of C_t into the equilibrium equation gives a difference equation in Y,

$$Y_t = C_0 + bY_{t-1} + I_0$$

$$Y_t = bY_{t-1} + k_0 \qquad \text{where } k_0 = C_0 + I_0$$

This is a linear, first-order difference equation.

WORKED EXAMPLE 10.7

THE LAGGED INCOME MODEL

Assume a simple national income model

$$Y_t = C_t + I_t$$

where $C_t = 5000 + 0.8Y_{t-1}$ and $I_t = 5000$.

(a) Write the national income equation as a difference equation in Y.

(b) Solve the difference equation. Hence, describe the time path. Will the system stabilise?

(c) If $Y_0 = 100\,000$, find the particular solution. Plot the time path for $t = 0$ to $t = 10$.

Solution

(a)

$$\begin{aligned} Y_t &= C_t + I_t \\ Y_t &= 5000 + 0.8Y_{t-1} + 5000 \quad \text{substituting } C_t \text{ and } I_t \text{ as given} \\ Y_t - 0.8Y_{t-1} &= 5000 + 5000 \\ Y_t - 0.8Y_{t-1} &= 10\,000 \end{aligned}$$

(b) The general solution has the form

$$Y_t = \text{CF} + \text{PI}$$

$$\text{or } Y_t = Y_{t,c} + Y_{t,p}$$

Complementary function (CF):

General form: $Y_{t,c} = A(a)^t$, therefore $Y_{t-1,c} = A(a)^{t-1} = A.a^t.a^{-1}$

Substitute into the homogeneous equation

$$\begin{aligned} Y_t - 0.8Y_{t-1} &= 0 \\ Aa^t - 0.8Aa^ta^{-1} &= 0 \\ Aa^t(1 - 0.8a^{-1}) &= 0 \\ Aa^t \neq 0 \quad 1 - \frac{0.8}{a} &= 0 \\ a - 0.8 &= 0 \\ a &= 0.8 \end{aligned}$$

Therefore $Y_{t,c} = A(0.8)^t$.

Particular integral (PI):

The general form of the PI is

$$Y_{t,p} = k, \text{ therefore } Y_{t-1,p} = k$$

Substitute the general form of the PI into the full difference equation and solve for k:

$$\begin{aligned} Y_t - 0.8Y_{t-1} &= 10\,000 \\ k - 0.8k &= 10\,000 \\ 0.2k &= 10\,000 \\ k &= \frac{10\,000}{0.2} = 50\,000 \end{aligned}$$

Therefore $Y_{t,p} = 50\,000$.

So the general solution is

$$Y_t = Y_{t,c} + Y_{t,p}$$
$$Y_t = A(0.8)^t + 50\,000$$

As t increases, $(0.8)^t$ will tend to zero. The solution then becomes

$$Y_t = A(0) + 50\,000$$
$$Y_t = 50\,000$$

Since $0 < (a = 0.8) < 1$, Y_t will decline steadily to a stable value of 50 000.

(c) Since a condition $Y_0 = 100\,000$ is given, find the particular solution:

$$Y_0 = A(0.8)^0 + 50\,000$$
$$100\,000 = A + 50\,000$$
$$A = 50\,000$$

Therefore the particular solution is

$$Y_t = 50\,000(0.8)^t + 50\,000$$

The solution stabilises to a value of 50 000

Table 10.7 gives the values of Y_t for $t = 0$ to $t = 10$. Figure 10.6 plots the time path over the first ten years. Use Excel, if available.

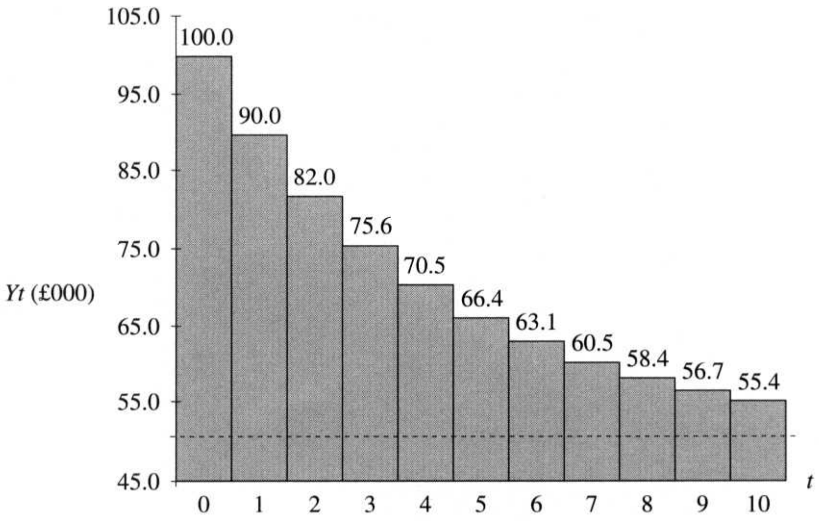


Figure 10.6 Solution decreases steadily to 50 (£000)

☐ The cobweb model

This is a microeconomic application of difference equations. Consider the demand and supply functions of a good over time given by

$$Q_{d,t} = a - bP_t \tag{10.6}$$

The quantity demanded of the good in time period, t, is a function of the price in that time period.

$$Q_{s,t} = c + dP_{t-1} \tag{10.7}$$

The quantity supplied of the good in time period t is a function of the price of the good during the previous period. This reflects the fact that a firm's output decision may lag behind the market price. For example, the quantity supplied of agricultural goods frequently depends on the price in the previous period.

The usual condition for market equilibrium is,

$$Q_{d,t} = Q_{s,t}$$

Substitution of the RHS of equations (10.6) and (10.7) into the condition for market equilibrium gives rise to a difference equation in P:

$$a - bP_t = c + dP_{t-1}$$
$$-bP_t - dP_{t-1} = (c - a)$$

WORKED EXAMPLE 10.8

THE COBWEB MODEL

The demand and supply functions for a good at time t are given as

$$Q_{d,t} = 125 - 2P_t$$
$$Q_{s,t} = -50 + 1.5P_{t-1}$$

(a) State the equilibrium condition, hence deduce a difference equation in P.
(b) Solve the difference equation to find the equilibrium price and quantity.
(c) Given that $P = 60$ at $t = 0$, plot and describe the time path to stability.

Solution

(a) At equilibrium, $Q_{d,t} = Q_{s,t}$, therefore

$$125 - 2P_t = -50 + 1.5P_{t-1}$$
$$-2P_t - 1.5P_{t-1} = -50 - 125$$
$$2P_t + 1.5P_{t-1} = 175$$

(b) The general solution has the form

$$P_t = \text{CF} + \text{PI}$$
$$\text{or } P_t = P_{t,c} + P_{t,p}$$

Complementary function: This is the solution of the homogeneous equation

$$2P_t + 1.5P_{t-1} = 0$$

Substitute the general form of $P_{t,c} = A(a)^t$ and $P_{t-1,c} = A(a)^{t-1} = A.a^t.a^{-1}$ into the homogeneous equation

$$\begin{aligned} 2P_t + 1.5P_{t-1} &= 0 \\ 2Aa^t + 1.5Aa^ta^{-1} &= 0 \\ Aa^t(2 + 1.5a^{-1}) &= 0 \end{aligned}$$

$$Aa^t \neq 0 \qquad 2 + \frac{1.5}{a} = 0$$

$$\begin{aligned} 2 &= -\frac{1.5}{a} \\ 2a &= -1.5 \\ a &= -0.75 \end{aligned}$$

Therefore $P_{t,c} = A(-0.75)^t$.

Particular integral: Since the RHS is a constant, the general form of the PI is

$$P_{t,p} = k, \text{ and } P_{t-1,p} = k$$

Substitute the general form of the PI into the full difference equation and solve for k:

$$\begin{aligned} 2P_t + 1.5P_{t-1} &= 175 \\ 2k + 1.5k &= 175 \\ 3.5k &= 175 \\ k &= \frac{175}{3.5} = 50 \end{aligned}$$

Therefore $P_{t,p} = 50$.

So the general solution is

$$\begin{aligned} P_t &= P_{t,c} + P_{t,p} \\ P_t &= A(-0.75)^t + 50 \end{aligned}$$

(c) $A(-0.75)^t$ since $a = -0.75$ but P_t will oscillate to the stable value of $P_t = 50$, since a is negative. If $P_0 = 60$, then

$$\begin{aligned} P_0 &= A(-0.75)^0 + 50 \\ 60 &= A + 50 \\ A &= 10 \end{aligned}$$

Therefore the particular solution is

$$P_t = 10(-0.75)^t + 50$$

Note: Assuming an equilibrium price, P^*, then $P_t = P^* = P_{t-1} = 50$, and

$$Q_{d,t} = 125 - 2P_t = 125 - 2(50) = 25$$

$$Q_{s,t} = -50 + 1.5P_{t-1} = -50 + 1.5(50) = 25$$

Table 10.8 gives values of P_t for $t = 0$ to $t = 10$. These points are plotted in Figure 10.7.

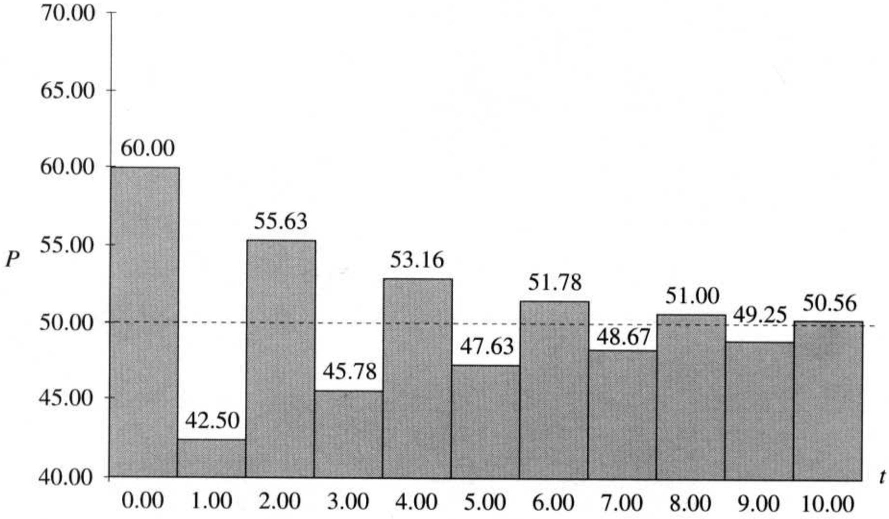


Figure 10.7 The cobweb model: solution oscillates to a stable value of 50

❐ The Harrod–Domar growth model

This is another macroeconomic application of difference equations. The Harrod–Domar growth model is a simple macroeconomic growth model which states that investment is proportional to the rate of change in income:

$$I_t = k(Y_t - Y_{t-1}) \tag{10.8}$$

and savings at time period t depends on income:

$$S_t = sY_t \tag{10.9}$$

where the constant k is the capital–output ratio, usually $k > 1$ and
the constant s is the marginal propensity to save, MPS, where $0 < s < 1$

At equilibrium, $I_t = S_t$, therefore,

$$k(Y_t - Y_{t-1}) = sY_t$$

$$(k-s)Y_t - kY_{t-1} = 0$$

thus reducing to a first-order homogeneous difference equation. On dividing by $(k - s)$ this may also be written as

$$Y_t - \left(\frac{k}{k-s}\right)Y_{t-1} = 0$$

which has the general solution

$$Y_t = A\left(\frac{k}{k-s}\right)^t$$

If $\left|\frac{k}{k-s}\right| < 1$, the system stabilises.

WORKED EXAMPLE 10.9

THE HARROD–DOMAR GROWTH MODEL

Given the equilibrium condition that

$$I_t = S_t$$

where $I_t = 2.5(Y_t - Y_{t-1})$ and $S_t = 0.1Y_t$:

(a) Write the equilibrium equation as a difference equation in Y_t.
(b) Solve the difference equation, given $Y_0 = £8\text{m}$.
(c) State whether the system will stabilise. Plot the time path for $t = 0$ to $t = 7$.

Solution

(a) Substitute the equations for I_t and S_t into the equilibrium equation:

$$I_t = S_t$$

$$2.5(Y_t - Y_{t-1}) = 0.1Y_t$$

$$2.4Y_t - 2.5Y_{t-1} = 0$$

$$Y_t - \frac{2.5}{2.4}Y_{t-1} = 0 \qquad \text{dividing across by 2.4}$$

$$Y_t - 1.04Y_{t-1} = 0$$

(b) Since this is a homogeneous difference equation, the solution has the general form

$$Y_t = A(a)^t, \text{ hence } Y_{t-1} = A(a)^{t-1} = A.a^t.a^{-1}$$

Substituting the general form into the given equation

$$Y_t - 1.04Y_{t-1} = 0$$

$$Aa^t - 1.04Aa^t a^{-1} = 0$$

$$Aa^t(1 - 1.04a^{-1}) = 0$$

$Aa^t \neq 0$, otherwise we have a trivial solution, hence

$$1 - 1.04a^{-1} = 0$$

$$1 - \frac{1.04}{a} = 0$$

$$1 = \frac{1.04}{a}$$

$$a = 1.04$$

Therefore the general solution is $Y_t = A(1.04)^t$.

Since $Y_0 = £8\text{m}$ is given, this condition may be used to find a value for the constant A,

$$Y_0 = A(1.04)^0$$

$$8 = A \qquad \text{since } (1.04)^0 = 1$$

Therefore the particular solution is

$$Y_t = 8(1.04)^t$$

(c) Since $a = 1.04 > 1$, the solution is unstable. Y_t will increase steadily without bound as t increases. This is shown in Table 10.9 which is then plotted in Figure 10.8.

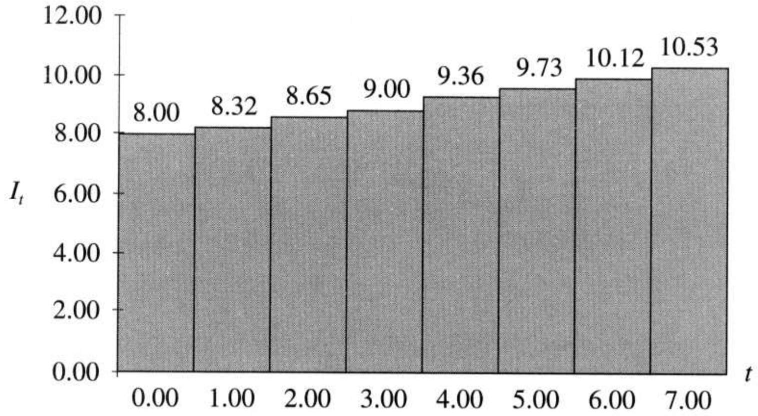


Figure 10.8 The Harrod–Domar growth model

Progress Exercises 10.2 First-order Difference Equations: Applications

1. An amount of money, A, is invested at $r\%$ compounded annually.
(a) Show that the value of the investment at the end of any year, t, is given by the difference equation,

$$P_{t+1} = (1 + r)P_t$$

(b) Show that the general solution of the equation is $P_t = P_0(1 + r)^t$.

2. Solve each of the following difference equations and plot the time path for $t = 0$ to $t = 20$ (use Excel). State which systems are stable and state the stable value.

(a) $Y_t = C_t + I_t$ where $C_t = 100 + 0.8Y_{t-1}$ and $I_t = 500$.
(b) $Y_t = C_t + I_t$ where $C_t = 500(1.05)^t + 0.75Y_{t-1}$ and $I_t = 0$

3. The quantity of cabbage supplied to a market depends on the price during the previous season, while the quantity demanded depends on the present price

$$Q_{s,t} = -4 + 2P_{t-1}: \quad Q_{d,t} = 80 - 2P_t$$

(a) Write the equilibrium condition as a difference equation in P and solve it.
(b) Solve the equation to obtain an expression for price in terms of t, given $P_5 = 10.5$. Hence, determine whether prices will stabilise.

4. The supply and demand functions for a good at time t are given by the equations

$$Q_{d,t} = 150 - 0.3P_t: \qquad Q_{s,t} = 0.1P_t - 30$$

The price in any season depends on the excess supplied during the previous season as described by the equation

$$P_{t+1} = P_t - 0.3(Q_{s,t} - Q_{d,t})$$

(a) Deduce a difference equation which relates the present price to the price in the previous season.
(b) Solve for an expression for the price at any time t in terms of t. Hence
(c) Determine whether prices will stabilise.

5. Investment at time t depends on the excess income over the previous year, as given by the difference equation $I_t = 4.5(Y_t - Y_{t-1})$ while savings are 12% of income, $S_t = 0.12Y_t$. Given the equilibrium condition $I_t = S_t$,
(a) Write the equilibrium condition as a difference equation in Y.
(b) Solve the difference equation, given $Y_0 = £314\text{m}$.
(c) Plot the time path for (i) Y_t, (ii) I_t for $t = 0$ to 4. Will investment stabilise? Explain.

6. Investment at time t depends on the excess income over the previous year, as given by the difference equation $I_t = 0.9(Y_t - Y_{t-1})$, while dissavings are 25.5% of income, $S_t = -0.255Y_t$. Given the equilibrium condition, $I_t = S_t$,
(a) Write the equilibrium condition as a difference equation in Y.
(b) Solve the difference equation, given $Y_0 = £800\text{m}$.
(c) Plot the time path for (i) Y_t, (ii) I_t over $t = 0$ to $t = 12$. Will investment stabilise? Explain.

10.4 Summary

1. A difference equation relates the value of a dependent variable which changes at discrete points in time only (such as income) to its values during previous time periods. For example,

$$Y_{t+1} = \frac{120}{100} Y_t$$

In words, the difference equation states 'income in any year is 20% higher than income in the previous year'.

2. The order of a difference equation is the number of time periods spanned by the equation. The order is 1 in the above example.
3. Solution of difference equations over a reasonably small number of time intervals by iteration, provided conditions are given. Excel is very useful here.
4. Mathematically, the solution of an homogeneous difference equation if of the form: $Y_t = A(a)^t$. To determine the actual solution, substitute the trial solution $Y_t = A(a)^t$ into the difference equation and solve for the base a. The constant A is determined if one condition is given, such as $Y_0 = 30$.
5. The solution of a non-homogeneous difference equation is the sum of two parts: the complementary function (CF) and a particular integral (PI)

CF is the solution of the corresponding homogeneous equation
PI is determined by inspection.

Stability of the solution of a difference equation: $Y_t = A(a)^t$ depends on a^t: behaviour of $(a)^t$ as t increases for $t > 0$

$-\infty < a < -1$	$-1 < a < 0$	$0 < a < 1$	$1 < a < \infty$
$(a)^t \to \pm\infty$	$(a)^t \to = 0$	$(a)^t \to 0$	$(a)^t \to \infty$
Solution unstable: time path alternates	Solution stable: time path alternates	Solution stable: time path steady to zero	Solution unstable: time path steady to infinity

☐ Applications

1. Lagged income model: equilibrium condition: $Y_t = C_t + I_t$

   where $C_t = C_0 + bY_{t-1}: \; I_t = I_0$

2. The cobweb model: equilibrium condition: $Q_{d,t} = Q_{s,t}$

   where $Q_{d,t} = a - bP_t: \quad Q_{s,t} = c + dP_{t-1}$

(The quantity of the good supplied in time period t is a function of the price in the previous time period: $Q_{s,t} = c + dP_{t-1}$.)

3. Harrod–Domar growth model: equilibrium condition: $I_t = S_t$

where $I_t = k(Y_t - Y_{t-1})$: $S_t = sY_t$

Test Exercises 10

1. (a) Solve the difference equation by iteration for Y_1 to Y_{10}:

$$Y_{t+1} = 4Y_t + 3, \text{ given } Y_1 = 12.$$

(b) Solve the difference equation in (a) by algebraic methods.

2. Determine the general solution of the difference equation

$$Y_{t+1} - 0.4Y_t = 0$$

State whether the solution is stable or unstable.

3. Given the national income model $Y_t = C_t + I_t$, where $C_t = 220 + 0.4Y_t$, $I_t = 0.35Y_{t-1}$,
(a) Write the equilibrium equation as a difference equation in Y_t.
(b) Determine the general expression for the equilibrium level of national income.
(c) Determine the particular solution given $Y_0 = 2\,200$. Graph the solution for $t = 0$ to 5. Comment on its stability.

4. The relationship between the number of whales inhabiting a given area to the numbers inhabiting that area during the previous seasons is given by the difference equation $P_n = 0.8P_{n-1} + 800$.
(a) Determine the general solution and discuss its stability
(b) If $P_0 = 5000$ determine the particular solution.
Graph the whale population for eight seasons.

5. Given the equation for equilibrium national model $Y_t = C_t + I_t$, where $C_t = 120 + 0.3Y_{t-1}$; $I_t = 5 + 0.2Y_{t-1}$,
(a) Write the equilibrium equation as a first-order difference equation.
(b) Solve the difference equation, given $Y_0 = 100$. Is the solution stable?
(c) Calculate the number of years it will take for equilibrium income to reach 450.

6. The price of a good is set each season according to the difference equation $P_t = P_{t-1} + 0.2(Q_{s,t} + Q_{d,t})$, where the quantities demanded and supplied are given by the equations

$$Q_{d,t} = 8 - 0.5P_t, \quad Q_{s,t} = 1.3P_{t-1} - 7$$

(a) Write the price-setting equation as a second-order difference equation.
(b) Solve for price, hence determine whether the price will stabilise.

Solutions to Progress Exercises

Chapter 1

1.2

1. $x = 3$
2. $x = 1$
3. $x = 2, x = -4$
4. $x = \sqrt{8}, x = -\sqrt{8}$
5. $x = 0, x = -2$
6. $x = 0, x = 2, x = -4$
7. $x = 0, x = 2$
8. $x = y$: infinitely many solutions
9. $y = -2, x = -2$
10. $x = 0.5, y = -2$
11. $x = 0$
12. $x = \sqrt{-4}, x = -\sqrt{-4}$
13. $x = 0$
14. $x = 0, x = \sqrt{-2}, x = -\sqrt{-2}$
15. $x = -4$
16. $x = 0$
17. $x = 0.5$
18. $x = 0, x = 4, x = -3.8$
19. $P = -20$
20. No solution
21. $x = 5, x = -5$
22. P may be any value, except -2. $Q = 4$
23. $Q = \dfrac{1}{100}$
24. (a) 297.1768 (b) 32 255.5721
25. 912.2007

26. 975.5194

27. Hong Kong dollar exchange rates

British pound	0.0905
US dollar	0.1282
Japanese yen	15.8687
Danish krone	1.0881
Swedish krona	1.3385

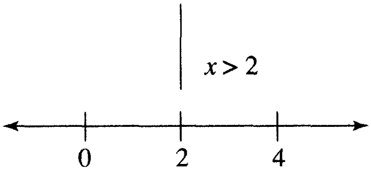
28. 8800.7112

29. (a) 258.1234 (b) 396.1620

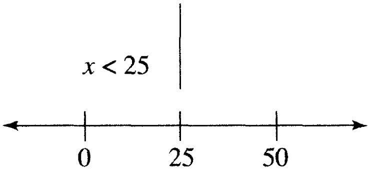
30. 82.0254

1.3

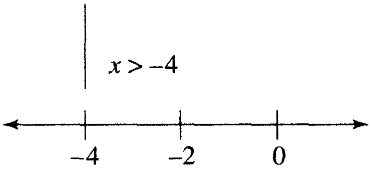
1. (a)



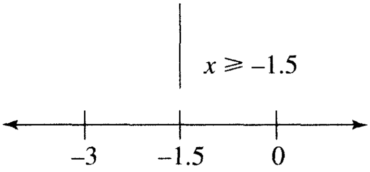
(b)



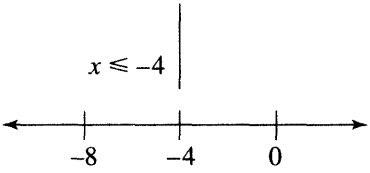
(c)



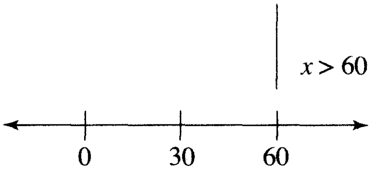
(d)


$x \geqslant -1.5$
−3
−1.5
0

(e) $-4 \geqslant x$ or $x \leqslant -4$


$x \leqslant -4$
−8
−4
0

(f) $60 < x$ or $x > 60$



2. (a) $x - 25 > 7$

$x > 32$ add 25 to both sides

x is greater than 32.

(b) $5 < 2x + 15$

$-10 < 2x$ add -15 to both sides

$-5 < x$ dividing both sides by 2

-5 is less than x or x is greater than -5 ($x > -5$)

(c) $\frac{25}{x} < 10$

$25 < 10x$ multiply both sides by x (x is positive)

$\frac{25}{10} < x$ dividing both sides by 10

2.5 is less than x or x is greater than 2.5 ($x > 2.5$)

(d) $\frac{x}{2}+\frac{x}{3}\geq\frac{17}{6}$

$\frac{5x}{6}\geq\frac{17}{6}$

$5x\geq 17$ multiply both sides by 6

$x\geq 3.4$ dividing both sides by 5

x is greater than or equal to 3.4

(e) $3x-29\leq 7x+11$

$-40\leq 4x$

$-10\leq x$ dividing both sides by 4

-10 is less than or equal to x or x is greater than or equal to -10 ($x\geq -10$)

3. (a) 651.924 (b) 20.1025 (c) 0.7475 **4.** (a) £0.791 (b) £6.441

5. £1400 **6.** 763, 809, 858 **7.** 320, 256, 205, 164, 131, 105

8. £383.15 **9.** 37.08% **10.** £458.832 **11.** £642.8 **12.** 12.5%

1.4

1. (a) −50.1407 (b) 2.5049 (c) 16.0084

2. (a) 3.061828 (b) 9.02986 (c) 44.6069

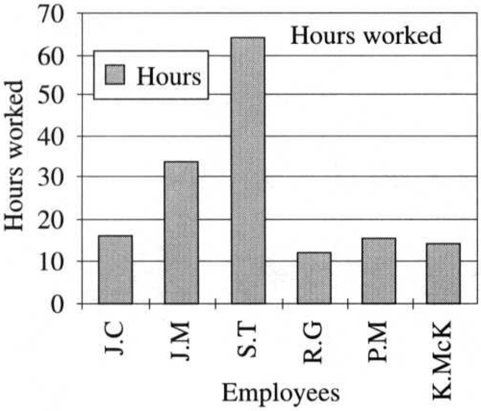
3.

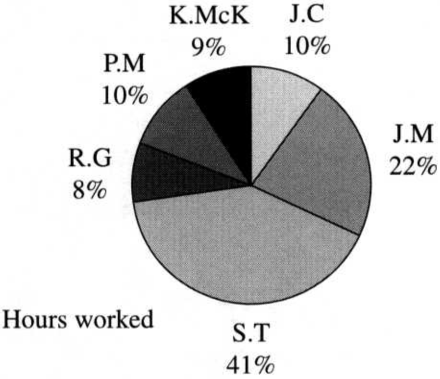
Data

Initials	J.C.	J.M.	S.T.	R.G.	P.M.	K.McK
Hours	16	34	64	12.5	15.5	14.5
Rate	£12.60	£38.80	£102.50	£15.50	£32.50	£45.00
Expenses	£120.00	£55.00	£0.00	£25.00	£12.00	£155.00

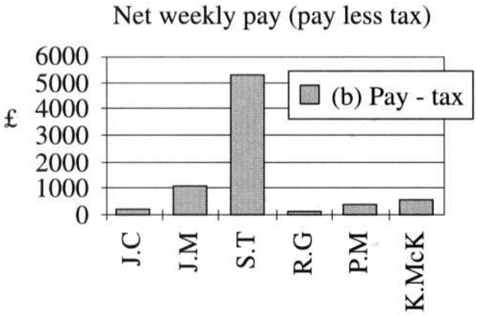
For each staff member							**Total pay for all staff**
(a) Pay	£201.60	£1319.20	£6560.00	£193.75	£503.75	£652.50	£9430.80
(b) Pay − tax	£161.28	£1055.36	£5248.00	£155.00	£403.00	£522.00	£7544.64
(c) Pay − tax + exps.	£281.28	£1110.36	£5248.00	£180.00	£415.00	£677.00	£7911.64

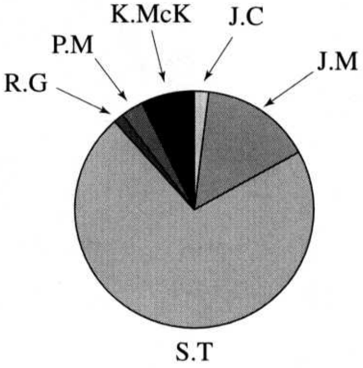
(d) (i)


Hours worked
Hours
Hours worked
70
60
50
40
30
20
10
0
J.C
J.M
S.T
R.G
P.M
K.McK
Employees

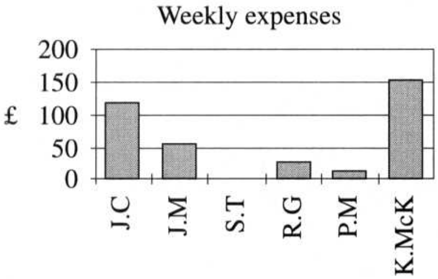

K.McK 9%
J.C 10%
P.M 10%
J.M 22%
R.G 8%
Hours worked
S.T 41%

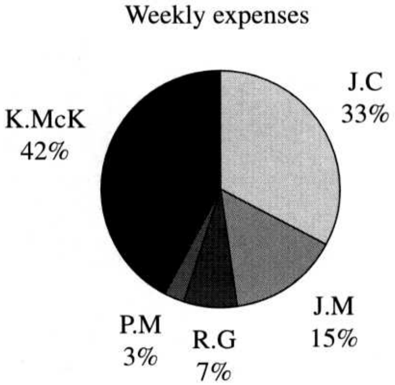
(d) (ii)


Net weekly pay (pay less tax)
6000
5000
4000
3000
2000
1000
0
£
(b) Pay - tax
J.C
J.M
S.T
R.G
P.M
K.McK


K.McK
J.C
P.M
J.M
R.G
S.T

(d) (iii)


Weekly expenses
200
150
100
50
0
£
J.C
J.M
S.T
R.G
P.M
K.McK


Weekly expenses
J.C 33%
K.McK 42%
J.M 15%
P.M 3%
R.G 7%

Chapter 2

2.1

1. Points lie on a line: slope = 1; vertical intercept = 2.

2. Points lie on a line: slope = −1; vertical intercept = 6.

3.

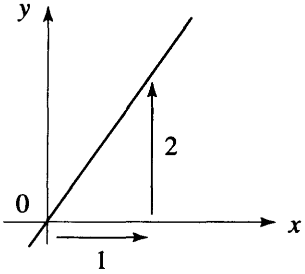


Figure PE 2.1 Q3

4.

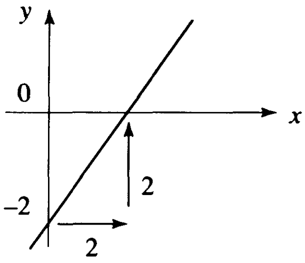


Figure PE 2.1 Q4

5.

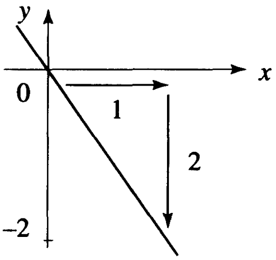


Figure PE 2.1 Q5

6.

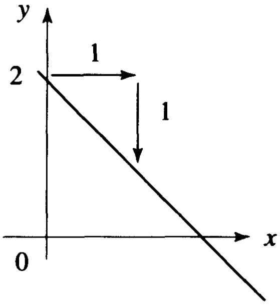


Figure PE 2.1 Q6

2.2

1. (3) $y = 2x$ (4) $y = x - 2$ (5) $y = -2x$ (6) $y = -x + 2$

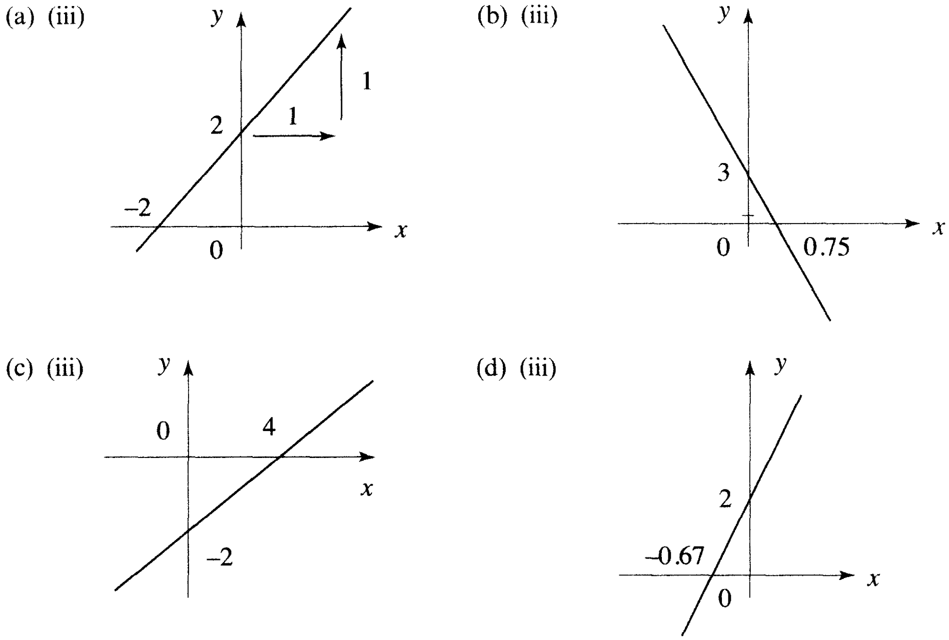
2. (a) (i) $m = 1$; $y(0) = 2$; $x(0) = -2$ (ii) 0, 2, 4, 6, 8

(b) (i) $m = -4$; $y(0) = 3$; $x(0) = 0.75$ (ii) 11, 3, −5, −13, −21

(c) (i) $m = 0.5$; $y(0) = -2$; $x(0) = 4$ (ii) −3, −2, −1, 0, 1

(d) Rearrange: $y = 3x + 2$.

(i) $m = 3$; $y(0) = 2$; $x(0) = -0.67$ (ii) −4, 2, 8, 14, 20



Figures PE 2.2 Q2(iii)

3. (a) $m = 0$; $y(0) = 2$, horizontal line. (b) $m = \infty$; $x(0) = -2$, vertical line.
(c) $y = -5x - 4$: $m = -5$; $y(0) = -4$; $x(0) = -0.8$ (d) $y = x$: $m = 1$; $y(0) = 0$; $x(0) = 0$: the 45° line. (e) $y = x + 5$: $m = 1$; $y(0) = 5$; $x(0) = -5$.

4. (a) (i) $y = 2.5x - 5$ (ii) $y(0) = -5$; $x(0) = 2$ (iii) $m = 2.5$, slope is positive, line is rising or y is increasing as x increases.

(b) (i) $y = -0.5x + 5$ (ii) $y(0) = 5$; $x(0) = 10$ (iii) $m = -0.5$, slope is negative, line is falling or y is decreasing as x increases.

(c) (i) $y = -5x + 15$ (ii) $y(0) = 15$; $x(0) = 3$ (iii) $m = -5$, slope is negative; line is falling or y is decreasing as x increases.

5. (a) (i) $x = 0.4y + 2$ (ii) $x(0) = 2$, this is the vertical intercept; $y(0) = -5$; $m = 0.4$

(b) (i) $x = -2y + 10$ (ii) $x(0) = 10$, this is the vertical intercept; $y(0) = 5$; $m = -2$

(c) (i) $x = -0.2y + 3$ (ii) $x(0) = 3$, this is the vertical intercept; $y(0) = 15$; $m = -0.2$

In question 5, the vertical intercept is the corresponding horizontal intercept in question 4 and vice versa and the slope in question 5 is the slope in question 4 inverted.

6. (a) A, B, C (b) A, C (c) A, C

7. (a) $y = 4$ (b) $x = 2$

8.

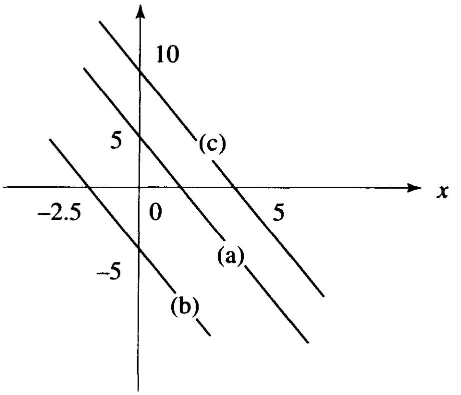


Figure PE 2.2 Q8

All lines have the same slope, $m = -2$, but different intercepts.

2.3

1. (a) (i) $m = -2.5$; $y(0) = 10$; $x(0) = 4$

(ii) $m = 10; y(0) = -2.5; x(0) = 0.25$

(iii) $m = -0.5$; $y(0) = 15$; $x(0) = 30$

(b) (i) y decreases by 2.5 units

(ii) y increases by 10 units

(iii) y decreases by 0.5 units

2. (a)

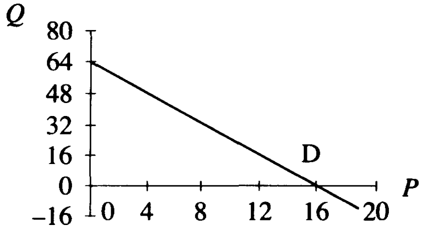


Figure PE 2.3 Q2

(b) Number of flights demanded drops by 4.

(c) When $P = 0$, $Q = 64$. Number of flights demanded when $P = £0$.

(d) When $Q = 0$, $P = 16$. When price is £1600, no flights are demanded.

3. (a) The demand function is horizontal. Its slope = 0, or no slope, since $P = 65 - 0Q$. The price is not influenced by a change in demand. It remains at $P = 65$.

(b)

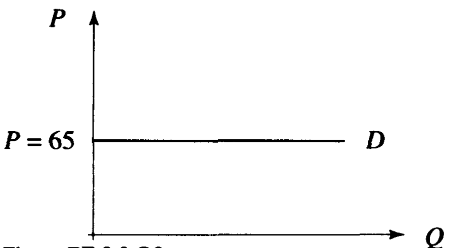


Figure PE 2.3 Q3

4. (a) The vertical intercept is at $P =$ FF500 when $Q = 0$. Below this price, no wine will be supplied.

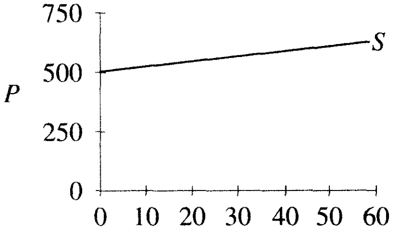


Figure PE 2.3 Q4

(b) When $P =$ FF600, $Q = 50$ litres of wine.

(c) When $Q = 20$, $P =$ FF540. When consumers are willing to pay FF540, producers supply 20 litres of wine.

5. (a) $Q = -250 + 0.5P$

(b) $m = 0.5$. When price increases by 1 FF, the producer will supply 0.5 extra litre.

6. (a) The supply function is horizontal. (b) Its slope $= 0$, or no slope, since $P = 50 + 0Q$. The price is not influenced by a change in supply. It remains at $P = 50$.

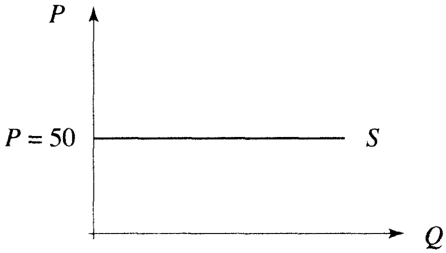


Figure PE 2.3 Q6

7. (a) The supply function is vertical (when plotted as $P = f(Q)$). Its slope $= \infty$. The quantity supplied is fixed and is independent of price.

(b)

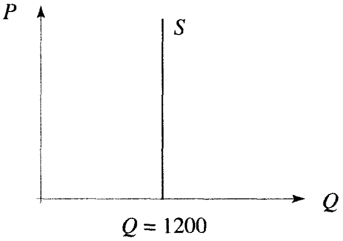


Figure PE 2.3 Q7

2.4

1. (a) $FC = 0$

(b)

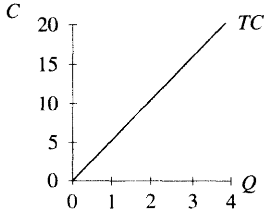


Figure PE 2.4 Q1

(c) $TC = 50$.

(d) Each additional unit costs £5.

2. (a) $P = 10$ (b)

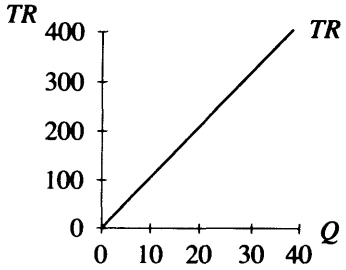


Figure PE 2.4 Q2

3. (a) $TC = 250 + 25Q$

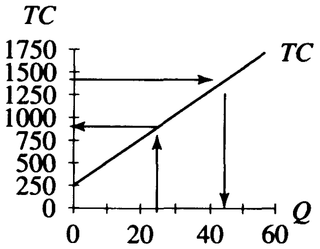


Figure PE 2.4 Q3

(b) When $Q = 28$, $TC = 950$.

(c) When $TC = 1400$, $Q = 46$.

(d) See arrows on graph

4. (a) $TR = 32Q$ (b) 32 (c) When $Q = 44$, $TR = 1408 > TC = 1350$.

2.5

1. (a) (i)

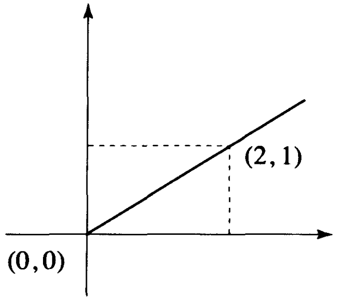


Figure PE 2.5 Q1(a)(i)

(ii)

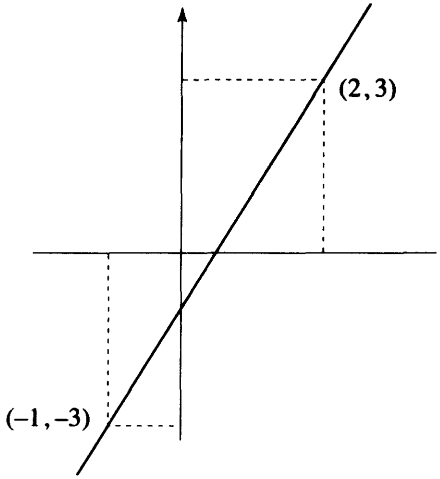


Figure PE 2.5 Q1(a)(ii)

(b) (i) $m = \frac{1-0}{2-0} = \frac{1}{2}$ (ii) $m = \frac{-3-3}{-1-2} = \frac{-6}{-3} = 2$

2. (a) $y = x + 2$ (b) $y = 2x$ (c) $y = -0.5x + 4$

3. (a) $TC = 55 + 1.5Q$

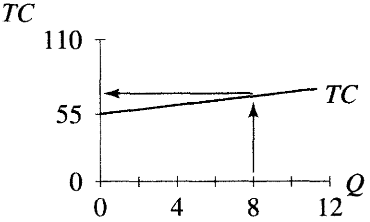


Figure PE 2.5 Q3

(b) When $Q = 8$, $TC = 67$, that is,

$TC = 55 + 1.5(8) = 67$

4. (a) $Q = 105 - 5P$

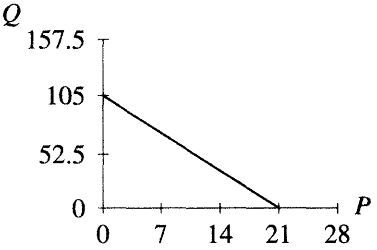


Figure PE 2.5 Q4

(b) (i) Q decreases by 15 units when P increases by £3.

(ii) Q increases by 10 units when P decreases by £2.

Confirm graphically.

(c) P decreases by £3 when Q increases by 15.

5. (a) $P = 0.125Q - 0.25$

(b) $Q = 8P + 2$. When P increases by £1, an additional 8 scarves are supplied.

(c) When $P = £8.5$, $Q = 8(8.5) + 2 = 70$.

(d) When $Q = 120$, $P = 0.125(120) - 0.25 = £14.75$.

(e) No scarves are supplied when the $P \leq -£0.25$ (positive quadrant only).

2.7

1. (a) Slope of demand function is negative, $P > 0$, $Q > 0$. Therefore

$$\varepsilon_{\mathrm{d}} = \frac{\Delta Q}{\Delta P}\frac{P}{Q} < 0$$

(b) $(\%\,\Delta Q) = \varepsilon_{\mathrm{d}}(\%\,\Delta P)$

(i) $(\%\,\Delta Q) = (-0.7)(5\%) = -3.5\%$ (ii) $(\%\,\Delta Q) = (-0.7)(-8\%) = 5.6\%$

2. $Q = 250 - 5P \rightarrow P = 50 - 0.2Q$

(a) $\varepsilon_{\mathrm{d}} = \dfrac{P}{P-a} = \dfrac{P}{P-50}$ or $\varepsilon_{\mathrm{d}} = -\dfrac{1}{b}\dfrac{P}{Q} = -\dfrac{1}{0.2}\dfrac{P}{250-5P} = -\dfrac{P}{50-P} = \dfrac{P}{P-50}$

(b) $P = 20$, $\varepsilon_{\mathrm{d}} = -\dfrac{2}{3}$; $P = 25$, $\varepsilon_{\mathrm{d}} = -1$; $P = 30$, $\varepsilon_{\mathrm{d}} = -1.5$.

3. (a) -3 (b) $-\frac{1}{3}$

4. (a) (i) $P = a - bQ \to \varepsilon_d = 1 - \frac{a}{b}\frac{1}{Q}$. (ii) $Q = c - dP \to \varepsilon_d = 1 - \frac{c}{Q}$

(b) (i) $\varepsilon_d = 1 - \frac{1800}{Q}$. $\varepsilon_d = -17; -2.6; -1.$

(ii) $\varepsilon_d = 1 - \frac{120}{Q}$. $\varepsilon_d = -0.2; 0.76; 0.867.$

5. (a) (i) $\varepsilon_d = \frac{P}{P-90}$, (ii) $\varepsilon_d = 1 - \frac{1800}{Q}$

(b) $\varepsilon_d = -0.2857, -0.5, -3.5.$ (c) $Q = 900$; $Q = 1800$.

6. (a) $\text{Slope} = \frac{\text{change in quantity}}{\text{change in price}} = \frac{\Delta Q}{\Delta P}$.

Same at every point on a linear demand function

$\varepsilon_d = \frac{\text{\% change in quantity}}{\text{\% change in price}} = \frac{\Delta Q}{\Delta P}\frac{P}{Q}$. Different at every point.

(b)

$P =$	20	30	45	70	90
$m =$	-0.05	-0.05	-0.05	-0.05	-0.05
$\varepsilon_d =$	-0.2857	-0.5	-1	-3.5	$-\infty$
$(\%\Delta P) = 10\% \to (\%\Delta Q) =$	-2.85%	-5%	-10%	-35%	$-\infty$

7. (a) $\frac{5}{3}$

(b) (i) at $Q = 40$, $\varepsilon_d = 2 \to (\%\Delta Q) = (2)(\%\Delta P) = (2)(10\%) = 20\%$

(ii) $(P = 40, Q = 40) \to (P = 44, Q = 48) \to \%\Delta Q_s = 20\%$.

2.9

1. (a) Points (x, y)

(0, 2), (1, 3), (2, 4), (3, 5), (4, 6)

(c) (i) $y = 2 + x$ (ii) $P = 2 + Q$

(b)

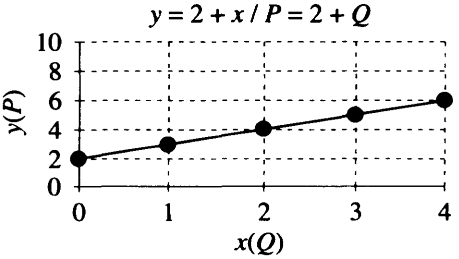


Figure PE 2.9 Q1(b)

2. (a) Demand (Q) decreases by 2.5 for each unit increase in price (b) 42.5, 35, 30

(c)

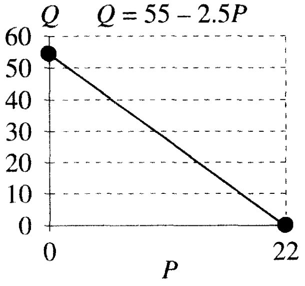


Figure PE 2.9 Q2(c)

(d) $Q = 55 - 2.5P$: $P = 22 - 0.4Q$

3. (a) Points (x, y) (0, 65), (17, 31), (34, –3)

Note: A line may be plotted from two points, use equally spaced x-values to plot from Excel.

(b) Slope = –2: y drops by two units when x increases by one unit. The vertical intercept is 65: $y = 65$ when $x = 0$. The horizontal intercept is $x = 32.5$: $x = 32.5$ when $y = 0$.

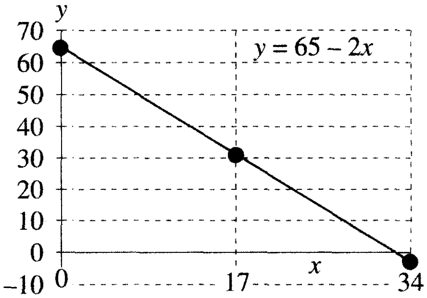


Figure PE 2.9 Q3(a)

4. (a) To define the graph in the first quadrant, find the horizontal and vertical intercepts. Vertical intercept: $Q = 0$, $P = 80$. Horizontal intercept: $P = 0$, $Q = 20$

(b)

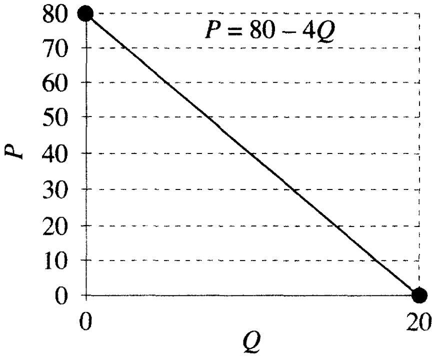


Figure PE 2.9 Q4(b)

(c) Slope = –4, vertical intercept = 80: horizontal intercept = 20

(d) $Q = 20 - 0.25P$

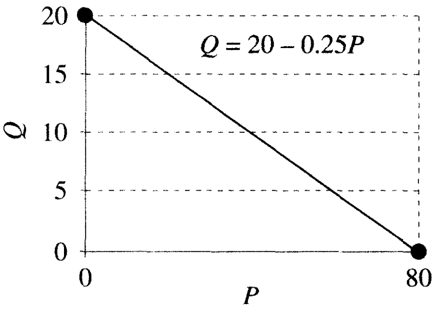


Figure PE 2.9 Q4(d)

5. (a) $Q = 0$, $P = 4$. No positive horizontal intercept, so select any other point to plot the graph, e.g. $Q = 10$, $P = 29$

(b)

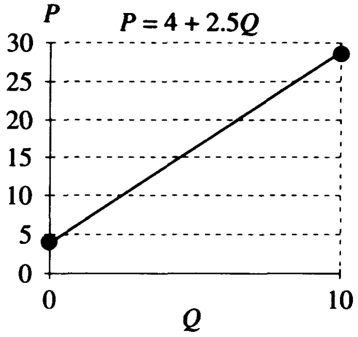


Figure PE 2.9 Q5(b)

(c) Slope = 2.5: vertical intercept at $P = 4$, horizontal intercept at $Q = -1.6$

(d) $Q = 0.4P - 1.6$

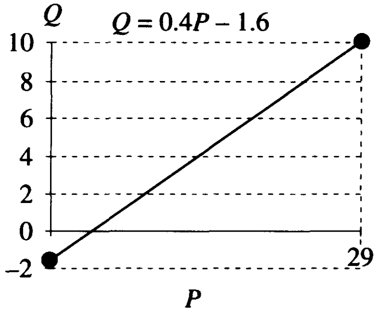


Figure PE 2.9 Q5(d)

6. $P = 60 - 3.5Q$: $Q = 40 - 0.5P$ are demand functions, $P = 5 + 1.2Q$ is a supply function

(a) There are several approaches. For example,

(i) Write each equation in the form $P = a - bQ$: set up a table containing several P-values in the first row, then use the formula $\varepsilon_d = P/(P - a)$ to calculate the corresponding value of elasticity.

(ii) Write each equation in the form $Q = c - dP$; set up a table containing several Q-values in the first row, then use the formula

$$\varepsilon_d = \frac{Q - c}{Q} = 1 - \frac{c}{Q}$$

to calculate the corresponding value of elasticity. Or

(iii) Use the direct method:

$$\varepsilon_d = -\frac{1}{b}\frac{P}{Q}$$

The following tables are examples for the three functions respectively

P	2	10	20	30
ε_d	−0.034	−0.2	−0.5	−1

Q	2	10	20	30
ε_d	−19	−3	−1	−0.333

P	2	10	20	30
ε_d	−0.667	2	1.3333	1.2

(b) Set up a table of P-values 0, 5, 10, 15, 20, 25 in the first row. Calculate the corresponding values of Q in the second row. Use the arc elasticity formula for calculations, placing the results in the third row beneath the G, P values at the start of the interval:

$P = 60 - 3.5Q$

P	0.00	5.00	10.00	15.00	20.00	25.00
Q	17.14	15.71	14.29	12.86	11.43	10.00
ε_d	−0.04	−0.14	−0.26	−0.41	−0.60	

$Q = 40 - 0.5P$

P	0.00	5.00	10.00	15.00	20.00	25.00
Q	40.00	37.50	35.00	32.50	30.00	27.50
ε_d	−0.03	−0.10	−0.19	−0.28	−0.39	

$P = 5 + 1.2Q$

P	0.00	5.00	10.00	15.00	20.00	25.00
Q	−4.17	0.00	4.17	8.33	12.50	16.67
ε_d	−1.00	3.00	1.67	1.40	1.29	

7. Let x = number of ice-creams; y = number of soft drinks

(a) $12x + 20y = 500 \rightarrow y = 25 - 0.6x$:

Vertical intercept = 25: horizontal intercept 41.67. See figure PE 2.9 Q7(a).

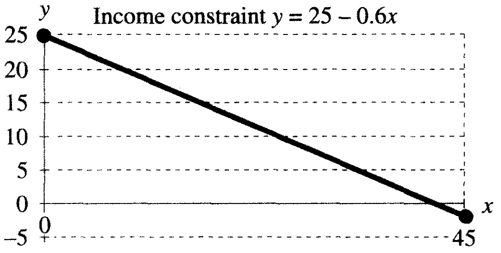


Figure PE 2.9 Q7(a)

Note: the original constraint is shown by the heavy line in this and the following diagrams.

(b) $9x + 20y = 500 \rightarrow y = 25 - 0.45x$

Vertical intercept = 25; horizontal intercept = 55.56 (increased).
See figure PE 2.9 Q7(b).

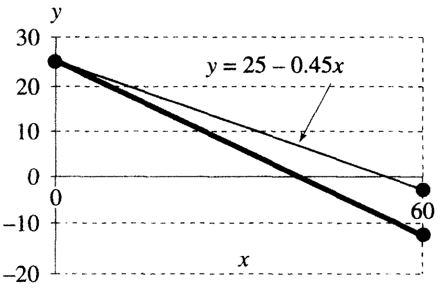


Figure PE 2.9 Q7(b)

(c) $12x + 25y = 500 \rightarrow y = 20 - 0.48x$.

Vertical intercept = 20 (decreased): horizontal intercept = 41.67.

See figure PE 2.9 Q7(c).

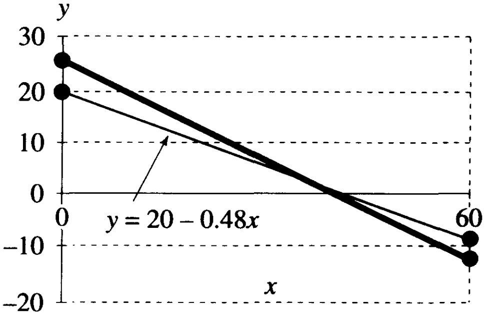


Figure PE 2.9 Q7(c)

(d) $12x + 20y = 750 \rightarrow y = 37.5 - 0.6x$

Vertical intercept = 37.5 (increased), horizontal intercept = 62.5 (increased).

See figure PE 2.9 Q7(d).

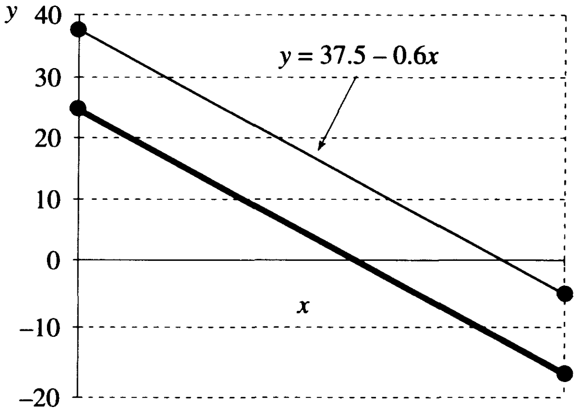


Figure PE 2.9 Q7(d)

8. When $P_X = 25$, $Y = 80$, then $Q_X = 94 + 5P_Y$. If the values of P_Y are entered in row 1 of Table PE 2.9 Q8, then the values of Q_X are calculated in row 2 of the table. In row 3, the values for arc elasticity (called cross-elasticity) are calculated from the data in rows 1 and 2 using the formula

$$\varepsilon_c = \frac{\Delta Q}{\Delta P}\frac{P_1 + P_2}{Q_1 + Q_2}.$$

Table PE 2.9 Q8

P_Y	10		20		30		40		50		60		65
Q_X	144		194		244		294		344		394		419
Arc elasticity		0.4438		0.5708		0.6506		0.7053		0.7453		0.7688	

Chapter 3

3.1

1. $x = 1.5, y = 1.5$ **2.** $x = 7, y = 3$ **3.** $x = 18, y = 1$

4. $x = 1, y = 1$ **5.** $x = 3, y = 2$ **6.** $x = 1, y = 5$

7. $x = 0, y = 4$ **8.** Infinitely many solutions: $y = 5 - 0.5x$

9. $x = 7.04, y = 16.16$

10. No unique solution. Infinitely many: $y = \dfrac{5x - 15}{2}$ **11.** $x = -0.75, y = -1.7$

12. $P = 5.44, Q = 5.92$ **13.** $P = \dfrac{55}{23} = 2.39, Q = 1.63$

14. $x = 1, y = 8, z = 7.$ **15.** $P_1 = 6, P_2 = 2, P_3 = 0$

3.2

1. (a) $y = 1.5x + 6$ (b) $Q = 3P - 45$

2. (a) $Q = 8, P = 26$ (b) $Q_s - Q_d = 16 - 4 = 12$

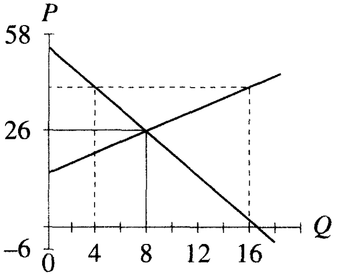


Figure PE 3.2 Q2

3. (a) $Q_d - Q_s = 10 - 4 = 6$

(b) Black market profit = £72

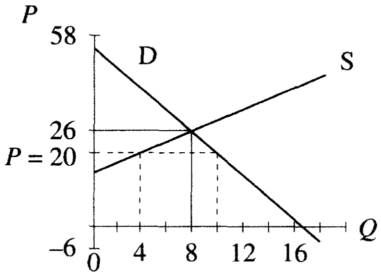


Figure PE 3.2 Q3

4. (a) $L = 10$, $w = 30$ (b) $L_d - L_s = 12.5 - 5 = 7.5$

   (c) $L_s - L_d = 15 - 7.5 = 7.5$

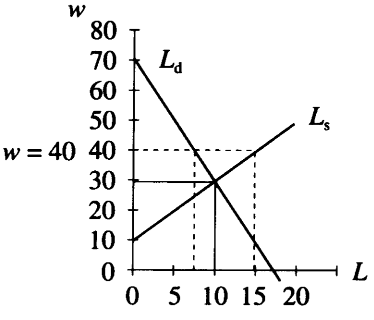


Figure PE 3.2 Q4(c)

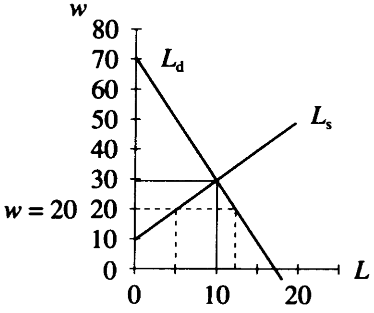


Figure PE 3.2 Q4(b)

3.3

1. (a) $P = 1400$, $Q = 11$

   (b) $P_d = 1620 - 20Q$

   $P_s = 960 + 40Q$

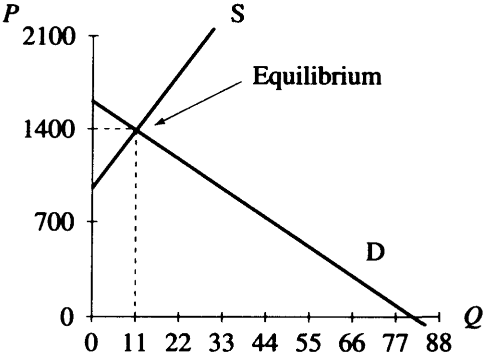


Figure PE 3.3 Q1

2. (a) $(P_s - 120) = 960 + 40Q$

   $P_s = 1080 + 40Q$

   (b) $Q = 9$, $P = 1440$

   (c) Consumer pays £40, company pays £80.

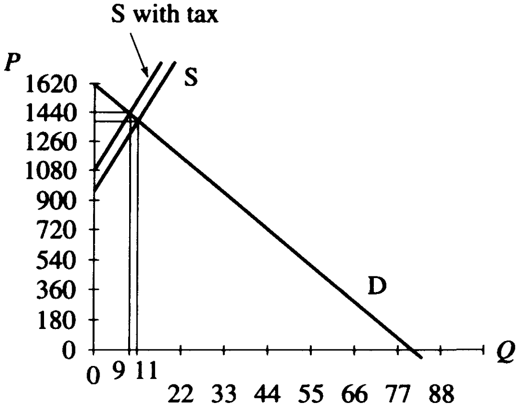


Figure PE 3.3 Q2(a)

3. Pitchers, X: $P_X = 27.5$, $Q_X = 45$. Putters, Y: $P_Y = 45$, $Q_Y = 5$

4. (a) $Q = 12$, $P = £140$.

   (b) (i) $(P_s - 9) = 92 + 4Q$

   $$P_s = 101 + 4Q$$

   (ii) $Q = 11$, $P = £145$

   (iii) Consumer pays £5, supplier pays £4.

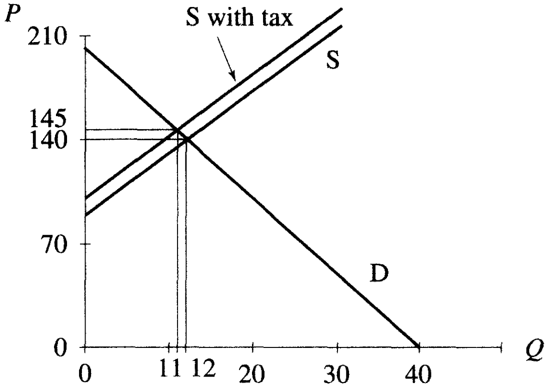


Figure PE 3.3 Q4

5. (a) $TR = 30Q$; $TC = 200 + 5Q$: therefore $TR = TC \rightarrow 30Q = 200 + 5Q \rightarrow Q = 8$.

   (b) $TR = 30(8) = £240$; $TC = 200 + 5(8) = £240$.

6. (a) $Q = 75$, $P = 50$.

   (b) (i) $P_s + 4 = 20 + 0.4Q \rightarrow P_s = 16 + 0.4Q$ (ii) $Q = 80$, $P = 48$

   (iii) Producer and consumer each receive £2.

7. (a) $TR = 6.6Q$; $Q = 125$ at break-even

   (b) $P(Q) = 800 + 0.2(Q)$

   $$P(160) = 800 + 0.2(160)$$

   $$P = 5.2$$

   (c)

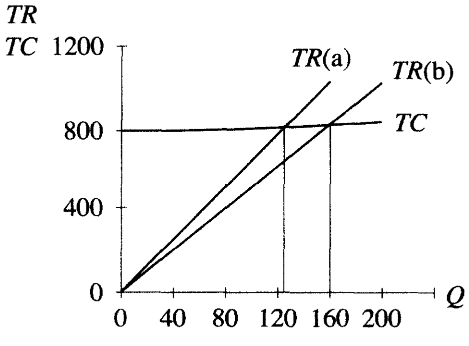


Figure PE 3.3 Q7

8. $P_X = 56$, $Q_X = 6$, $P_Y = 68$, $Q_Y = 14$.

3.4

1. (a) See text.

2. (a) $Q = 180$, $P = 22$

   (b) (i) Consumers pay: $180 \times 22 = 3960$ (ii) Willing to pay: $3960 + 3240 = 7200$
   (iii) $CS = 7200 - 3960 = 3240$

   (c) (i) Producers receive: $180 \times 22 = 3960$

   (ii) Willing to accept: $(4 \times 180) + \frac{1}{2}(18 \times 180) = 2340$

   (iii) $PS = 3960 - 2340 = 1620$

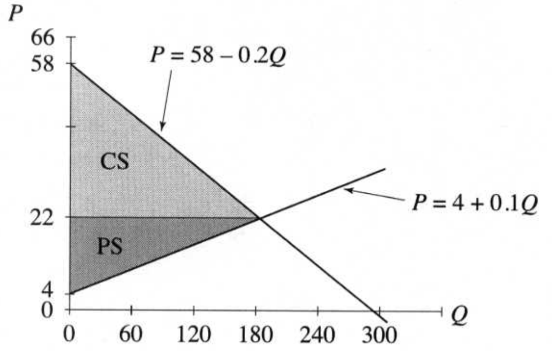


Figure PE 3.4 Q2(a)

3. $P = 500 - 10Q$

   $P = 100 + 10Q$

   (a) $P = 300$, $Q = 20$

   (b) $CS = 2000$

   (c) $PS = 2000$

   (d) $TS = 4000$

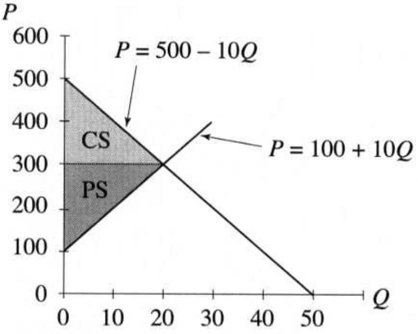


Figure PE 3.4 Q3(a)

4. (a) Price decrease: $P = 250$, $Q = 25$

   (b) $CS = 3125$. Without price decrease, $CS = 2000$,
   $\therefore$ increase in $CS = 1125$.

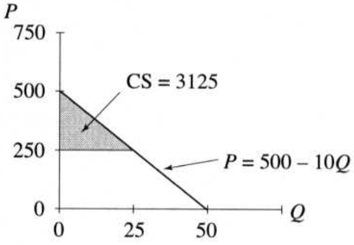


Figure PE 3.4 Q4(b)

5. (a) Price decrease: $P = 250$, $Q = 15$

   (b) $PS = 1125$. Without price decrease, $PS = 2000$, $\therefore$ decrease in $PS = 875$.

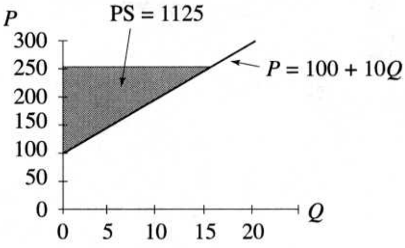


Figure PE 3.4 Q5(b)

3.5

1. See text.
2. (a) $E = 360 + 0.6Y$.

   Intercept = 360,
   slope = 0.6

   (b) Algebraically,

   When $E = Y$

   $360 + 0.6Y = Y$

   $\rightarrow Y_e = 900$

   (c) If MPC increases to 0.9,

   Y_e increases to

   $$\frac{360}{1-0.9} = 3600$$

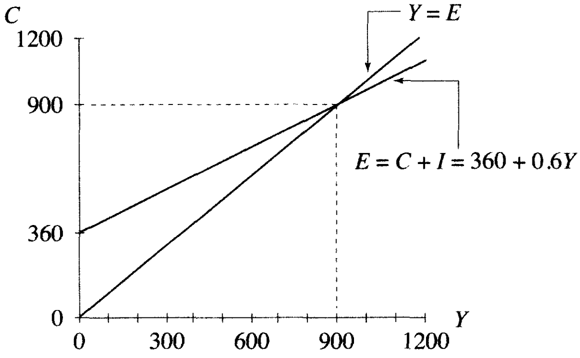


Figure PE 3.5 Q2(b)

3. (a) $E = C + I = 280 + 0.6Y_d + 80 = 360 + 0.48Y$

   Intercept = 360, slope = 0.48

   (b) Algebraically, when $E = Y \rightarrow 360 + 0.48Y = Y \rightarrow Y_e = 692$

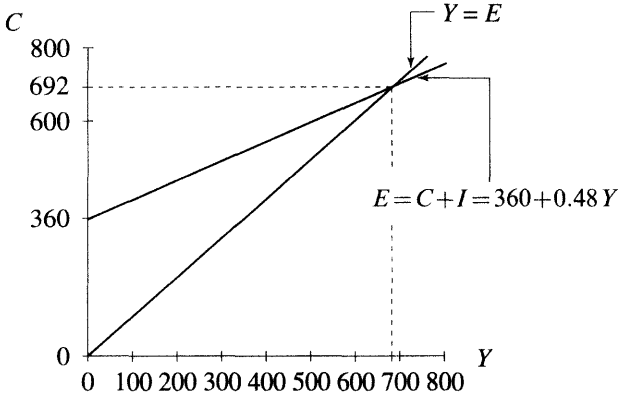


Figure PE 3.5 Q3(b)

(c) If MPC increases to 0.9, then

$$\begin{aligned} E &= C_0 + b(1-t)Y + I_0 \\ &= 280 + 0.9(1-0.2)Y + 80 \\ &= 360 + 0.72Y \end{aligned}$$

Hence $Y = E \rightarrow Y_e$ increases to $\dfrac{360}{1-0.72} = 1287.5$

4. $\Delta Y_e = \dfrac{1}{1-b}\Delta I_e = £1250\text{m}.$ $\quad Y_e + \Delta Y_e = 800 + 1250 = 2050 = \text{new } Y_e$

3.7

1. (a) $x = 6, y = -3$

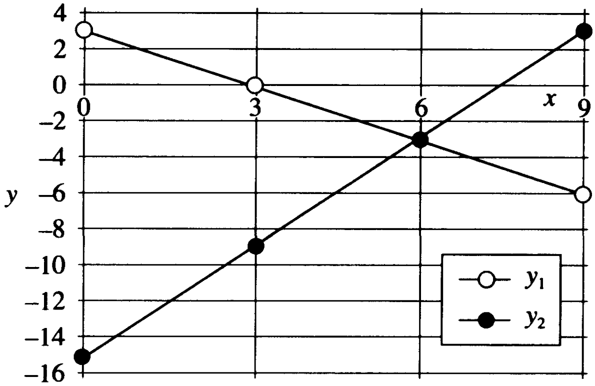


Figure PE 3.7 Q1(a)

(b) $Q = 1.91$: $P = 3.53$

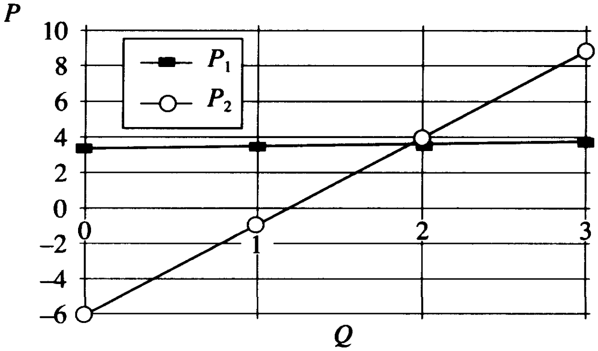


Figure PE 3.7 Q1(b)

2. $P = 200$, $Q = 10$

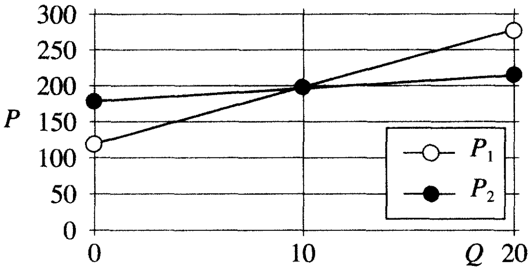


Figure PE 3.7 Q2

3. (a) $Q = 14$, $P = 61$

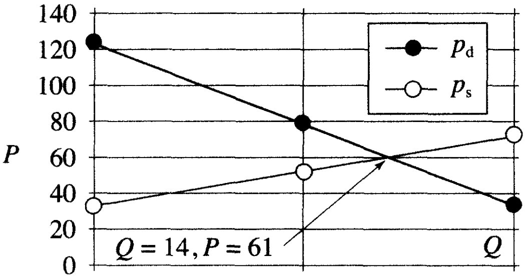


Figure PE 3.7 Q3(a)

(b) (i) $Q = 9.38$, $P = 81.79$, (ii) consumer pays 20.79, producer pays 9.21 (working correct to 2D)

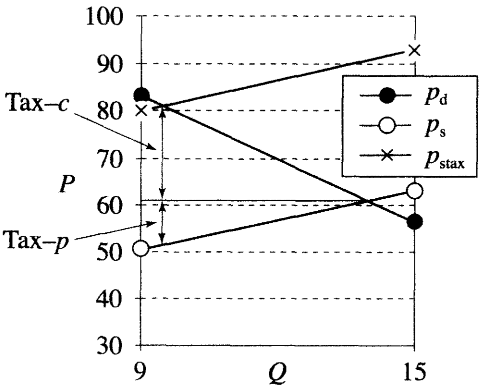


Figure PE 3.7 Q3(b)

4. (a) $Y_e = 414.29$, see Figure below.

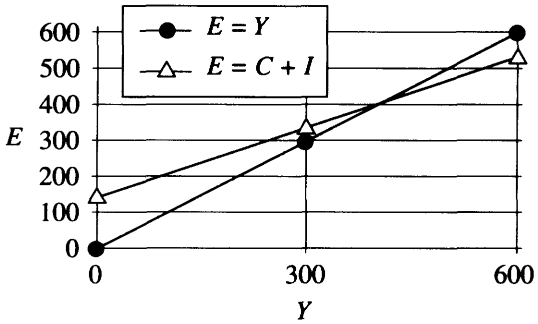


Figure PE 3.7 Q4(a)

(b) Assume $C = 125 + 0.65Y_d$: where $Y_d = Y - T$. Then $Y_e = 302.08$ (see figure below)

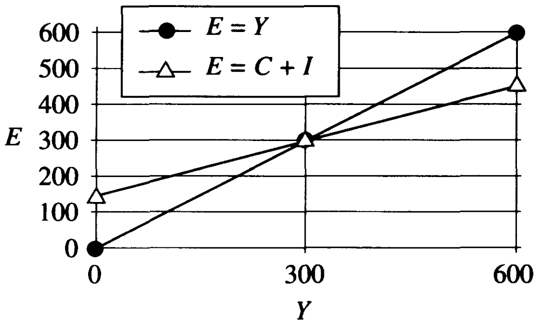


Figure PE 3.7 Q4(b)

Chapter 4

4.1

1. $x = 1, x = 5$

2. $Q = 1, Q = 2.5$

3. $Q = 1, Q = 5$

4. $Q = -1, Q = -5$

5. $P = \sqrt{7} = 2.6458, P = -\sqrt{7} = -2.6458$

6. $Q = 3$

7. $Q = -1.2426, Q = 7.2426$

8. $Q = 0, Q = 6$

9. $x = -0.7202, x = 9.7202$

10. $P = \pm\sqrt{-9}$, imaginary numbers, $3i$ and $-3i$

11. $P = 1$, $P = -0.8182$

12. $Q = 0.25$

13. $P = 0$, $P = 2$

14. $P = -1.5$, $P = 1.5$

15. $P = -1$, $P = 0.25$

4.2

1.

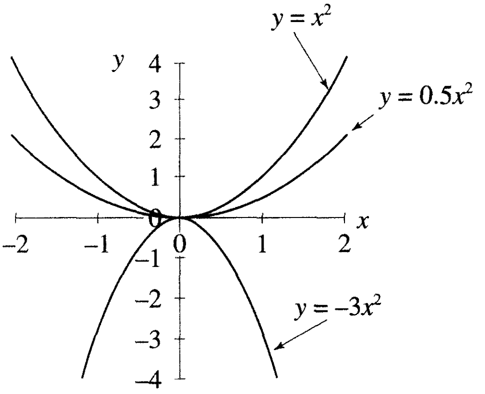


Figure PE 4.2 Q1

2. (a) is a horizontal translation to the left by two units.

(b) is a horizontal translation to the right by one unit.

(c) is a vertical translation upwards by one unit.

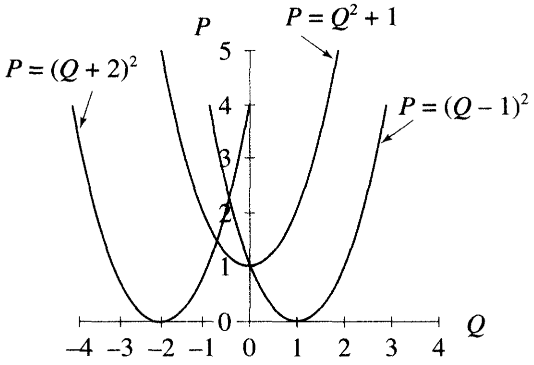


Figure PE 4.2 Q2

3. (i)

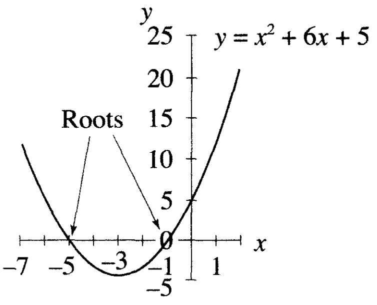


Figure PE 4.2 Q3(a)

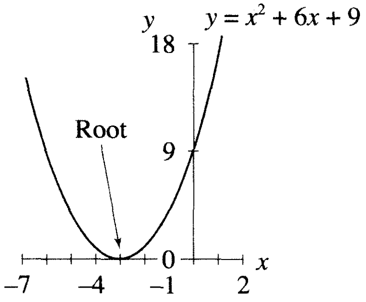


Figure PE 4.2 Q3(b)

(ii) Graph cuts x-axis at:

(a) $x^2 + 6x + 5 = 0$, at $x = -1$, $x = -5$. Turning point at $x = -3$

(b) $x^2 + 6x + 9 = 0$, at $x = -3$. Turning point at $x = -3$

(c) $x^2 + 6x + 10 = 0$, no real roots, therefore, no intersection.

(iii) Graphs are all vertical translations of each other.

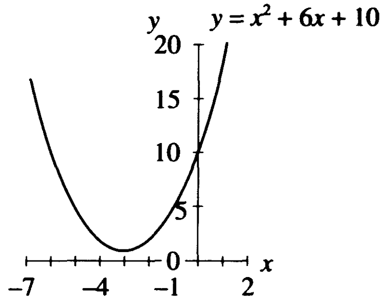


Figure PE 4.2 Q3(c)

4. (a)

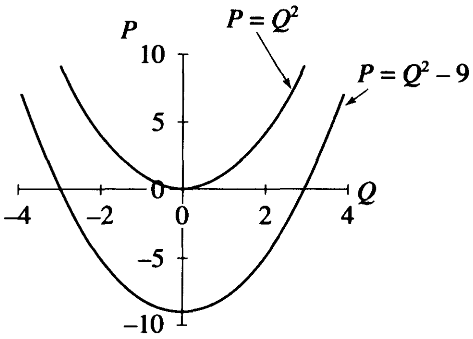


Figure PE 4.2 Q4

(b) $P = Q^2 - 9$. Point of intersection with x-axis: $Q = 3$, $Q = -3$; y-axis, $P = -9$

5. (a)

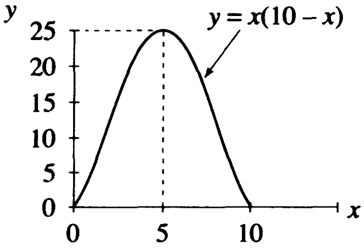


Figure PE 4.2 Q5

(b) Roots: $x = 0$, $x = 10$.
Maximum at $x = 5$.
Maximum value, $y = 25$

6. (i)

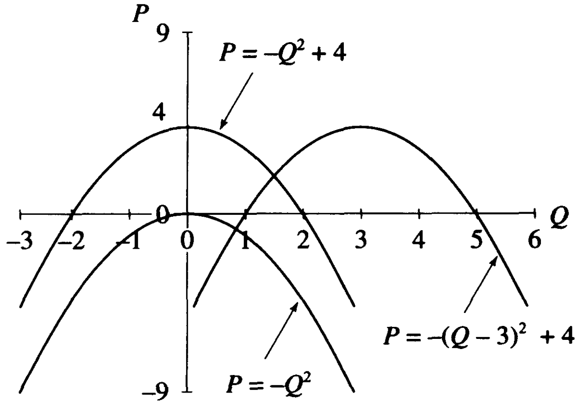


Figure PE 4.2 Q6

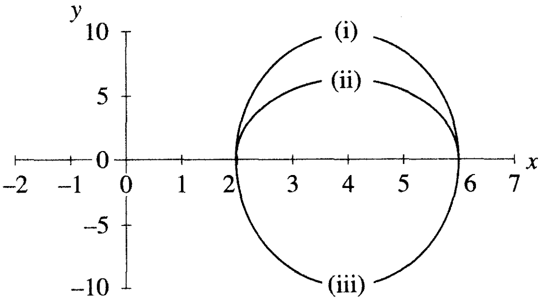
(ii)

(a) $P = -Q^2$

(b) $P = -Q^2 + 4$: roots at $Q = -2$, $Q = 2$.

(c) $P = -(Q - 3)^2 + 4$: roots at $Q = 1$, $Q = 5$.

7. (a)



Graphs (i) and (ii) are maximum type while graph (iii) is a minimum type. Each type may go to any height above or below the axis.

(b) $y = A(x - 2)(x - 6)$

Figure PE 4.2 Q7

4.3

1. (a)

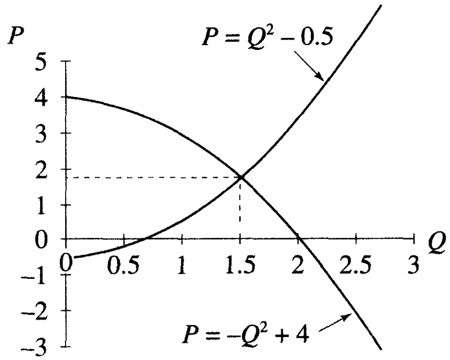


Figure PE 4.3 Q1

(b) Equilibrium price and quantity:

$$P_s = P_d$$

$$Q^2 - 0.5 = -Q^2 + 4$$

$$2Q^2 = 4.5$$

$$Q^2 = 2.25$$

$Q = \pm 1.5$. As $Q = -1.5$ is not economically meaningful, then $Q = 1.5$, $P = 1.75$.

2. (a) $TR = (12 - Q)Q = 12Q - Q^2$

(b) $TR = 0$ at $Q = 0$, $Q = 12$.

Maximum at $Q = 6$, $TR = 36$.

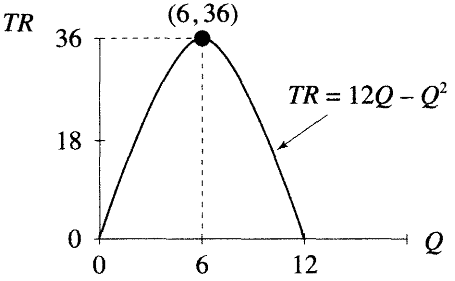


Figure PE 4.3 Q2

3.

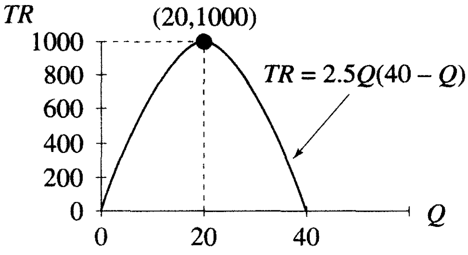


Figure PE 4.3 Q3

$TR = AQ(40 - Q)$
At $Q = 20$, $TR = 1000$, therefore,

$1000 = A(20)(40 - 20) \rightarrow A = 2.5$

Therefore, $TR = 2.5Q(40 - Q)$

4. (a) $TR = PQ = (100 - 2Q)Q = 100Q - 2Q^2$: At $Q = 10$, $TR = 800$

   (b) $TR = PQ = P(50 - 0.5P) = 50P - 0.5P^2$ At $P = 10$, $TR = 450$

5. (a)

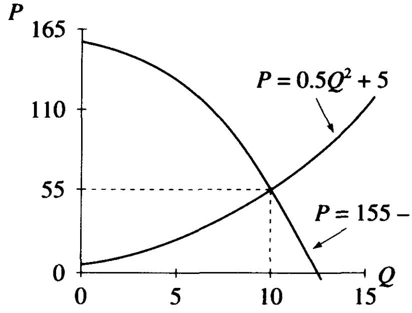


Figure PE 4.3 Q5

   (b)

$$\begin{aligned} P_s &= P_d \\ 0.5Q^2 + 5 &= 155 - Q^2 \\ 1.5Q^2 &= 150 \\ Q^2 &= 100 \end{aligned}$$

$Q = \pm 10$. As $Q = -10$ is not economically meaningful, then, $Q = 10$, $P = 55$.

6. (a) $TR = (107 - 2Q)Q$

   (b) $Q = 26.75$, $TR = 1431$

   (c) Break-even: $Q = 2$, $Q = 50$.

   (d) Between $Q = 2$ to $Q = 50$.

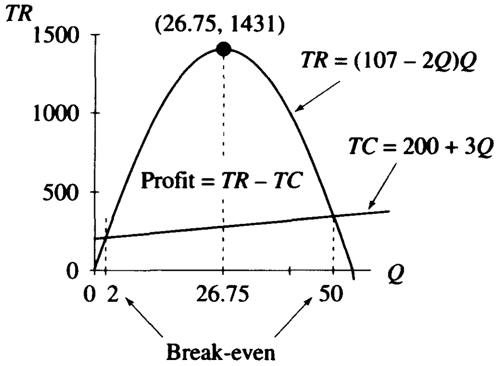


Figure PE 4.3 Q6

7. $\pi = TR - TC = -2Q^2 + 104Q - 200$

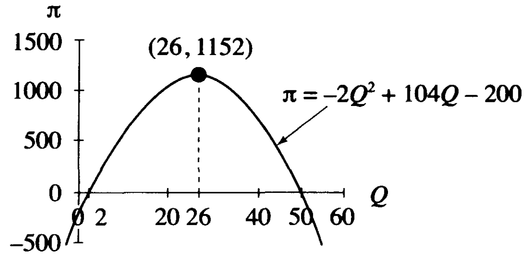


Figure PE 4.3 Q7

(a) $\pi = 0$ at $Q = 2$ and $Q = 50$

(b) Maximum profit at $Q = 26$, $\pi = 1152$

8. (a)

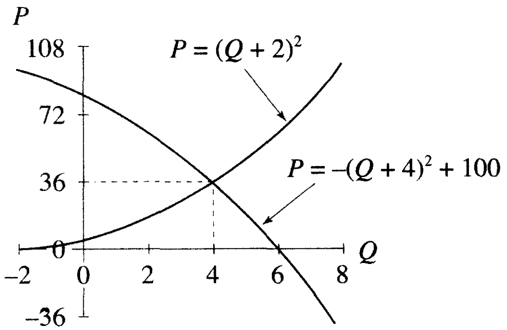


Figure PE 4.3 Q8

(b) Solve

$P_s = P_d \rightarrow Q^2 + 6Q - 40 = 0$:

therefore $Q = 4$, $Q = -10$

At $Q = 4$, $P = 36$

4.4

1.

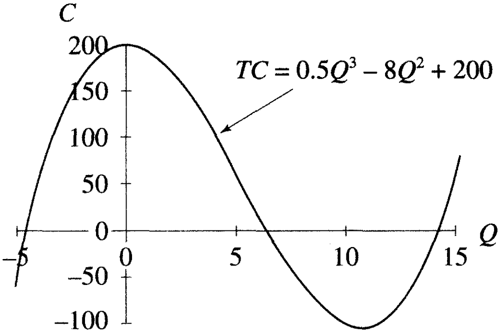


Figure PE 4.4 Q1

Roots at $Q = -4.4, 6.48, 13.94$
max. at $Q = 0$, $C = 200$
min. at $Q = 10.67$, $C = -103.4$

2.

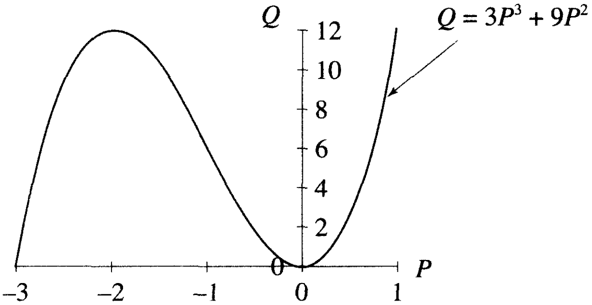
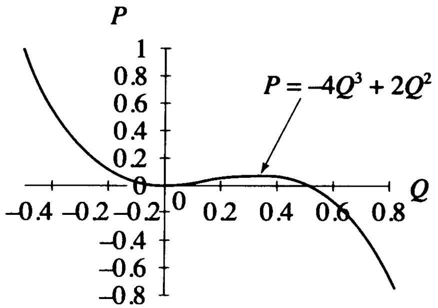


Figure PE 4.4 Q2

Roots at $Q = -3$, $Q = 0$
max. at $P = -2$, $Q = 12$
min. at $P = 0$, $Q = 0$

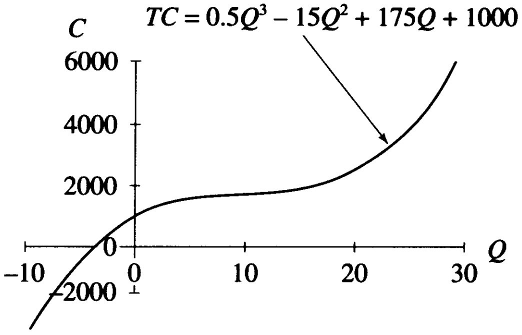
3.



Roots at $Q = 0$, $Q = 0.5$
max. at $Q = \frac{1}{3}$, $P = 0.07$
min. at $Q = 0$, $P = 0$

Figure PE 4.4 Q3

4.



Roots at $Q = -4.09$
no maximum or
minimum point

Figure PE 4.4 Q4

4.5

1.	(a) 36	(b) 27	(c) 5	(d) 125	(e) 9	(f) 16
	(g) 1	(h) 0.2	(i) 0.0278	(j) 0.008	(k) 1.5811	(l) 0.1317

2. (a) $\frac{1}{2}$ (b) $(4)(3^{4.1})$

3. (a) $\frac{3^5}{2^3 5^7}$ (b) 2^y

4. (a) a^y (b) $5^x 2^{y-x}$

5. (a) $L^{\alpha-1}K^{\beta}$ (b) $16L^8$

6. (a) $4L^4K^3$ (b) $2L^2$ (c) $\frac{5^{0.5}L^{1.5}}{K^{0.5}}$

7. $\frac{0.6K^2}{0.4L^2} = \frac{3}{2}\left(\frac{K}{L}\right)^2$

8. $\dfrac{L^2}{K^2}$

9. (a) $2Q^{1.5}$ (b) $\dfrac{2Q}{3P^{1.5}}$

10. (a) $\dfrac{K^{0.75}}{2L^{0.75}} = 0.5\left(\dfrac{K}{L}\right)^{0.75}$ (b) $\dfrac{1}{4L^4}$

11. e^{x+2}

12. $5e^{6x}$

13. e^{6-3x}

14. $\dfrac{120}{e^{1.5t}}$

15. $e^{0.75t} - e^{1.75t}$

16. $\dfrac{e^{4t}}{25}$

17. $e^{1.25t}$

18. $1 + 2e^t + e^{2t}$

19. $5e^{2t} + 5e^{3t}$

20. $\dfrac{1}{e^{3t}} + \dfrac{1}{e^{2t}}$

4.6

1. $x = -2$

2. $Q = 8$

3. $x = -12$

4. $x = 3$

5. $Q = 0$

6. $x = 3$

7. $K = 0.125$

8. $K = 0.25$

9. $x = 2$

10. $t = 32$

11. $t = 0$

12. $t = 71.5$

13. $t = 3, t = -5$

14. $t = -\dfrac{4}{3}$

15. $t = \dfrac{2}{3}$

16. $x = 0$

17. $t = 7$

18. $t = 1$

19. $K = 14$

20. $L = -2$

21. $x = 7.5$

22. $x = \dfrac{5}{6}$

23. $t = -0.5$

24. $t = -3$

4.7

1.

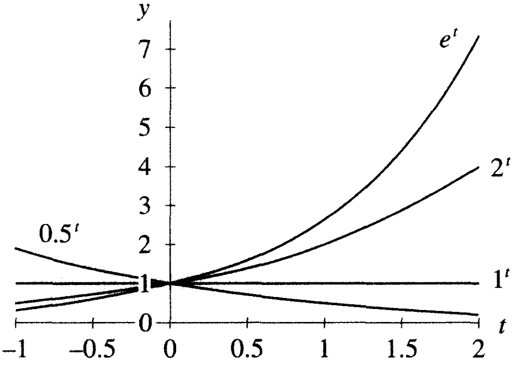


Figure PE 4.7 Q1

Note

$(0.5)^t = \left(\dfrac{1}{2}\right)^t = (2^{-1})^t = 2^{-t}$

For $a > 1$, $x > 0$, unlimited growth, increasing more rapidly for larger a. For $a = 1$, curve is constant. For $a > 1$, decay curve tends to a limit $y = 0$, i.e. the x-axis

2.

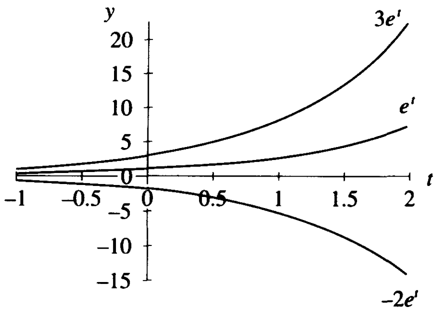


Figure PE 4.7 Q2

For equations of the form $y = ke^t$, unlimited growth when the constant, k, multiplying the exponential is positive; unlimited decay when k is negative. Multiplying the RHS of $y = f(t)$ by a negative number flips the graph through the x-axis

3.

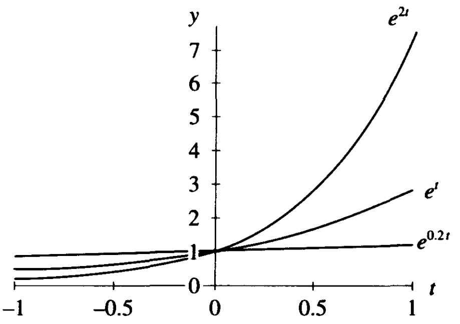


Figure PE 4.7 Q3

Equations of the form $y = e^{\alpha t}$, $\alpha > 0$, increase more rapidly, without limit for larger α

4.

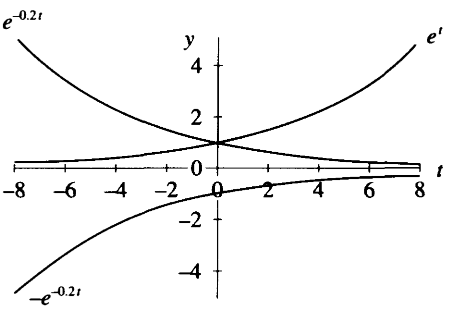


Figure PE 4.7 Q4

The negative exponentials $e^{-0.2t}$ and $-e^{-0.2t}$ are reflections of each other in the x-axis, one growing, the other decaying to the horizontal axis; e^t grows without limit

5.

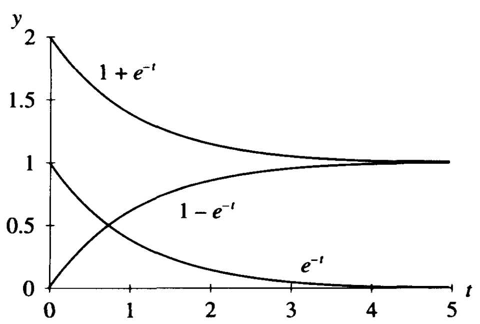


Figure PE 4.7 Q5

All graphs display limited growth or decay. $1 + e^{-t}$ decays to a limit of 1 while $1 - e^{-t}$ grows to a limit of 1; e^{-t} decays to zero as t increases

6.

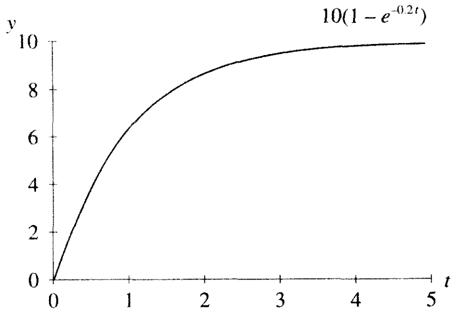


Figure PE 4.7 Q6

Growth curve, tending to a limit of 10

4.8

1. (a) 125 500

(b)

t	10	30	60	90	100
S	142	180	258	370	417

(c)

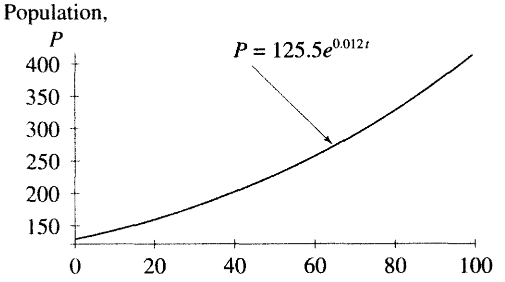


Figure PE 4.8 Q1

2. (a) 9 754

(b)

t	5	20	35	45	50	52
S	44	126	165	179	184	185

(c)

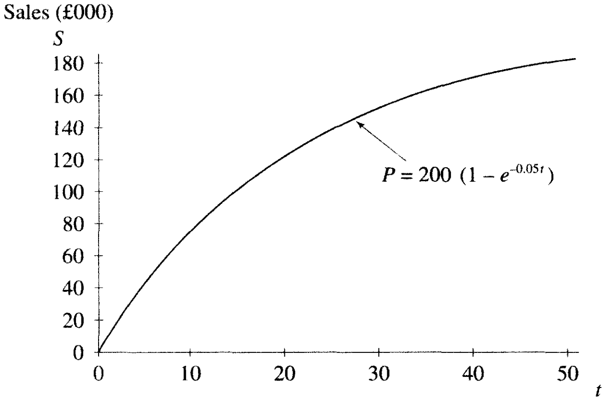


Figure PE 4.8 Q2

3. (a) 1 (b)

t	0	20	40	60	80	100
N	1	7	51	268	631	772

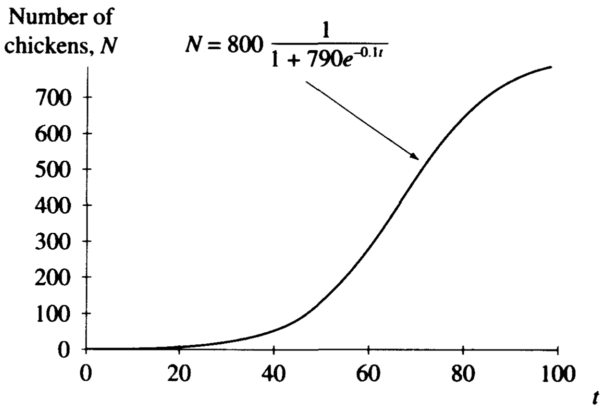


Figure PE 4.8 Q3

(c) As t increases $e^{-0.1t}$ in the denominator will approach zero, therefore $N = 800$ infected chickens.

4.9

1. (a) 1.3617 (b) 1.7633 (c) −0.6576 (d) 4.0601 (e) 1.3315
(f) −0.5500 (g) 1.3315 (h) 3.0659 (i) 1.2060

2. (a) $\log_a(N) = \text{power}$ (b) $\log_3 10 = 2.0959$ (c) $\log_{10} 3 = 0.4771$
(d) $\log_e 7.389 = 2$ (e) $\log_2 10 = x$ (f) $\log_3 8 = x$
(g) $\log_2 4 = x$ (h) $\log_2 15 = 3x$ (i) $\log_5 4 = Q$
(j) $\log_5 10 = Q + 1$ (k) $\log_{10} 25 = 4x + 3$ (l) $\log_P 12 = 0.3$
(m) $\log_K 5.6 = -0.5$ (n) $\log_L(0.0493) = 0.9$ (o) $\log_{(K+10)} 2 = 0.8$

4.10

1. $x = 1.4472$ **2.** $x = 0.2218$ **3.** $x = 4.6990$

4. $t = 4.8283$ **5.** $t = -0.6098$ **6.** $t = 0.6098$

7. $t = -4.8847$ **8.** $t = 0.9316$ **9.** $x = 0.6990$

10. (a) (i) $t = 0 \rightarrow S = 0$ (ii) $t = 1 \rightarrow S = 73.93$ (iii) $t = 5 \rightarrow S = 275.34$
(b) Similar to Q2, Progress Exercises 4.8

When $S = 400\,000$, $t = 10.1$ weeks

Maximum sales (never reached) = £500 000

11. (a) $t = 0 \rightarrow P = 200$; $t = 4 \rightarrow P = 875.27$; $t = 10 \rightarrow P = 3918.61$

(b) Similar to Q3, Progress Exercises 4.8

(c) $P = 1000$ in 4.39 years; $P = 3000$ in 8.42 years; $P = 4000$ in 10.15 years

12. (a) $t = 0 \rightarrow P = 5.2$m (year 1998); $t = 10 \rightarrow P = 5.15$m (year 2008); $t = 50 \rightarrow P = 4.95$m (year 2048).

(b) 262.36 years, that is, in AD 2260.36

4.11

1. $(x + 5) = 10^{1.2}$ **2.** $(x + 2) = 10^{3.5/3}$ **3.** $(t - 5) = e^{0.5}$

4. $(x + 1) = 10^{-0.5(2.5)}$ **5.** $2 - 2x = 10^0$ **6.** $x^2 + 1 = 10^{0.3}$

7. $10^{-0.65} = 0.5x$ **8.** $x(x + 1) = 10^1$ **9.** $x - 4.5 = e^2$

10. $\dfrac{x + 1}{x} = e^{3.5}$

11. (1) 10.8489 (2) 12.6780 (3) 6.6487 (4) −0.9438 (5) 0.5

(6) ±0.9976 (7) 0.4477 (8) 2.7016, 3.7016 (9) 11.8891 (10) 0.0311

12. (a) $x = -0.368$ (b) $x = -0.8755$ (c) $x = -17.9435$ (d) $x = -4.4859$

(e) $t = 0.8473$ (f) $x = 9.8833$ (g) $t = 0.6931$ (h) $x = -5.6711$

13. $\log(12)^2(20) = 3.4594$

14. $\ln\dfrac{(125)(28)}{(80)} = 3.7785$ **15.** $\ln\dfrac{(138)^3}{(95)^2} = 5.6740$

16. $\ln\dfrac{(2)^8(8)^3}{(3)^2} = 9.5863$ **17.** $\ln\dfrac{(4)(80)}{(2)^2} = 4.3820$

24. (a) $\dfrac{\log(27)}{\log(3)} = 3$ (b) $\dfrac{\log(51)}{\log(5)} = 2.4430$ (b) $2\left(\dfrac{\log(30)}{\log(3.5)}\right) = 5.4299$

25. (a) $x = 25.2982$ (b) $Q = 5.3096$

4.12

1.

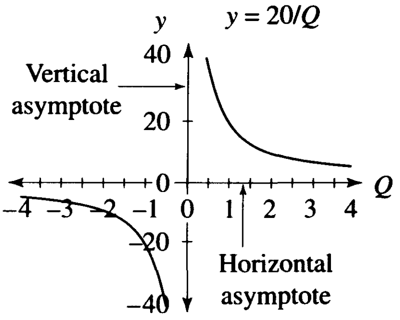


Figure PE 4.12 Q1

2.

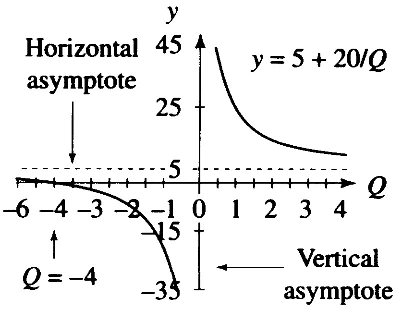


Figure PE 4.12 Q2

3.

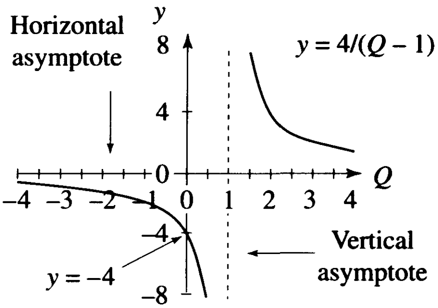


Figure PE 4.12 Q3

4.

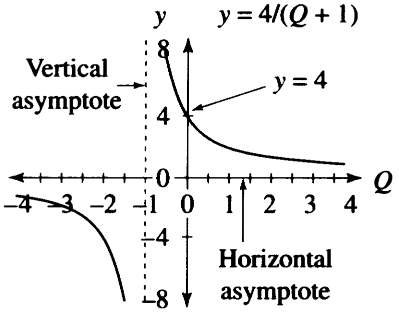


Figure PE 4.12 Q4

5.

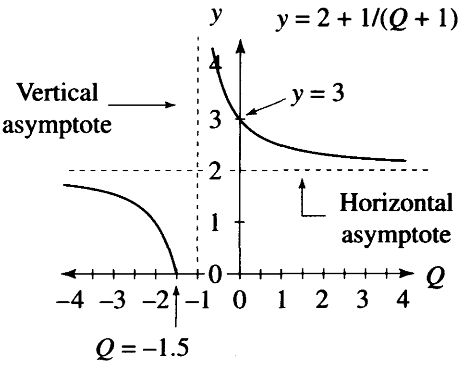


Figure PE 4.12 Q5

6.

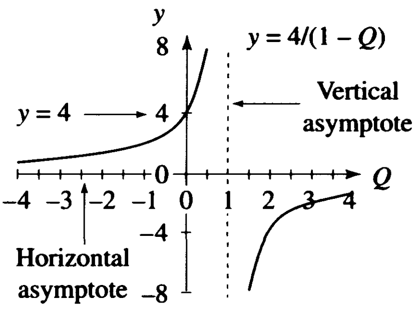


Figure PE 4.12 Q6

7.

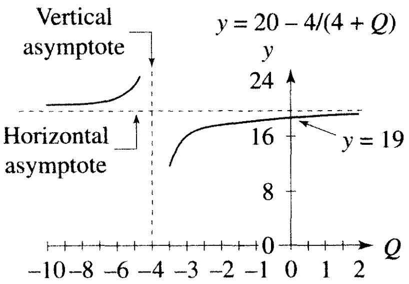


Figure PE 4.12 Q7

8.

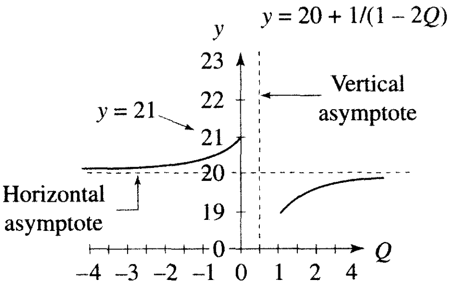


Figure PE 4.12 Q8

9.

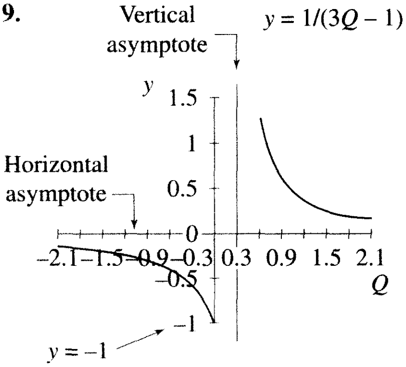


Figure PE 4.12 Q9

4.13

1. $x = -0.5$ 2. $x = -6$ 3. $x = 1, x = 4$ 4. $Q = 4$

5. $Q = 6.5311, Q = -1.5311$ 6. $Q = 0.375$

7. (a)

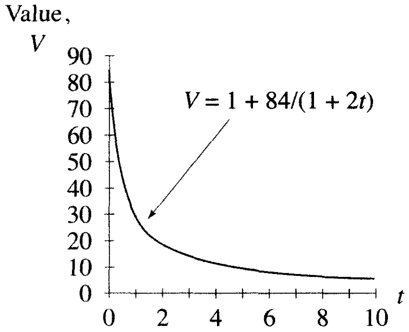


Figure PE 4.13 Q7

(b) 1.71 years

8. (a) $Q = 11.694$, $P = 39.388$

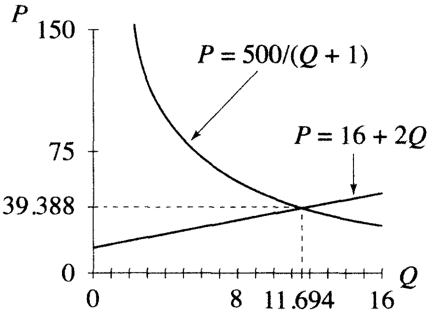


Figure PE 4.13 Q8(b)

9. (a)

t	0	5	10
S_{old}	3000	1125	692
S_{new}	1714	2250	3273

(b) Sales are equal at 1.8 months

$S_{\text{old}} = S_{\text{new}} = 1875$

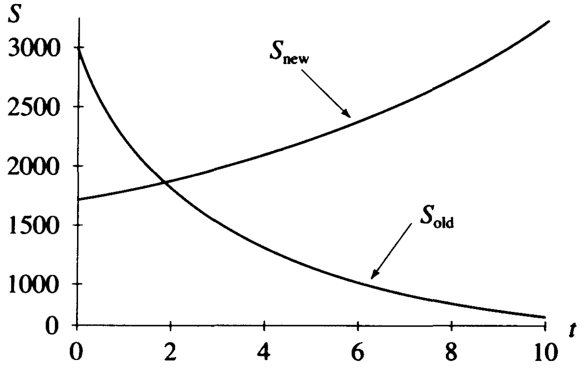


Figure PE 4.13 Q9

10. (a)

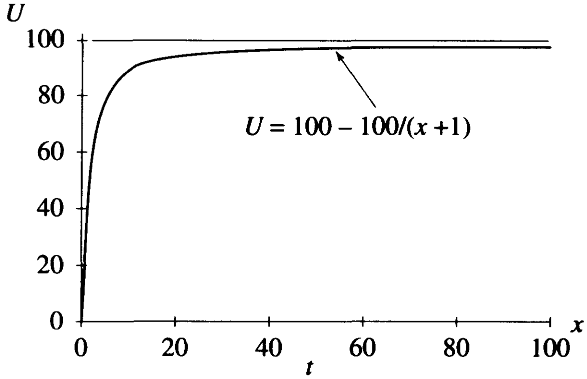


Figure PE 4.13 Q10

(b)

U	10	50	80	100
x	0.11	1	4	∞

Chapter 5

5.1

1. 96 **2.** 25 **3.** 0.5 **4.** $ar^{n-1} = 1.03$

5. $a = 1, d = 1, n = 20$ **6.** $a = 2, d = 2, n = 20$ **7.** $a = 90, d = -3, n = 5$ or $n = 56$

8. $a = 2, d = 3$: series $2 + 5 + 8 + 11$ **9.** Proof

10. $a = 28\,672,\ r = \frac{1}{8},\ S_\infty = 32\,768$

11. (a) $a + a(1.1) + a(1.1)^2 + \ldots;\ r = \frac{110}{100} = 1.1$

(b) $S_8 = 11.436a - 8a = 3.436a$

(c) $a = 800$, $S_8 = 9148.8$, number of tins sold. Profit = £823.392. Cost of campaign for 8 weeks is £1600. Therefore, advertising campaign not worth while.

5.2

1. (a) 594 (b) 6474 (c) $n = 11.47$, the twelfth month.

2. (a) (i) $n = 11$, $T_n = a + (n - 1)d = 38\,000$. **Note:** $d = -15\,200$ (ii) 1 064 000

(b) 15 833 per week

3. (a) 900 (b) 8250

4. (a)

Week	1	5	10	15
Output A	1000	1800	2800	3800
Output B	500	1037	2580	6420

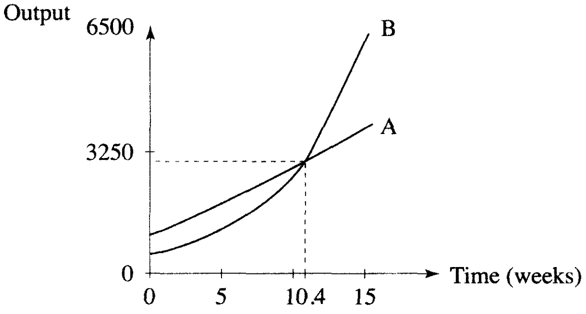


Figure PE 5.2 Q4

Output for both firms is approximately the same in week 11.

(b) Company A: total production during first 15 weeks: 36 000
Company B: total production during first 15 weeks: 36 018

5. $n = 3$ in 1999; $n = 8$ in 2004.

(a) Forecast 1: 60 000; 85 000 Forecast 2: 56 180; 75 182

(b) Forecast 1: 540 000 Forecast 2: 494 873

5.3

1. (a) £6381.41 (b) £7012.76 (c) £8052.55

2. 13400.96

3. (a) 2625 (b) 3113.81 (c) 4072.24 (d) 6633.24

4. £4898 **5.** (a) £5425 (b) £5897.7 **6.** £5572.7

7. £8024.51 **8.** (a) 5.95% (b) 9.05% (c) 14.87% **9.** 4.78 years

10. (a) 4.78 years (b) 9.43 years **11.** (a) 11.90 years (b) 23.45 years

12. $r = 15.83\%$ **13.** $t = 25.89$ years

14.

	$t = 0$	$t = 5$	$t = 10$	$t = 15$	$t = 20$
Simple interest	1	1.5	2.0	2.5	3
Compound interest	1	1.61	2.59	4.18	6.73

15. 7.699%

5.4

1. (a) (i) 52 750, 55 500, 58 250, 61 000, 63 750 (ii) 18.18 years

(b) (i) 52 750, 55 651.25, 58 712.07, 61 941.23, 65 348 (ii) 12.946 years

(c) (i) 52 807.24, 55 772.0929, 58 903.406, 62 210.527, 65 703.325

(ii) 12.689 years

(d) (i) 52 827.03, 55 813.904, 58 969.656, 62 303.837, 65 826.534 (ii) 12.603 years

2. (a) £6420.13 (b) £7095.34 (c) £8243.61

3. (a) 6.136% (b) 6.168% (c) 6.184%

4. (a) £4072.24 (b) £4109.05 (c) £4120.81 (d) £4120.81

5. 6.174% **6.** 8.243% **7.** 8.33% **8.** 3.045%

9. 9.667% **10.** 12.055 years

11. (a) 6567.29 (b) 6581.74 (c) 6584.60 (d) 6584.70

12. Bank A: $APR = 8.99\%$; Bank B: $APR = 9.04\%$; Bank B charges most interest

5.5

1. £65 158.3

2. (a) See text

(b) Project A: (i) $NPV = £2104$ (ii) $IRR \cong 11.6\%$

Project B: (i) $NPV = £483$ (ii) $IRR \cong 8.3\%$

3. Project A: @6%, $NPV = £22\,331.68$; @8%, $NPV = -£5964.2$; $IRR = 7.56\%$

Project B: @6%, $NPV = £6457.72$; @8%, $NPV = -£1549.35$; $IRR = 7.6\%$

At 6% project A is more profitable, at 8% project B: according to the *IRR* project B is more profitable.

5.6

1. (a) 2444.49 (b) 409.08

2. (a) 10 172.09 (b) 24 807.07

3. 26 398.97

4. (a) 16 274.54 (b) 62 745.39 (c) interest repaid 10 000 (year 1) 9372.55 (year 2)

5. (a) 3021.07 (b) 10 421.48 (c) 937.5, 898.43, 858.63, 818.09 paid in quarters 1, 2, 3, 4

6. (a) 156 000 (b) 98 677.17 (c) 57 322.8255

7. 11 716.44

8. (a) 69 207.64 (b) 182.89

9. (a) 10 533.25 (b) 109.95 (c) 334.90

10. (a) 165.04 (b) 717.16

11. (a) buy since leasing exceeds the purchase price (b) lease

5.7

1. (a) Annual payment = $r \times$ price of bond = $0.2 \times 5000 = 1000$.

(b) and (c)

End of year	**Cash flow**	**0.05**	**0.10**	**0.15**	**0.20**	**0.25**	**0.30**
1	1000.00	952.38	909.09	869.57	833.33	800.00	769.23
2	1000.00	907.03	826.45	756.14	694.44	640.00	591.72
3	1000.00	863.84	751.31	657.52	578.70	512.00	455.17
4	1000.00	822.70	683.01	571.75	482.25	409.60	350.13
5	1000.00	783.53	620.92	497.18	401.88	327.68	269.33
	NPV	4329.48	3790.79	3352.16	2990.61	2689.28	2435.57
NPV of redeemed bond		3917.63	3104.61	2485.88	2009.39	1638.40	1346.65
Total *NPV*		8247.11	6895.39	5838.04	5000.00	4327.68	3782.22

2. (a) Annual payment = $r \times$ price of bond = $0.09 \times 10\,000 = 900$.

(b)

End of	**Cash**		**Discount rates**				
year	**flow**	**0.065**	**0.09**	**0.12**	**0.065**	**0.09**	**0.12**
1	900.00	845.07	825.69	803.57			
2	900.00	793.49	757.51	717.47			
3	900.00	745.06	694.97	640.60			
4	900.00	699.59	637.58	571.97		(c)	
5	900.00	656.89	584.94	510.68			
6	900.00	616.80	536.64	455.97	845.07	825.69	803.57
7	900.00	579.16	492.33	407.11	793.49	757.51	717.47
8	900.00	543.81	451.68	363.49	745.06	694.97	640.60
9	900.00	510.62	414.39	324.55	699.59	637.58	571.97
10	900.00	479.45	380.17	289.78	656.89	584.94	510.68
	NPV	6469.95	5775.89	5085.20	3740.11	3500.69	3244.30
NPV of redeemed bond		5327.26	4224.11	3219.73	7298.81	6499.31	5674.27
Total *NPV*		11 797.21	10 000.00	8304.93	11 038.92	10 000.00	8918.57

3. (a) £400 (b) £6400 − £5000 = £1400 (c) £4000 − £5000 = −£1000

4.

	Bond		**Bank**					
	Rate = 0.2		**(a) Rate = 0.2**		**(b) Rate = 0.05**		**(c) Rate = 0.3**	
End of year	**Cash flow**	***PV* of cash flow**	**Balance −1000**	***PV* of cash flow**	**Balance −1000**	***PV* of cash flow**	**Balance −1000**	***PV* of cash flow**
1	1000.00	833.33	5000	833.33	4250.00	952.38	5500.00	769.23
2	1000.00	694.44	5000	694.44	3462.50	907.03	6150.00	591.72
3	1000.00	578.70	5000	578.70	2635.63	863.84	6995.00	455.17
4	1000.00	482.25	5000	482.25	1767.41	822.70	8093.50	350.13
5	1000.00	401.88	5000	401.88	855.78	783.53	9521.55	269.33
NPV of cash flow		2990.61		2990.61		4329.48		2435.57
PV of redeemed bond		2009.39	*PV* of final balance	2009.39		670.52		2564.43
NPV cash flow+ of final balance		5000.00		5000.00		5000.00		5000.00

Value of redeemed bond is £5000. Final balance in bank is (a) 5000, (b) 855.78, (c) 9521.55.
When the interest rate is 30% the final balance is £9521.55 (greater than the value of the bond), but at 5% the final balance is only £855.78.
However, the *NPV* of the cash flow combined with the *PV* of the final balance is £5000 in all cases.

Chapter 6

6.1

1. (a) and (b) Draw tangents at $x = -1.5$; 0; 2. Measure slopes. For convenience, $\Delta x = 1$ in the sketch below.

(c) $y = x^2$, $\dfrac{dy}{dx} = 2x = \text{slope}$

at $x = -1.5$, slope $= -3$
at $x = 0$, slope $= 0$
at $x = 2$, slope $= 4$

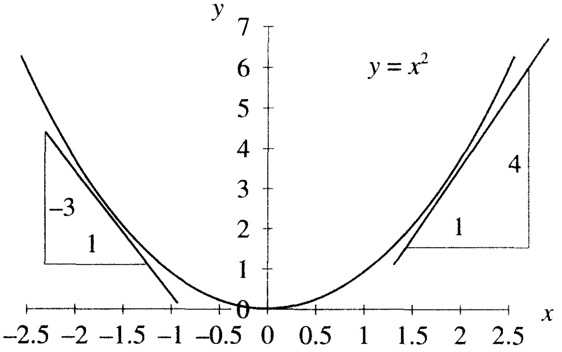


Figure PE 6.1 Q1

2. (a) $5x^4$ (b) $5.5Q^{4.5}$ (c) $80Q^7$ (d) 0 (e) $6Q$ (f) $\frac{3.04}{Q^{0.8}}$ (g) $-\frac{1.4}{Y^{0.3}}$

3. (a) $-\frac{2}{x^3}$ (b) $-\frac{10}{x^3}$ (c) $5-\frac{2}{x^3}$ (d) $-\frac{12}{x^3}$ (e) $\frac{\mathrm{d}P}{\mathrm{d}Q}=Q^2+70-30Q$

4. (a) $\frac{5}{2x^{1/2}}$ (b) $-\frac{1}{2x^{3/2}}$ (c) $-\frac{1}{2Q^{3/2}}$ (d) $-\frac{7}{2Q^{3/2}}$ (e) $\frac{15}{2}Q^{3/2}$ (f) $\frac{1}{9Q^{2/3}}$

(g) 1 (h) $-\frac{2}{Q^2}$

5. (a) Q^8 (b) $4Q^{-3/2}$ (c) $x^{-1}+x^{-2}$ (d) $5^{1/2}Q^{3/2}$ (e) $3Q+8$

6. (a) $8Q^7$ (b) $-\frac{6}{Q^{5/2}}$ (c) $-\frac{1}{x^2}-\frac{2}{x^3}$ (d) $5^{1/2}\left(\frac{3}{2}Q^{1/2}\right)$ (e) 3

7. (a) $5x^4$; $20x^3$ (b) $6Q$; 6 (c) $10+0.5Q^{-0.5}$; $-0.25Q^{-1.5}$

(d) $\frac{1}{3}Q^{-2/3}$; $-\frac{2}{9}Q^{-5/3}$ (e) $-\frac{1}{x^2}$; $\frac{2}{x^3}$ (f) $4Y^3+\frac{4}{Y^5}$; $12Y^2-\frac{20}{Y^6}$

8. $(TC)'=\frac{3Q^2}{5}-16Q+\frac{5}{2}$; $(TC)''=\frac{6Q}{5}-16$; $(TC)'''=\frac{6}{5}$

9. $Q=-10P+27$, $\frac{\mathrm{d}Q}{\mathrm{d}P}=-10$; $P=-\frac{1}{10}Q+2.7$, $\frac{\mathrm{d}P}{\mathrm{d}Q}=-\frac{1}{10}$

10. $y=\frac{1}{25}x^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{25}x$; $\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{25}{2x}$

11. $\frac{\mathrm{d}S}{\mathrm{d}i}=960i+200i^4-48$

12. $t=\frac{18}{6x^3}=\frac{3}{x^3}=3x^{-3}$: $\frac{\mathrm{d}t}{\mathrm{d}x}=-\frac{9}{x^4}$; $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{x^4}{9}$

6.3

1. $Q=120-3P\rightarrow P=40-\frac{1}{3}Q$

(a) $TR=40Q-\frac{1}{3}Q^2$;

$MR=40-\frac{2}{3}Q$;

$AR=40-\frac{1}{3}Q=P$

At $Q = 15$, $TR = 525$; total revenue from the sale of 15 units.

$MR = 30$; revenue accrued from sale of 1 extra unit when $Q = 15$.

$AR = 35$; price per unit for each of the first 15 units.

(b) $AR = 40 - \frac{1}{3}Q$; when $AR = 0$

$$0 = 40 - \frac{1}{3}Q \rightarrow Q = 120.$$

When $Q = 120$, $MR = -40$. Stop selling when $MR = 0$, that is, at $Q = 60$.

(c) Use Excel, if available.

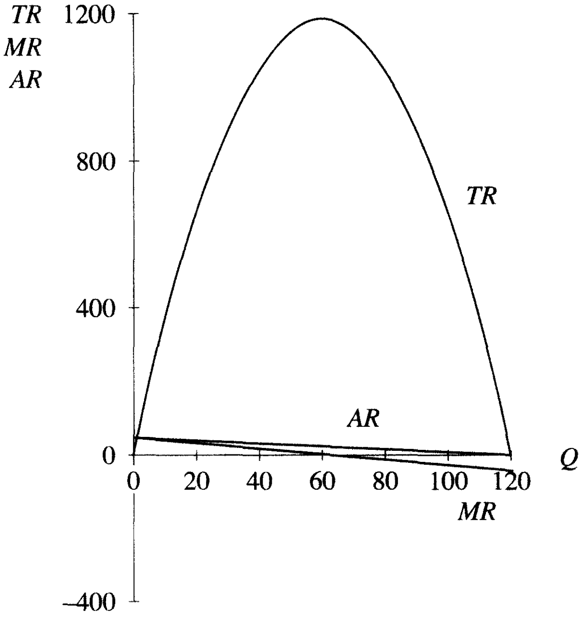


Figure PE 6.3 Q1

2. (a) $TR = 125Q - Q^{2.5}$; $MR = 125 - 2.5Q^{1.5}$; $AR = 125 - Q^{1.5}$

No. Slope $AR = -1.5Q^{0.5}$; slope of $MR = -3.75Q^{0.5}$

(b) At $Q = 10$, $TR = 933.77$; $MR = 45.94$; $AR = 93.377$

Comment: AR for first 10 units (per unit) $= 93.377$. $MR = 45.94$, the sale of one extra unit when $Q = 10$ should add £45.94 to TR.

At $Q = 25$, $TR = 0$; $MR = -187.5$; $AR = 0$

Comment: $AR > MR$. When $P = 0$, $TR = 0$ as expected. $MR = -187.5$, the sale of one extra unit causes a decrease of £187.5 in TR.

(c) $MR = 125 - 2.5Q^{1.5}$; when $MR = 0 \rightarrow 0 = 125 - 2.5Q^{1.5} \rightarrow Q = 13.57$

$AR = 125 - Q^{1.5}$; when $AR = 0 \rightarrow 0 = 125 - Q^{1.5} \rightarrow Q = 25$

Stop selling when $Q = 13.57$. When $Q > 13.57 \rightarrow MR < 0$.

(d) Use Excel, if available.

3. (a) $TC = FC + VC \rightarrow TC = 1000 + 3Q$. When $Q = 20$, $TC = 1060$.

(b) $MC = \frac{\mathrm{d}(TC)}{\mathrm{d}Q} = 3$. When $Q = 20$, $MC = 3$.

MC is the slope of the linear TC function. It is the amount by which TC increases for each extra unit produced. Slope is constant.

4. (a) If AC decreases, then $\frac{\mathrm{d}(AC)}{\mathrm{d}Q} < 0$. $\frac{\mathrm{d}(AC)}{\mathrm{d}Q} = -\frac{10}{Q^2} < 0$ for $Q > 0$

The average cost decreases as the number of units produced increases (economies of scale).

(b) Since $AC = \frac{TC}{Q} \rightarrow TC = Q(AC)$.

$TC = 50Q + 10$. $TVC = 50Q$; $AVC = 50$; $FC = 10$.

(c) $MC = 50$. MC is the amount by which TC increases for each extra unit produced. This is the same, whether it is for the first unit or the 900th!

5. (a) $TC = Q^3 - 9Q^2 + 150 + 75Q$. When $Q = 15$, $TC = 2625$

(b) $TVC = Q^3 - 9Q^2 + 75Q$ and $FC = 150$.

(c) $MC = 3Q^2 - 18Q + 75$. $MC = 480$ when $Q = 15$.

MC is a quadratic curve. If MC does not cross the Q-axis, then the curve lies entirely above the Q-axis. Solve

$$MC = 0, \quad 3Q^2 - 18Q + 75 = 0 \rightarrow Q = \frac{18 \pm \sqrt{-576}}{6}$$

These are imaginary values (confirm these findings by plotting MC using Excel, if available).

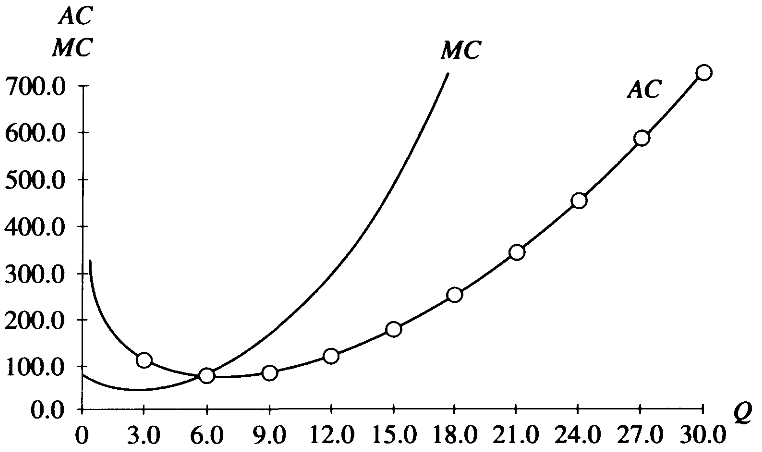


Figure PE 6.3 Q5(c)

6. (a) (i) $TR = Q(AR) = 180Q - 12Q^2$ (ii) $TR = 12Q$

(b) (i) $MR = 180 - 24Q$ (ii) $MR = 12$

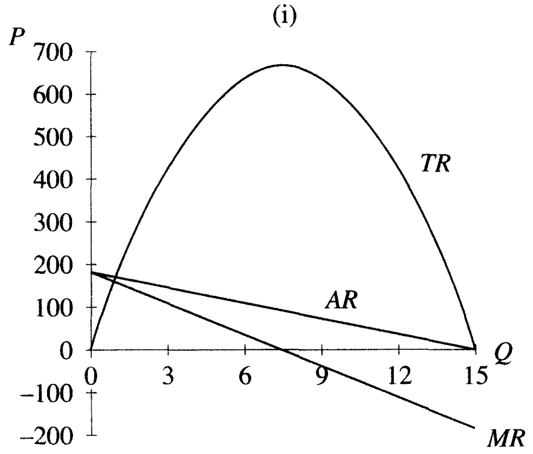


Figure PE 6.3 Q6d(i)

(c) (i) Slope of *MR* is twice slope of *AR* (monopolist)

(ii) Slope of *MR* is equal to slope of *AR* (perfectly competitive firm).

(d) Use Excel, if available.

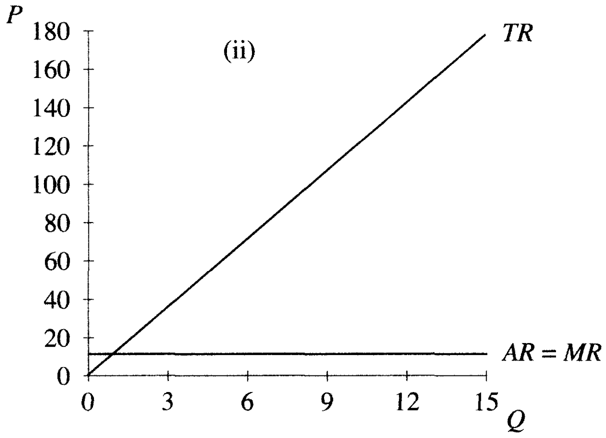


Figure PE 6.3 Q6d(ii)

6.4

1. (a) (i) $TLC = wL = 180L$; $ALC = \dfrac{TLC}{L} = 180$; $MLC = \dfrac{\mathrm{d}(TLC)}{\mathrm{d}L} = 180$

(ii) $TLC = 200L + 5L^2$; $ALC = 200 + 5L$; $MLC = 200 + 10L$

(b) (i) $ALC = MLC$ (ii) $MLC > ALC$ for all values of $L > 0$.

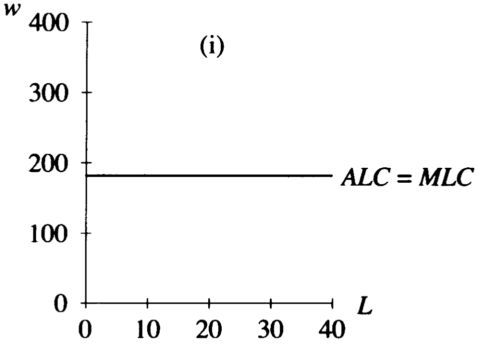


Figure PE 6.4 Q1(i)

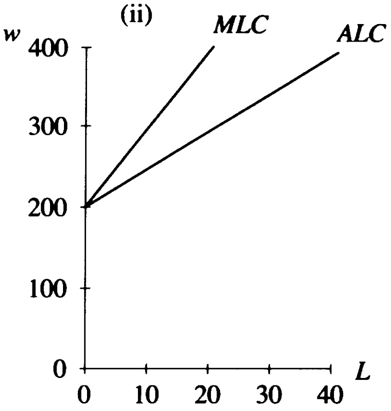


Figure PE 6.4 Q1(ii)

2. (a) (i) $Q = 20L - L^2, \quad MP_L = \dfrac{dQ}{dL} = 20 - 2L.$

This is a linear function which decreases (negative slope) as L increases.

(ii) $Q = 225L - \dfrac{1}{3}L^3, \quad MP_L = \dfrac{dQ}{dL} = 225 - L^2$

This is a quadratic function which decreases as L increases.

(b) (i) $APL = \dfrac{Q}{L} = \dfrac{20L - L^2}{L} = 20 - L$. Also a linear function with negative slope.

(ii) $APL = \dfrac{225L - \frac{1}{3}L^3}{L} = 225 - \dfrac{1}{3}L^2$

(c) (i) Slope of MP_L is double the slope of the APL.
(ii) $MP_L < APL$

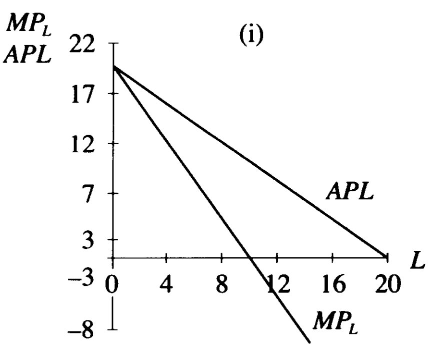


Figure PE 6.4 Q2(i)

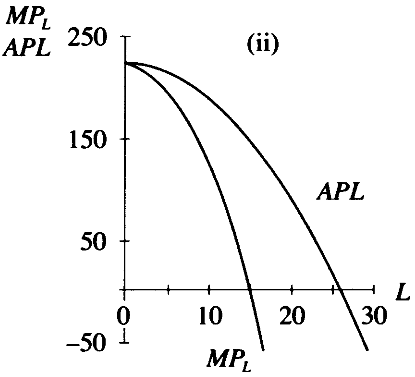


Figure PE 6.4 Q2(ii)

3. (a) $APL = \dfrac{Q}{L} \rightarrow Q = L(APL)$ (b) $MP_L = \dfrac{dQ}{dL}$

(i) $Q = 150L$ (ii) $Q = 30L - 2L^2$ (i) $MP_L = 150$ (ii) $MP_L = 30 - 4L$

(c) (i) $APL = MP_L$

(ii) Slope of MP_L is double slope of APL, both are linear decreasing functions.

4. (i) $C = 40 + 0.8Y$

(a) $MPC = 0.8$; $APC = \dfrac{40}{Y} + 0.8$

$APC > MPC$

$\dfrac{40}{Y} + 0.8 > 0.8$ True

At $Y = 10$, $APC = 4.8$; $MPC = 0.8$

At $Y = 20$, $APC = 2.8$; $MPC = 0.8$

(b) $MPS = 1 - MPC = 1 - 0.8 = 0.2$

$APS = 1 - APC = 0.2 - \dfrac{40}{Y}$

$MPS > APS$

$0.2 > 0.2 - \dfrac{40}{Y}$ True

At $Y = 10$, $MPS = 0.2$; $APS = -3.8$

At $Y = 20$, $MPS = 0.2$; $APS = -1.8$

(ii) $C = 50 + 0.5Y^{0.8}$

(a) $MPC = \dfrac{0.4}{Y^{0.2}}$; $APC = \dfrac{50}{Y} + \dfrac{0.5}{Y^{0.2}}$

$APC > MPC$

$$\frac{50}{Y} + \frac{0.5}{Y^{0.2}} > \frac{0.4}{Y^{0.2}}$$

$\dfrac{50}{Y} + \dfrac{0.1}{Y^{0.2}} > 0$ True

At $Y = 10$, $APC = 5.3$; $MPC = 0.25$

At $Y = 20$, $APC = 2.77$; $MPC = 0.22$

(b) $MPS = 1 - \dfrac{0.4}{Y^{0.2}}$

$APS = 1 - \dfrac{50}{Y} - \dfrac{0.5}{Y^{0.2}}$

$MPS > APS$

$$1 - \frac{0.4}{Y^{0.2}} > 1 - \frac{50}{Y} - \frac{0.5}{Y^{0.2}}$$

$1 + \dfrac{0.1}{Y^{0.2}} + \dfrac{50}{Y} > 0$ True

At $Y = 10$, $MPS = 0.75$; $APS = -4.3$

At $Y = 20$, $MPS = 0.78$; $APS = -1.8$

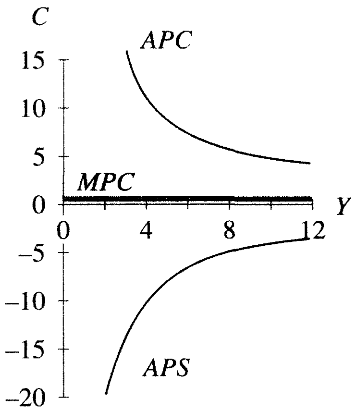


Figure PE 6.4 Q4(i)

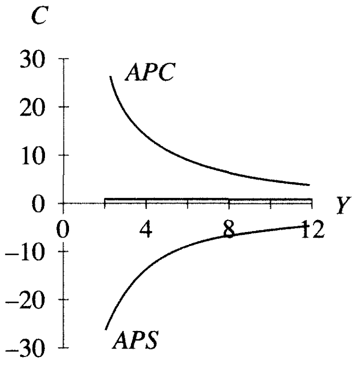


Figure PE 6.4 Q4(ii)

5. (a) $U = 5x^{0.8}$ $\quad MU = \dfrac{dU}{dx} = 4x^{-0.2} = \dfrac{4}{x^{0.2}}$

(b) If MU declines, then, $\dfrac{d(MU)}{dx} < 0$. Therefore, $\dfrac{d(MU)}{dx} = -\dfrac{0.8}{x^{1.2}}$ which is negative for $x > 0$.

(c) $MU \to 0$, but never becomes negative. There is always benefit to be gained from study.

6.5

1. $\dfrac{dy}{dx} = 2x - 6 = 0$: turning point at $x = 3$.

2. $\dfrac{dy}{dx} = 6x(x - 1)$: turning point at $x = 0, y = 0$ and $x = 1, y = -1$.

3. $\dfrac{dP}{dQ} = -4Q + 8$: turning point at $Q = 2$, $P = 8$.

4. $\dfrac{dC}{dY} = 1$: no turning point.

5. $\dfrac{dQ}{dP} = -2.5$: no turning point.

6. $\dfrac{dy}{dx} = 3x^2 - 6x - 9$: turning point at $x = 3, y = -27$; and $x = -1, y = 5$.

7. $\dfrac{d(TC)}{dQ} = 144 - \dfrac{1}{Q^2}$: turning point at $Q = \pm\dfrac{1}{12}$.

8. $\dfrac{dy}{dx} = -2x + 1$: turning point at $x = \dfrac{1}{2}$, $y = 8.25$.

9. $\dfrac{dU}{dx} = \dfrac{4.8}{x^{0.4}}$: no turning point.

10. $\dfrac{dy}{dx} = 4x(x^2 - 1)$: turning point at $x = 0, y = 0; x = 1, y = -1; x = -1, y = -1$.

6.6

1. $y' = 2x - 6$, $y'' = 2 > 0$ (a minimum).

2. $y' = 6x^2 - 6x$, $y'' = 12x - 6$.

At $x = 0$, $y'' = -6 < 0$ (a maximum). At $x = 1$, $y'' = 6 > 0$ (a minimum).

3. $P' = -4Q + 8$, $P'' = -4 < 0$ (a maximum).

4. $C' = 1$, $C'' = 0$, no turning point.

5. $Q' = -2.5$, $Q'' = 0$, no turning point.

6. $y' = 3x^2 - 6x - 9$, $y'' = 6x - 6 = 0$, at $x = 3$, $y'' = 12 > 0$ (a minimum);
at $x = -1$, $y'' = -12 < 0$ (a maximum).

7. $(TC)' = 144 - \dfrac{1}{Q^2}$, $(TC)'' = \dfrac{2}{Q^3}$
< 0 when $Q = -\dfrac{1}{12}$ (a maximum), > 0 when $Q = \dfrac{1}{12}$ (a minimum).

8. $y' = -2x + 1$, $y'' = -2 < 0$ (a maximum).

9. No turning point

10. $y' = 4x^3 - 4x$, $y'' = 12x^2 - 4$.
At $x = 0$, $y'' = -4 < 0$ (a maximum). At $x = 1$, $y'' = 8 > 0$ (a minimum).
At $x = -1$, $y'' = 8 > 0$ (a minimum).

6.7

To find the intervals along which a function is increasing or decreasing, first find the turning point, then find the type of turning point.

1. y increases for $0 < x < \infty$
$y'' = 2 > 0$ (a minimum).

2. $(AC)' = 2Q - 20$, turning point at $Q = 10$.
$(AC)'' = 2 > 0$, turning point is a minimum.
AC decreases until $Q = 10$ and increases after $Q = 10$.

3. $(TR)' = 50 - 2Q$, turning point at $Q = 25$.
$(TR)'' = -2 < 0$, turning point is a maximum.
TR increases until $Q = 25$ and decreases after $Q = 25$.

4. $(AR)' = -2$, AR decreases for all Q. $(AR)'' = 0$

5. $(MC)' = 2Q - 18$, turning point at $Q = 9$.
$(MC)'' = 2 > 0$, turning point is a minimum.
MC decreases until $Q = 9$ and increases after $Q = 9$.

6. $P' = -\dfrac{0.8}{Q^{0.6}}$, P' is decreasing, no turning point.

$P'' = \dfrac{0.48}{Q^{1.6}} > 0$, slope is increasing (becoming less negative).

7. $(TC)' = Q^2$, $(TC)' = 0$ at $Q = 0$.

$(TC)'' = 2Q$, $(TC)'' = 0$ (no max. or min.).

$(TC)' > 0$, TC is increasing for all Q.

8. C increases for $Y > 0$.

6.8

1.

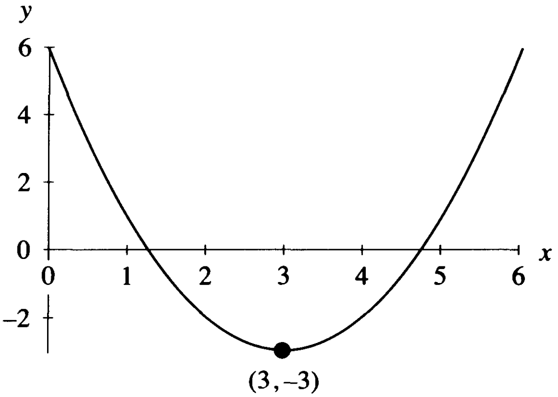


Figure PE 6.8 Q1

2.

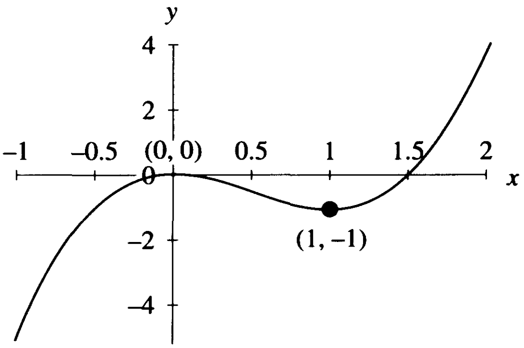


Figure PE 6.8 Q2

3.

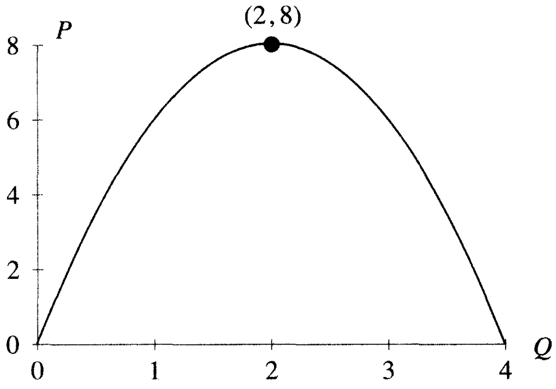


Figure PE 6.8 Q3

4.

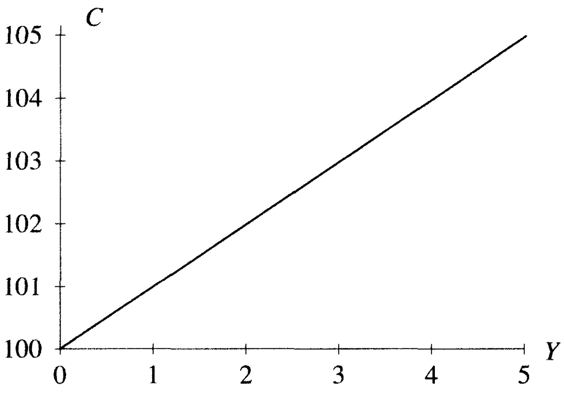


Figure PE 6.8 Q4

5.

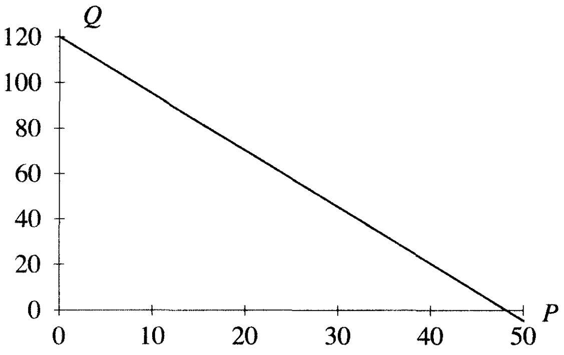


Figure PE 6.8 Q5

6.

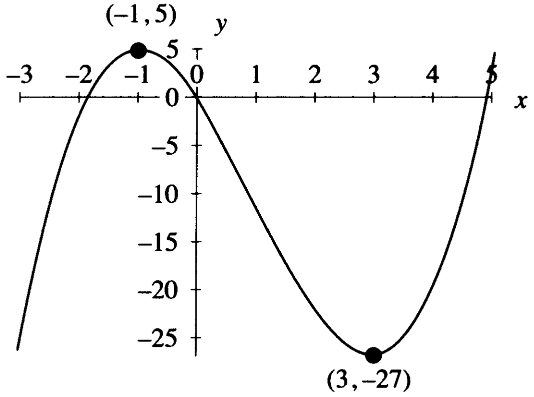


Figure PE 6.8 Q6

7.

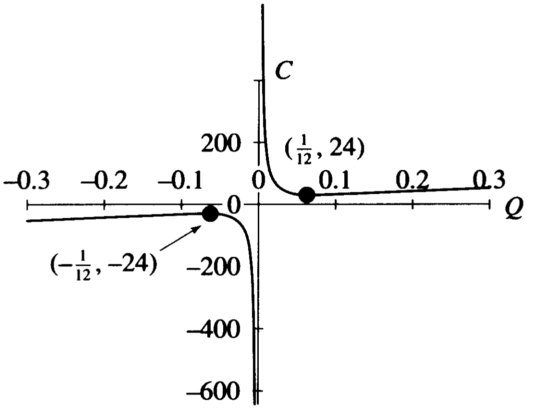


Figure PE 6.8 Q7

8.

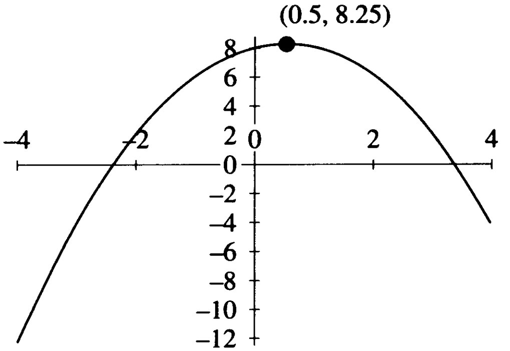


Figure PE 6.8 Q8

9.

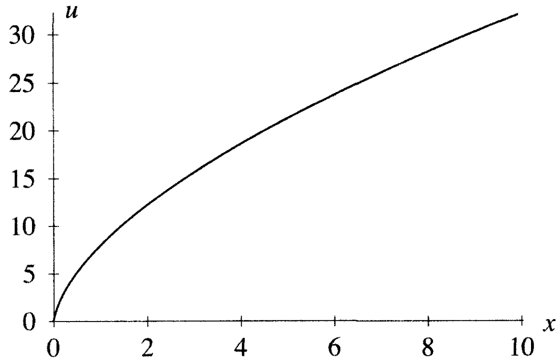


Figure PE 6.8 Q9

10.

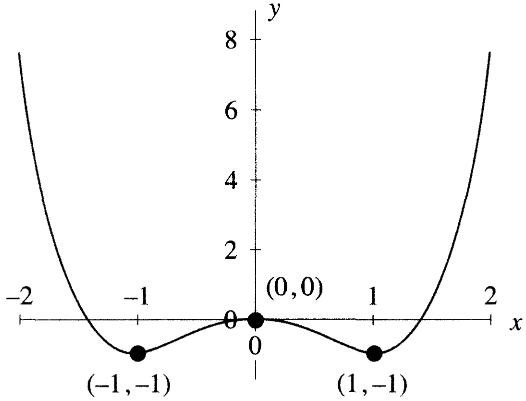


Figure PE 6.8 Q10

6.9

1. (a) TR is a maximum at $Q = 15$ and maximum $TR = £675$.

(b) $AR = 90 - 3Q$, $MR = 90 - 6Q$

For $Q < 15$, AR and MR are positive, but declining. At $Q = 15$, $MR = 0$.

For $Q > 15$, $MR < 0$ while AR remains positive until $Q = 30$.

(c) When $TR = 0$, $AR = 90$ and $MR = 90$.

When TR is a maximum, $AR = 45$ and $MR = 0$.

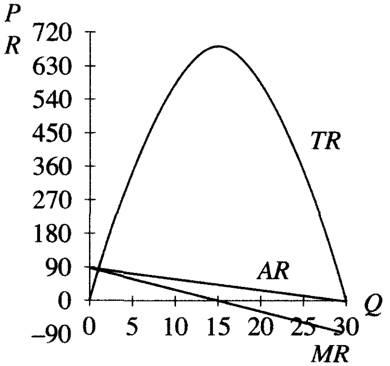


Figure PE 6.9 Q1

2.

	(i) $Q = 150 - 0.5P$	(ii) $P = 80 - 2Q$	(iii) $P = 45$
(a) Total revenue	$300Q - 2Q^2$	$80Q - 2Q^2$	$45Q$
(b) TR maximum at	$Q = 75$	$Q = 20$	No maximum
Maximum TR	$TR = 11\,250$	$TR = 800$	–
(c) $MR = 0$ at	$Q = 75$	$Q = 20$	MR is never zero
$AR = 0$ at	$Q = 150$	$Q = 40$	AR is never zero

3. (a) $TR = 240Q - 10Q^2$. $\pi = TR - TC = 240Q - 10Q^2 - 120 - 8Q$
$= 232Q - 10Q^2 - 120$

(b) (i) $\pi' = -20Q + 232 = 0$. Maximum profit at $Q = 11.6$
(ii) Maximum TR at $Q = 12$

(c) $MR = 240 - 20Q$ and $MC = 8$. $MC = MR$ at $Q = 11.6$

(d) (i) from the graphs and algebraically, break-even at $Q = 0.5$ and $Q = 22.67$

(d) (ii) MC curve intersects MR curve at the point of maximum profit, i.e. $Q = 11.6$.

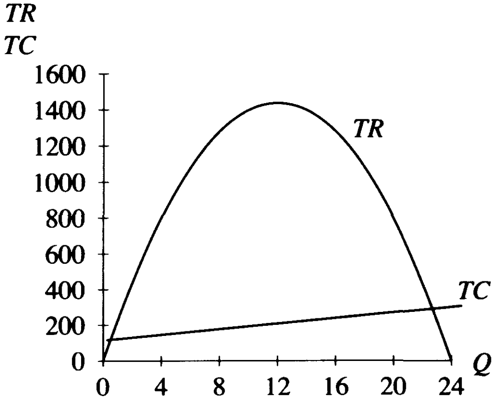


Figure PE 6.9 Q3(d)(i)

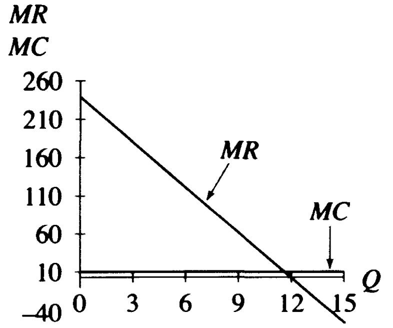


Figure PE 6.9 Q3(d)(ii)

4. (a) $TR = 25Q$; $TC = 15Q + 8000$; $MR = 25$; $MC = 15$

(b) Break-even when $TR = TC$: $25Q = 15Q + 8000 \rightarrow Q = 800$

(c) $\pi = TR - TC = 25Q - 15Q - 8000 = 10Q - 8000$.

Maximum profit when $\frac{d\pi}{dQ} = 0$, but $\frac{d\pi}{dQ} = 10 \neq 0$.

$MR = 25$ and $MC = 15$, therefore, $MR \neq MC$, since $MR - MC = 10$.

Marginal profit is 10 for each unit produced and sold etc.

(d)

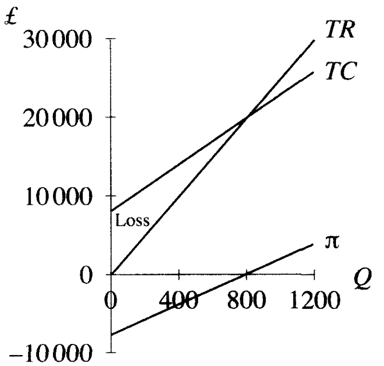


Figure PE 6.9 Q4(d), *TR*, *TC*

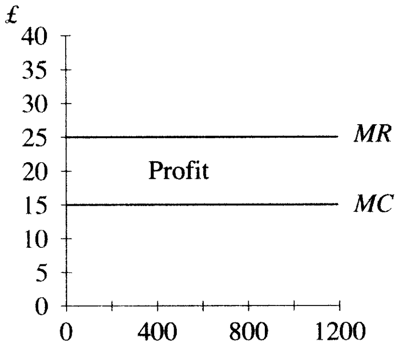


Figure PE 6.9 Q4(d), *MR*, *MC*

5. Pricing strategy I

(a) $MR = 2374$; $MC = 120$;
$\pi = 2254Q - 608\,580$

(i) No maximum profit since profit function is linear.

(ii) $\pi' = 2254$, therefore can never be zero, so no maximum or minimum.

Comment: since $\pi = 2254Q - 608\,580$ profit increases indefinitely by 2254 for each unit produced.

(b) At $Q = 3365$, $\pi = £6\,976\,130$

(c) Break-even at $Q = 270$

Pricing strategy II

(a) $MR = 5504 - 1.6Q$; $MC = 120$;
$\pi = 5384Q - 0.8Q^2 - 608\,580$

(i) $MR = MC$:
$5504 - 1.6Q = 120 \rightarrow Q = 3365$

At $Q = 3365$,
$P = 5504 - 0.8(3365) = 2812$

(ii) $$\frac{d\pi}{dQ} = 5384 - 1.6Q$$
$$= 0 \rightarrow Q = 3365$$
$$\frac{d^2\pi}{dQ^2} = -1.6 < 0,$$

thus confirming a maximum.

(b) At $Q = 3365$, $\pi = £8\,450\,000$

(c) Break-even at $Q = 115$ and $Q = 6615$

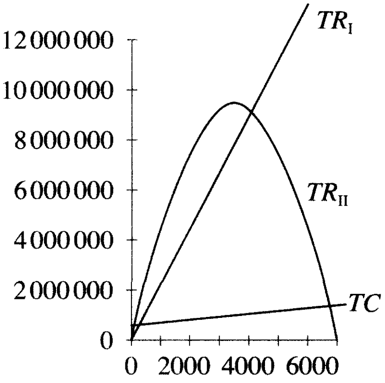


Figure PE 6.9 Q5(c)

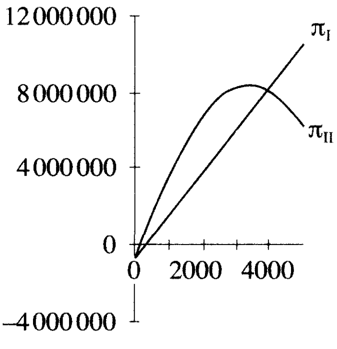


Figure PE 6.9 Q5(d)

Same total revenue occurs at $Q = 0$ and $Q = 3912.5$.

(d) Profits are equal at $Q = 0$ and $Q = 3912.5$.

II is more profitable than I up to the point at which total revenues are equal. After this point, I becomes more profitable than II.
I makes a loss until break-even at $Q = 270$. II makes a loss until break-even at $Q = 115$ and after $Q = 6615$. Choice of pricing strategy depends on the quantity produced.

6.10

Point of inflection (PoI) at $y'' = 0$; plus change in sign before and after.
curvature: $y'' > 0$, concave up.
curvature: $y'' < 0$ concave down.

1. $y' = 2x + 2$
$y'' = 2$ (no point of inflection)
$y'' > 0$ (concave up)

2. $y' = -3x^2 + 5$
$y'' = -6x$; $y'' = 0$ at $x = 0$ (PoI)
$y'' < 0$ for $x > 0$ (concave down)

3. $y' = 12x^3 + 2x$
$y'' = 36x^2 + 2$ (no point of inflection)
$y'' > 0$ (concave up)

4. $L' = \dfrac{0.125}{K^{0.75}}$
$L'' = -\dfrac{0.09375}{K^{1.75}} < 0$ for $K > 0$ (no PoI)
$L'' < 0$ (concave down)

5. $(TC)' = 3Q^2 - 12Q + 3$
$(TC)'' = 6Q - 12$ (PoI at $Q = 2$)
$(TC)'' > 0$ for $Q > 2$ (concave up)
$(TC)'' < 0$ for $Q < 2$ (concave down)

6. $(TC)' = -4Q^3$
$(TC)'' = -12Q^2$ (no PoI)
$(TC)'' < 0$ for all Q (concave down)

7. $(AC)' = 2Q - \dfrac{2}{Q^2}$
$(AC)'' = 2 + \dfrac{4}{Q^3}$ (PoI at $Q = -1.26$)
$(AC)'' > 0$ for $Q > 0$ (concave up)

8. $(TR)' = 120 - 1.6Q$
$(TR)'' = -1.6$ (No PoI)
$(TR)'' < 0$ (concave down)

9. Simplify: $y = \dfrac{1}{x} + \dfrac{2}{x^2} - \dfrac{2}{x^2} = \dfrac{1}{x}$
$y' = -\dfrac{1}{x^2}$
$y'' = \dfrac{2}{x^3}$ (no point of inflection)
$y'' > 0$ for $x > 0$ (concave up)

10. $(AFC)' = -\dfrac{100}{Q^2}$
$(AFC)'' = \dfrac{200}{Q^3}$ (no PoI)
$(AFC)'' > 0$ for $Q > 0$ (concave up)

6.11

1. (a) $MP_L = \dfrac{dQ}{dL} = -6L^2 + 24L$. $APL = -2L^2 + 12L$

At $L = 1.5$, $MP_L = 22.5 > APL = 13.5$. See text for further description.

(b) MP_L is a maximum at $L = 2$, $MP_L = 24$. APL is maximum at $L = 3$, $APL = 18$. When $L = 3$, $MP_L = 18$, point of intersection.
Graph similar to Figure 6.36(b)

(c) Turning points: $L = 0$ (min.); $L = 4$ (max.). Point of inflection: $L = 2$. MP_L increases when $L < 2$, decreases when $L > 2$ etc.
Graph similar to Figure 6.36(b)

(d) Left to the reader to describe the results in (a), (b) and (c) in terms of quantity of sandwiches and number of labour-hours hired.

2. (a) $MC = Q^2 - 60Q + 2800$. $AC = \dfrac{Q^2}{3} - 30Q + 2800 + \dfrac{900}{Q}$

$AVC = \dfrac{Q^2}{3} - 30Q + 2800$. $AFC = \dfrac{900}{Q}$

(b) $\dfrac{\mathrm{d}(MC)}{\mathrm{d}Q} = 2Q - 60 = 0 \to Q = 30$. $\dfrac{\mathrm{d}^2(MC)}{\mathrm{d}Q^2} = 2 > 0$,

minimum $MC = 1900$ at $Q = 30$.

(c) $\dfrac{\mathrm{d}(AVC)}{\mathrm{d}Q} = \dfrac{2}{3}Q - 30 = 0 \to Q = 45$. $\dfrac{\mathrm{d}^2(AVC)}{\mathrm{d}Q^2} = \dfrac{2}{3} > 0$,

minimum $AVC = 2125$ at $Q = 45$. At $Q = 45$, $AVC = MC = 2125$.

(d) graphs similar to those in text (Figure 6.38), MC and AVC intersect at $Q = 45$, the minimum point on AVC.

3. (a) Plot the graph (Excel or otherwise). From the graph, TR curve $>$ TC curve between $Q = 6$ and $Q = 41$ (approx.). Therefore, the company makes a profit when output is between $Q = 6$ and $Q = 41$.

(b) $\pi = -\dfrac{1}{3}Q^3 + 13Q^2 + 56Q - 750$

(i) $\dfrac{\mathrm{d}\pi}{\mathrm{d}Q} = -Q^2 + 26Q + 56$; $\dfrac{\mathrm{d}^2\pi}{\mathrm{d}Q^2} = -2Q + 26$

Maximum π at $Q = 28$, $\pi = 3692.67$; minimum π at $Q = -2$, $\pi = -807.33$ (loss).
(ii) At maximum (or minimum) profit $MR = MC$ and

$$(MR)' > (MC)' \text{ for a minimum}$$
$$(MR)' < (MC)' \text{ for a maximum}$$

(c) Marginal profit, $\dfrac{\mathrm{d}\pi}{\mathrm{d}Q} = -Q^2 + 26Q + 56$. Maximum $\pi = 225$ at $Q = 13$.

Marginal π is a max. at $Q = 13$ (PoI), increasing for $Q < 13$ and decreasing for $Q > 13$.

4. (a) Graph similar to Figure 6.27, hence the firm makes a profit when $TR > TC$ between $Q = 21$ and $Q = 101$ (approx.).

(b) $\pi = -\dfrac{1}{3}Q^3 + 36Q^2 - 140Q - 9900$

(i) Maximum π at $Q = 70$, $\pi = £42\,366.67$;
Minimum π at $Q = 2$, $\pi = -£10\,038.67$
(ii) Optimum profit when $MR = MC$, but use second-order conditions to classify points as max./min.

6.12

1. $10e^x$

2. $\dfrac{e^x}{10}$

3. $-\dfrac{243}{x^2}+\dfrac{1}{x}$

4. $\dfrac{20}{Q}$

5. $\dfrac{1}{3L^{2/3}}$

6. $-150e^t$

7. $104.4e^t$

8. $\dfrac{1}{Q}-\dfrac{2}{Q^2}$

9. $\dfrac{1}{12P}+\dfrac{3}{P^{3/2}}$

10. $10.85e^t$

11. $-\dfrac{150}{Y^2}+0.8e^Y$

12. $2x-5e^x+\dfrac{1}{x}$

13. $e^Y-\dfrac{12}{Y^2}$, $e^Y+\dfrac{24}{Y^3}$

14. $y=\ln x-12$, $\dfrac{dy}{dx}=\dfrac{1}{x}$

15. $\dfrac{dt}{dY}=\dfrac{1}{Y}\rightarrow\dfrac{dY}{dt}=Y=e^{15+t}$; $\dfrac{d^2Y}{dt^2}=e^{15+t}$

16. $Y=e^{x+5}-10\rightarrow e^x.e^5-10$; $\dfrac{dy}{dx}=e^x.e^5$, $\dfrac{d^2y}{dx^2}=e^x.e^5$

6.13

1. $14(2x-5)^6$

2. $-15(4-5x)^2$

3. $\dfrac{4}{(1-0.8x)^6}$

4. $-\dfrac{5}{2(5x+12)^{3/2}}$

5. $\dfrac{Q}{\sqrt{Q^2+12}}$

6. $\dfrac{(5Q^4+6Q)}{2\sqrt{Q^5+3Q^2}}$

7. $\dfrac{0.85(2L+3)}{3(L^2+3L)^{2/3}}$

8. $-\dfrac{1200}{(P+5)^2}$

9. $9.72e^{0.81x}$

10. $1333.07e^{1.09t}$

11. $-1120e^{-1.4Q}$

12. $222e^{-1.2t}$

13. $-2.5e^{2.5t}$

14. $-\dfrac{320}{e^{4t}}$

15. $-\dfrac{15e^t}{(1+e^t)^2}$

16. $26.4e^{0.88Y}$

17. $\dfrac{2Y}{Y^2+4}$

18. $\dfrac{500(3K^2+8)}{K^3+8K}$

19. $-\dfrac{2}{(x+2)^2}-\dfrac{2}{2x+2}$

20. $P=\ln\left(\dfrac{Q^2-2Q}{Q}\right)=\ln(Q-2)$, $\dfrac{dP}{dQ}=\dfrac{1}{Q-2}$

21. $-5014.375e^{-0.5t}$ **22.** $C = 425 + 1.2[\ln Y - \ln(Y+2)],\ \dfrac{dC}{dY} = 1.2\left[\dfrac{1}{Y} - \dfrac{1}{Y+2}\right]$

23. $161.2e^{-0.65Y}$ **24.** $-\dfrac{2}{Q^{3/2}} + \dfrac{5}{Q}$

6.14

1. $1 + \ln x$ **2.** $50 - 3\sqrt{Q}$ **3.** $e^t(2t + t^2)$ **4.** $-\dfrac{1}{Q^2} - 1 - \ln Q$

5. $\sqrt{L+5} + L\left(\dfrac{1}{2\sqrt{L+5}}\right)$ **6.** Simplify: $y = x^{1.5}9^{1/2}x^{1/2} = 3x^2,\quad \dfrac{dy}{dx} = 6x$

7. $-e^Q - Qe^Q$ **8.** $\sqrt{1+2Q} + Q\left(\dfrac{1}{\sqrt{1+2Q}}\right)$ **9.** $100e^{-0.5Y}(1 - 0.5Y)$

10. $\dfrac{1}{Y}\cdot\dfrac{1}{(Y+8)} - \dfrac{1}{Y^2}\ln(Y+8)$ **11.** $\dfrac{\sqrt{Y}}{Y} + \dfrac{1}{2\sqrt{Y}}\ln(8Y)$ **12.** $\dfrac{3}{20}e^{x-5}(1+x)$

13. $\dfrac{Q^2}{5} + \sqrt{3Q-1} + \dfrac{3Q}{2\sqrt{3Q-1}} - 1$ **14.** $-\sqrt{5}\left(\dfrac{\sqrt{Q}}{Q} + \dfrac{1}{2\sqrt{Q}}\ln Q\right)$

15. $(3t-5)^{-4} - 12t(3t-5)^{-5}$

16. Simplify: $x = y^3 3\ln y = 3(y^3 \ln y),\quad \dfrac{dy}{dx} = 3\left(3y^2 \ln y + \dfrac{y^3}{y}\right) = 3y^2(3\ln y + 1)$

6.15

1. $\dfrac{1 - \ln x}{x^2}$ **2.** $\dfrac{-100}{(50+Q)^2}$

3. $\dfrac{2t(t^2-1) - 2t(t^2+1)}{(t^2-1)^2} = \dfrac{2t(t^2 - 1 - t^2 - 1)}{(t^2-1)^2} = \dfrac{-4t}{(t^2-1)^2}$

4. $\dfrac{1}{Q^2}\left(\dfrac{Q}{2\sqrt{Q+1}} - \sqrt{Q+1}\right)$ **5.** $-\left(\dfrac{(Q+1) - Q}{(Q+1)^2}\right) = -\left(\dfrac{1}{(Q+1)^2}\right)$

6. $919e^{-0.8Y}\left(\dfrac{0.8Y+1}{Y^2}\right)$ **7.** Simplify: $y = 2\ln(x) + \dfrac{1}{x};\quad \dfrac{dy}{dx} = \dfrac{2}{x} - \dfrac{1}{x^2}$

8. $205\left(\dfrac{1 - \ln Y}{(Y)^2}\right)$

6.16

1. (a) $TR = (24 - 6\ln Q)Q$. TR is maximised at $Q = 20.09$.

(b) $MR = 18 - 6\ln Q$. Rate of change of $MR = \dfrac{\mathrm{d}(MR)}{\mathrm{d}Q} = -\dfrac{6}{Q} < 0$.

$(MR)'$ is negative for $Q > 0$. Therefore MR is decreasing for $Q > 0$.

$(MR)'' = \dfrac{6}{Q^2} > 0$, therefore concave up.

2. (a) $MPC = 160e^{-0.2Y}$. Decreases as Y increases.

(b) Maximum consumption at $\mathrm{d}C/\mathrm{d}Y = 0$. That is, at $160e^{-0.2Y} = 0$. However, there is no value of Y to satisfy this condition; therefore, no max./min.

(c) Curvature: concave down. $C'' = -32e^{-0.2Y}$.

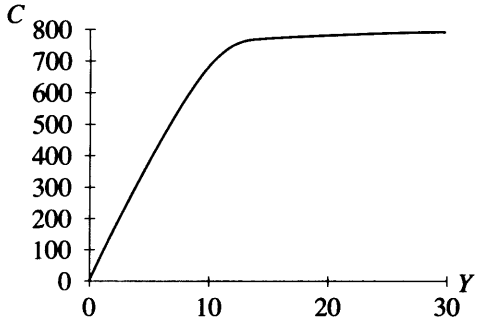


Figure PE 6.16 Q2

3. (a) $U = 20xe^{-0.1x}$. U is a maximum at $x = 10$ and $U = 73.58$.

(b) $MU_x = 20e^{-0.1x}(1 - 0.1x)$. $(MU)' = 4e^{-0.1x}(0.05x - 1)$.

When $x > 10$, MU_x is declining, enjoyment from wine is diminishing after 10 glasses.

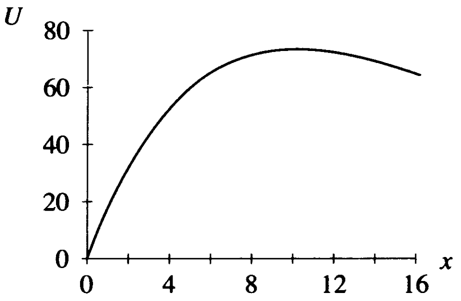


Figure PE 6.16 Q3

4. (a) $TR = \dfrac{50Q}{e^{0.05Q}} = 50Qe^{-0.05Q}$. $MR = 50(-0.05Q + 1)e^{-0.05Q}$

(b) Maximum TR when $Q = 20$, $P = 18.39$ and $TR = 367.88$.

5. (a) $U = 95 - \ln(t + 5)$.

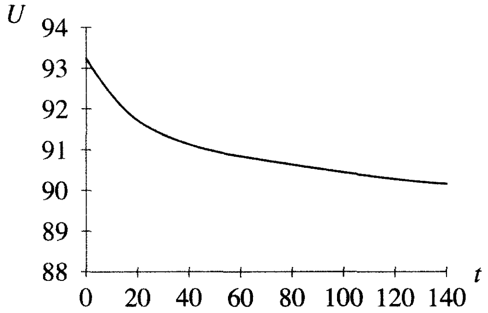


Figure PE 6.16 Q5

U (enjoyment) declines with time.

(b) $MU_x = -\dfrac{1}{(t+5)} < 0$; $(MU_x)' = \dfrac{1}{(t+5)^2} > 0$: enjoyment declines at an increasing rate.

6. $AC = \dfrac{100e^{0.05(Q-10)}}{Q+1}$ (a) Minimum AC at $Q = 19$ and $AC = 7.842$.

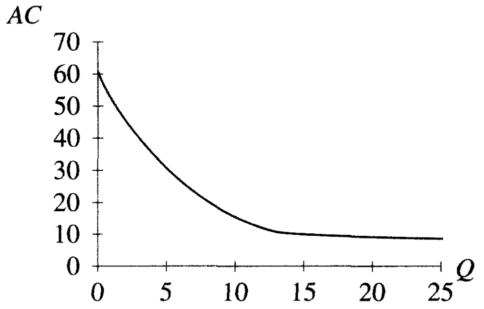


Figure PE 6.16 Q6

(b) $TC = \dfrac{100Qe^{0.05(Q-10)}}{Q+1}$

6.17

1. (a) $\varepsilon_d = \dfrac{\%\text{ change in quantity}}{\%\text{ change in price}} = \dfrac{(\Delta Q/Q) \times 100}{(\Delta P/P) \times 100} = \dfrac{\Delta Q}{\Delta P}\dfrac{P}{Q} \to \dfrac{dQ}{dP}\dfrac{P}{Q}$, at a point

(b) $\varepsilon_d = \dfrac{(\%\text{ change in } Q)}{(\%\text{ change in } P)} \rightarrow -0.8 = \dfrac{(\%\text{ change in } Q)}{(5\%)} \rightarrow \%$ change in $Q = -4\%$

If ε_d is constant, the demand function is of the form $Q = a/P^{0.8}$

2. (a) (i) $\varepsilon_d = -\dfrac{1}{2}\dfrac{P}{Q}, 1 - \dfrac{40}{Q}, \dfrac{P}{P-80}$: (ii) $\varepsilon_d = -4\dfrac{P}{Q}, 1 - \dfrac{120}{Q}, \dfrac{P}{P-30}$

(iii) $\varepsilon_d = \infty$ (iv) $\varepsilon_d = -\dfrac{1}{b}\dfrac{P}{Q}, \; 1 - \dfrac{a}{b}\dfrac{1}{Q}, \dfrac{P}{P-a}$

(b) (i) -1.67; -0.33 (ii) 2.5; -3 (iii) ∞; ∞ (iv) $\dfrac{50}{50-a}$; $\; 1 - \dfrac{a}{b}\dfrac{1}{30}$

3. (a) $\varepsilon_d = -\dfrac{100}{Q}$: when $Q = 100$, $\varepsilon_d = -1$: when $Q < 100$, $\varepsilon_d < -1$ so demand is elastic, the % change in Q is greater than the % change in P: when $Q > 100$, $-1 < \varepsilon_d < 0$ so demand is inelastic, the % change in Q is smaller than the % change in P.

(b) When $Q = 150$, $P = 111.57$. Since $\varepsilon_d = -\dfrac{100}{150}$ at $Q = 150$, the response to a 10% increase in P is a 6.67% decrease in Q.

4. (a) $\varepsilon_d = \dfrac{2P^2}{P^2 - 192}$. When $\varepsilon_d = -1$, $P = £8$, $Q = 128$ seats available

(b) Ten seats is an increase of 7.8125% in Q. P will drop by 7.8125% approx., new price = £7.348 approximately.

5. (a) See Worked Example 6.42 (b) (i) $\varepsilon_d = -1.2$, (ii) decrease by 6% approximately, (iii) increase by 6% approximately.

6. (a) $Q = 20\,260$ (b) If P increases by 5%, the new price = £31.5 with the corresponding demand, $Q = 19\,108$. Percentage decrease in Q is 5.686%. The approximate % decrease in 5 (b) (ii) is 6%. This is approximate because the elasticity as well as the values of P and Q change over the interval $P = £30$ to £31.5, but the formula uses the slope and the ratio P/Q at the start of the interval to calculate the % change in Q throughout the interval.

7. (a) $TR = 50Q - 0.5Q^2$: $MR = 50 - Q$: $AR = 50 - 0.5Q$. (b) Max. TR at $Q = 50$, $P = 25$ (c) $\varepsilon_d = -2\dfrac{P}{Q}$ (i) $\dfrac{P}{P-50}$ (ii) $1 - \dfrac{100}{Q}$ (d) See text for proof

8. (a) Graph similar to Figure 6.41. (b) $\varepsilon_d = -1$ at $P = 25$ (see Figure 6.41).

9. (a) $TR = 1500Qe^{-0.025Q}$ $MR = 1500(1 - 0.025Q)e^{-0.025Q}$ $AR = 1500e^{-0.025Q}$

(b) $Q = 40$, $P = 551.82$FF (c) $\varepsilon_d = -\dfrac{40}{Q}$: $\varepsilon_d = -\dfrac{1}{7.31 - \ln P}$

(d) Max. $TR = 22072.8$FF at $\varepsilon_d = -1$, hence $Q = 40$, $P = 551.82$

10. (a) No, $\varepsilon_d = -\dfrac{1}{b}\dfrac{P}{Q} = \dfrac{P}{P-a}$ changes as P changes. (b) (i) $MR = -5044.44$, (ii) $P = 95\,844.44 - 1056.05Q$, (iii) max. $TR = 2\,174\,710$ at $P = 47\,922.22$, $Q = 45.38$

11. (a) See equation (6.36) with $MR = MC$ for max. profit. (b) Max. profit at $P = 151$, $Q = 149$, $\varepsilon_d = -151/149$; substitute $P = 151$, $MC = 2$, $\varepsilon_d = -(151/149)$ into equation in (a) to confirm its validity: take any other point on the demand function, such as $P = 100$, $Q = 200$, $\varepsilon_d = -(100/200)$, $MC = 2$, to confirm that the equation in (a) is not true.

Chapter 7

7.1

1. $z_x = 2x + 2y;\ z_y = 2x$

2. $z_x = 10xy^3;\ z_y = 15x^2y^2$

3. $Q_x = \frac{1}{x};\ Q_y = \frac{3}{y}$

4. $Q_L = 8L^{-0.2}K^{0.2};\ Q_K = 2L^{0.8}K^{-0.8}$

5. $P_t = 96e^{0.8t}$

6. $U_x = 8xy^2;\ U_y = 8x^2y$

7. $z_x = 2x(1 + 2y);\ z_y = 2x^2$

8. $z_x = 5y - \frac{y}{x^2};\ z_y = 5x + \frac{1}{x}$

9. $Q_x = \frac{1}{x} + 3\ln(y);\ Q_y = \frac{3x}{y}$

10. $Q_L = 8L^{-0.2}K^{0.2} - 6;\ Q_K = 2L^{0.8}K^{-0.8} - 4$

11. $C_Q = -96e^{-0.8Q}$

12. $Z_L = 0.5L^{-0.5}K^{0.5} + 2\lambda;\ Z_K = 0.5L^{0.5}K^{-0.5} + 3\lambda;\ Z_\lambda = -(50 - 2L - 3K)$

13. Proof:

$$Q = LQ_L + KQ_K$$
$$Q = L(0.5L^{-0.5}K^{0.5}) + K(0.5L^{0.5}K^{-0.5})$$
$$Q = 0.5L^{0.5}K^{0.5} + 0.5L^{0.5}K^{0.5}$$
$$Q = L^{0.5}K^{0.5}$$

14. (a) $K = \frac{200}{L}$

(b) $L = 5$, $K = 40$; $L = 10$, $K = 20$; $L = 15$, $K = 13.33$; $L = 20$, $K = 10$.

e.g. $Q = 2LK = 2(5)(40) = 400$

(c)

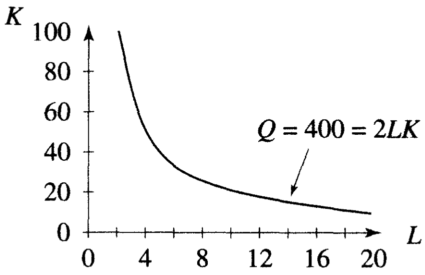


Figure PE 7.1 Q14

15. (a) $K = \dfrac{100}{L^2}$

(b) $L = 5$, $K = 4$; $L = 10$, $K = 1$;

$L = 15$, $K = 0.44$; $L = 20$, $K = 0.25$.

e.g. $Q = 2LK^{0.5} = 2(5)(4)^{0.5} = 20$

(c)

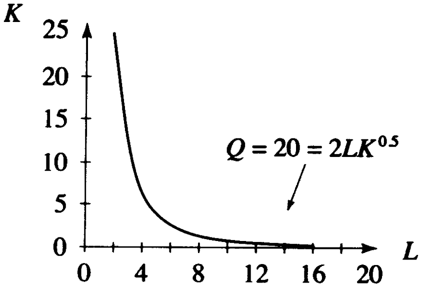


Figure PE 7.1 Q15

7.2

1. $z_{xx} = 2$; $z_{yy} = 0$; $z_{xy} = 2$

2. $z_{xx} = 10y^3$; $z_{yy} = 30x^2y$; $z_{xy} = 30xy^2$

3. $Q_{xx} = -\dfrac{1}{x^2}$; $Q_{yy} = -\dfrac{3}{y^2}$; $Q_{xy} = 0$

4. $Q_{LL} = -1.6L^{-1.2}K^{0.2}$; $Q_{KK} = -1.6L^{0.8}K^{-1.8}$; $Q_{LK} = 1.6L^{-0.2}K^{-0.8}$

5. $P_{tt} = 76.8e^{0.8t}$

6. $U_{xx} = 8y^2$; $U_{yy} = 8x^2$; $U_{xy} = 16xy$

7. $z_{xx} = 2(1 + 2y)$; $z_{yy} = 0$; $z_{xy} = 4x$

8. $z_{xx} = \dfrac{2y}{x^3}$; $z_{yy} = 0$; $z_{xy} = 5 - \dfrac{1}{x^2}$

9. $Q_{xx} = -\dfrac{1}{x^2}$; $Q_{yy} = -\dfrac{3x}{y^2}$; $Q_{xy} = \dfrac{3}{y}$

10. $Q_{LL} = -1.6L^{-1.2}K^{0.2}$; $Q_{KK} = -1.6L^{0.8}K^{-1.8}$; $Q_{LK} = 1.6L^{-0.2}K^{-0.8}$

11. $C_{QQ} = 76.8e^{-0.8Q}$

12. $Z_{LL} = -0.25L^{-1.5}K^{0.5}$; $Z_{KK} = -0.25L^{0.5}K^{-1.5}$; $Z_{\lambda\lambda} = 0$

$Z_{LK} = 0.25L^{-0.5}K^{-0.5}$; $Z_{KL} = 0.25L^{-0.5}K^{-0.5}$; $Z_{K\lambda} = 3$ $Z_{L\lambda} = 2$

7.3

1. $dz = (2x)dx$

2. $dz = (2x)dx + (2y)dy$

3. $dz = (3e^x)dx + dy$

4. $dz = (2L^{-0.8}K^{0.6})dL + (6L^{0.2}K^{-0.4})dK$

5. $dz = \left(\dfrac{1}{x}\right)dx + \left(\dfrac{5}{y}\right)dy$

6. $\Delta Q = 3\%Q$ **7.** $\Delta U = 4\%U$ **8.** $\Delta(TR) = 0.6\%TR$

9. $\Delta(TR) = 1.5\%TR$ **10.** $\Delta(TR) = 4.4\%TR$ **11.** $\Delta Q = 0.06\%Y + 0.9\%P_s$

12. Proof: $z = 0.5(e^{x+y} + e^{x+y}) = z$

7.4

1. (a) $MP_L = 5L^{-0.5}K^{0.5}$; $MP_K = 5L^{0.5}K^{-0.5}$

(b) $Q = L(5L^{-0.5}K^{0.5}) + K(5L^{0.5}K^{-0.5}) = 5L^{0.5}K^{0.5} + 5L^{0.5}K^{0.5} = 10L^{0.5}K^{0.5}$

2. $Q_L = 40L^{-0.5}K^{0.2}$; $Q_K = 16L^{0.5}K^{-0.8}$;

$Q_{LL} = -20L^{-1.5}K^{0.2}$; $Q_{KK} = -12.8L^{0.5}K^{-1.8}$; $Q_{LK} = 8L^{-0.5}K^{-0.8}$

(a) See text.

(b) Q_L and Q_K both decrease: their derivatives Q_{LL} and Q_{KK} are negative.

(c) When $Q = 80$: $80 = 80L^{0.5}K^{0.2} \rightarrow K^{0.2} = \frac{1}{L^{0.5}} \rightarrow K = \frac{1}{L^{2.5}}$

L	2	5	10	
K	0.18	0.018	0.003	
MP_L	20.07	8.01	3.96	decreases as L increases
MP_K	89.2	890	5277	decreases as K increases

3. (a) $MP_L = 40L^{-0.5}K^{0.2}$; $APL = 80L^{-0.5}K^{0.2}$: $APL > MP_L$

(b) $MP_K = 16L^{0.5}K^{-0.8}$; $APK = 80L^{0.5}K^{-0.8}$: $APK > MP_K$

4. (a) $K = \frac{64}{L^2}$

L: 2, 4, 6
K: 16, 4, 1.78

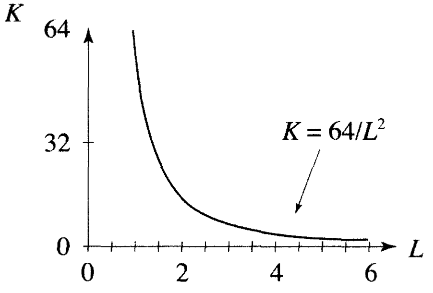


Figure PE 7.4 Q4(a)

(b) $Q_L = MP_L = 10K^{0.5}$
$Q_K = MP_K = 5LK^{-0.5}$

L	2	4	6	
K	16	4	1.78	
MP_L	40	20	13.34	decreases as L increases
MP_K	2.5	10	22.49	increases as L increases (decreases as K increases)

(c) $-MP_L/MP_K$ −16 −2 −0.59 magnitude decreasing, confirming law of diminishing $MRTS$

5. (a) See text: $\dfrac{dK}{dL} = -\dfrac{MP_L}{MP_K} = -2.5\,\dfrac{K}{L}$

(b) $K = \dfrac{1024}{L^{2.5}}$

L	4	9	16	
K	32	4.21	1	
$\dfrac{dK}{dL} = -2.5\,\dfrac{K}{L}$	−20	−1.17	−0.16	magnitude decreasing, confirming law of diminishing *MRTS*

6. $MRTS = \dfrac{dK}{dL} = -\dfrac{Q_L}{Q_K} = -\dfrac{15L^{-0.4}K^{0.5}}{12.5L^{0.6}K^{-0.5}} = -1.2\,\dfrac{K}{L}$

Given $\alpha = 0.6$, $\beta = 0.5$: $MRTS = -\dfrac{\alpha K}{\beta L} = -\dfrac{0.6}{0.5}\dfrac{K}{L} = -1.2\,\dfrac{K}{L}$, as above.

7. Q decreases by 0.5%.

8.

L	2	4	6	
$APL = 50L^{-0.7}K^{0.5}$	$30.77K^{0.5}$	$18.95K^{0.5}$	$14.26K^{0.5}$	decreases as L increases
$MP_L = 15L^{-0.7}K^{0.5}$	$9.23K^{0.5}$	$5.68K^{0.5}$	$4.28K^{0.5}$	decreases as L increases
				$MP_L < APL$

7.5

1. (a) $U_x = 2x^{-0.8}y^{0.8}$; $U_y = 8x^{0.2}y^{-0.2}$

(b) $U = xU_x + yU_y = 2x^{0.2}y^{0.8} + 8x^{0.2}y^{0.8} = 10x^{0.2}y^{0.8} = U$

2. (a) $y = \dfrac{50}{2 + \ln x}$

(b) $(x = 2, y = 18.57)$;

$(x = 10, y = 11.62)$

$(x = 25, y = 9.58)$

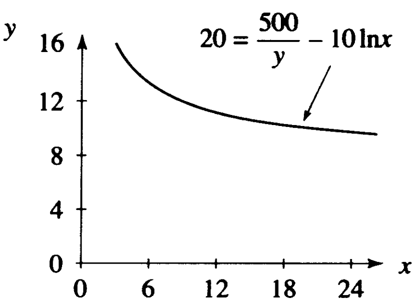


Figure PE 7.5 Q2(b)

(c) Slope $= \dfrac{dy}{dx} = \dfrac{-50}{x(2 + \ln x)^2}$

x	2	10	25	
Slope	−3.45	−0.27	−0.07	slope is decreasing in magnitude; diminishing *MRS*

3. (a) $U_x = -\dfrac{10}{x}$; $U_y = -\dfrac{500}{y^2}$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{U_x}{U_y} = -\frac{10}{x}\cdot\frac{y^2}{500} = -\frac{10}{x}\cdot\frac{1}{500}\left(\frac{50}{2+\ln x}\right)^2 = -\frac{50}{x(2+\ln x)^2}$, as above.

4. (a) $MU_x = 12.5x^{-0.5}y^{0.2}$; $MU_y = 5x^{0.5}y^{-0.8}$

(b) $MRS = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{U_x}{U_y} = -2.5\frac{y}{x}$

(c) $\alpha = 0.5, \beta = 0.2$: $MRS = -\frac{\alpha y}{\beta x} = -\frac{0.5}{0.2}\frac{y}{x} = -2.5\frac{y}{x}$

5. Utility decreases by -7.4%.

6. (a) Constant, $\alpha + \beta = 1$ (b) Increasing, $\alpha + \beta > 1$ (c) Decreasing, $\alpha + \beta < 1$

7. (a) $Q = 371.45$

(b) $L = 55$, $K = 220$, hence $Q = 420.44$. A 10% increase gives $Q = 408.6$. Therefore, the production function exhibits increasing returns to scale.

(c) Q increases by 9.5%.

7.6

1. $\varepsilon_{\mathrm{d}} = -0.25$; $\varepsilon_Y = 0.5243$; $\varepsilon_c = 0.076$

2. (a) $MP_L = 150L^{-0.4}K^{0.2}$; $APL = 250L^{-0.4}K^{0.2}$

(b) $\varepsilon_{QL} = 0.6$; (c) $\varepsilon_{QL} = \frac{MP_L}{APL} = \frac{150L^{-0.4}K^{0.2}}{250L^{-0.4}K^{0.2}} = 0.6$

3. (a) $Y_e = 800$

(b) (i) investment multiplier = 1.82, (ii) government expenditure multiplier = 1.82
(iii) income tax rate multiplier = -872.73

4. Production function I: $\varepsilon_{QL} = 0.6$; $\varepsilon_{QK} = 0.4$

(a) $\varepsilon_{QL} = \frac{\%\text{ change in } Q}{\%\text{ change in } L} \rightarrow \%\text{ change in } Q \cong 0.6(3) = 1.8\%$

(b) 1.8% $(\Delta K = 0)$

Production function II: $\varepsilon_{QL} = 0.4$; $\varepsilon_{QK} = 0.3$

(a) % change in $Q = \varepsilon_{QK}(5) = 1.5\%$

(b) Q increases by 1.5%. Answers (a) and (b) are the same.

5. (a) $MP_K = 50L^{0.6}K^{-0.8}$; $APK = 250L^{0.6}K^{-0.8}$

(b) $\varepsilon_{QK} = 0.2$ (c) $\varepsilon_{QK} = \frac{MP_K}{APK} = \frac{50L^{0.6}K^{-0.8}}{250L^{0.6}K^{-0.8}} = 0.2$

7.7

1. Min. $= -61$ at $(x = 5, y = 6)$ **2.** Max. $= 116$ at $(x = 10, y = 4)$

3. Min. $= -68$ at $(x = 6, y = 2)$ **4.** Min. $= -108$ at $(x = 6, y = 6)$

5. Max. $= 15.78$ at $(L = -50, K = 5)$

6. Min. $= 147$ at $(x = 3, y = 2)$; max. $= 280$ at $(x = -3, y = -2)$;
   Saddle points $= 172$ and 248 at $(x = 3, y = -2)$ and $(x = -3, y = 2)$

7.8

1. Maximum at $(x = 6, y = 23)$

2. (a) $TR = 320x + 106y - 4x^2 - 20y^2 + 4xy$
   (b) $x = 43.5$; $P_X = 160$; $y = 7$; $P_Y = 53$; $TR = 7331$

3. (a) $\pi = 280x + 80y - 4x^2 - 20y^2$
   (b) Maximum profit $= 4980$ when $x = 35$; $P_X = 184$; $y = 2$; $P_Y = 136$; $TR = 6712$

4. (a) $TR = 190P_1 + 120P_2 - 4P_1^2 - 3P_2^2 - 3P_1P_2$
   (b) $P_1 = 20$; $Q_1 = 100$; $P_2 = 10$; $Q_2 = 50$
   (c) Maximum revenue $= 2500$

5. Outline proof: $TC_{Q_1} = 4Q_2 + 10 = 0$; $TC_{Q_2} = 4Q_1 + 15 = 0$.
   First-order conditions give negative values of Q.

6. Minimum cost at $(x = 100, y = 7)$.

7. Maximum at $(x = 4, y = 2)$.

8. (a) $TR = 8x + 20y - 2x^2 - 0.5y^2$; $\pi = 4x + 14y - 2x^2 - 0.5y^2 - 10$
   (b) Maximum $TR = 208$ at $x = 2$; $P_X = 4$; $y = 20$; $P_Y = 10$
   (c) Maximum profit $= 90$ at $x = 1$; $P_X = 6$; $y = 14$; $P_Y = 13$ and $TR = 188$
   (d) $\varepsilon_{dX} = \dfrac{P}{P-a} = \dfrac{6}{6-8} = -3$; $\varepsilon_{dY} = \dfrac{P}{P-a} = \dfrac{13}{13-20} = -1.86$

9. (a) Maximum profit $= 722.5$ at $x = 15$; $P_X = 42.5$; $y = 6$; $P_Y = 65$
   (b) Maximum profit $= 682.0$ at $P = 47$, $x = 21$
   (c) $\varepsilon_{dX} = -1.13$; $\varepsilon_{dY} = -1.08$

7.9

1. Maximum utility at $= 47.5$ $x = 9$; $y = 72$

2. (a) $U_x = 0.2x^{-0.8}y^{0.8}$; $U_y = 0.8x^{0.2}y^{-0.2}$

   (b) $\dfrac{U_x}{U_y} = \dfrac{y}{4x} = \dfrac{72}{36} = \dfrac{2}{1} =$ ratio of prices; $P_X = 4$, $P_Y = 2$

3. (a) $MU_x = x^{-0.8}y^{0.8}$: $MU_y = 4x^{0.2}y^{-0.2}$

   (b) $\dfrac{MU_x}{MU_y} = \dfrac{P_X}{P_Y} \rightarrow \dfrac{y}{4x} = \dfrac{5}{2} \rightarrow 2y - 20x = 0$ (equation (1))

   Second equation is the constraint, $5x + 2y = 200$ (equation (2))
   Solution of equations (1) and (2) is $x = 8$, $y = 80$

   (c) Same as (b), $x = 8$, $y = 80$ when utility is maximised; that is, 8 hours skating, 80 hours bridge per month.

4. Maximum $Q = 846.932$ when $L = 60$, $K = 50$.

5. Minimum costs $= 193.62$ when $K = 32.27$, $L = 19.36$

6. Maximum output $= 39.48$ when $L = 1.875$, $K = 12.5$

7. (a) Maximum profit $= 136$ when $x_1 = 4$; $P_1 = 30$; $x_2 = 5$; $P_2 = 20$.

   (b) $\varepsilon_{d1} = -1.5$; $\varepsilon_{d2} = -2$

8. Cost constraint: $C = 200x_A + 400x_B$

   Maximum utility $= 26.88$ when $x_A = 6$; $x_B = 6$

9. (a) $P_L = 8$; $P_K = 4$ (b) $L = 30$, $K = 40$

   (c) $\dfrac{MP_L}{MP_K} = \dfrac{30K}{20L}$. At maximum this is reduced to $\dfrac{30(40)}{20(30)} = 2$, the ratio of prices

   $\dfrac{P_L}{P_K} = \dfrac{8}{4} = 2$

10. (a) $20x + 2y = 1450$

    (b) $x = 14.5$ units of strawberries, $y = 580$ units of blackberries.
    Maximum $Q = 5546.8$

    (c) $\dfrac{MP_x}{MP_y} = \dfrac{y}{4x}$

    At maximum Q,

    $\dfrac{MP_x}{MP_y} = \dfrac{580}{58}$

This is the ratio of prices:

$$\frac{P_X}{P_Y} = \frac{20}{2} = \frac{10}{1}$$

Chapter 8

8.1

Simplify before attempting to integrate

1. $\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^{4.5}}{4.5} + c$

2. Simplify $\frac{1}{x^{5/2}} = x^{-5/2}$. Answer: $\frac{x^{-3/2}}{-\frac{3}{2}} = -\frac{2}{3x^{3/2}} + c$

3. Simplify to $x^{-3/2}$. Answer: $\frac{x^{-1/2}}{-\frac{1}{2}} = -\frac{2}{x^{1/2}} + c$

4. Simplify to $1 + x^{-1/2}$. Answer: $x + 2x^{1/2} + c$ **5.** $x^2 + 2\ln(x) + c$

6. $x^2 + \frac{1}{2}\ln(x) + c$ **7.** $\frac{2}{5}x^{5/2} + 3x + c$ **8.** $\frac{5x^2}{2} + \sqrt{5}\,\frac{2}{3}\,x^{3/2} + c$

9. Simplify to $x^3 - 6x^2 + 9x$. Answer: $\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2} + c$

10. Simplify to $1 + \frac{7}{x}$. Answer: $x + 7\ln(x) + c$

11. Simplify to $x^2 + 1$. Answer: $\frac{x^3}{3} + x + c$

12. Simplify to $Q^2 - Q^{3/2}$. Answer: $\frac{Q^3}{3} - \frac{2}{5}Q^{5/2} + c$

13. Simplify to $t + 2$. Answer: $\frac{t^2}{2} + 2t + c$ **14.** $4t + 4\ln(t) + c$

15. $5Q + \ln(Q) + c$ **16.** Simplify to $2 - \frac{5}{Q}$. Answer: $2Q - 5\ln(Q) + c$

17. Simplify to $20Q - 0.5Q^2$. Answer: $10Q^2 - \frac{0.5Q^3}{3} + c$

18. Simplify to $\frac{1}{Q} + 4Q^{-2}$. Answer: $\ln(Q) - \frac{4}{Q} + c$. **19.** $\frac{1}{5}\ln(Q) - 5Q + c$

20. $1.25Y^2 + c$

8.2

1. $\dfrac{(x-5)^{1.2}}{1.2}+c$

2. $\dfrac{1}{2}\dfrac{(2x-5)^{1.2}}{1.2}+c$

3. $\dfrac{(3x-5)^{1.2}}{3(1.2)}+c$

4. $\dfrac{(9x-5)^{1.5}}{9(1.5)}+c$

5. $\dfrac{(9x-5)^{4/3}}{9(4/3)}=\dfrac{(9x-5)^{4/3}}{12}+c$

6. $10\ln(x-1)+c$

7. $\dfrac{1}{10}\ln(10x-1)+c$

8. Simplify to $10x^{-2}$; answer $-\dfrac{10}{x}+c$

9. $-\dfrac{2.5}{(x-1)^4}+c$

10. $\dfrac{(20-2Q)^{1.45}}{(-2)(1.45)}=-\dfrac{(20-2Q)^{1.45}}{2.9}+c$

11. $\dfrac{\ln(2+7Q)}{7}+c$

12. $\dfrac{4\ln(2+7Q)}{7}+c$

13. $-\dfrac{\ln(2-7Q)}{7}+c$

14. $-\dfrac{15}{6Q^6}+c$

15. $-\dfrac{5}{2(Q-5)^6}+c$

16. $-\dfrac{5}{4(2Q-5)^6}+c$

17. $\dfrac{2}{3}(Q+1)^{3/2}+2Q^{1/2}+c$

18. $\dfrac{10(4+Y)^{3/2}}{9}+c$

19. $-\dfrac{10(4-Y)^{3/2}}{9}+c$

20. $\dfrac{(1+0.8t)^{2.12}}{1.696}+c$

21. $-\dfrac{(1-0.8t)^{2.12}}{1.696}+c$

22. $\dfrac{e^{12x}}{12}+c$

23. $-2e^{-2x}+c$

24. $-58.8e^{-0.85Q}+c$

25. $100(Y+2e^{-0.5Y})+c$

26. Simplify to $e^{-2x}+2e^{-x}$; answer $-0.5e^{-2x}-2e^{-x}+c$

27. $3e^{1+Y}+c$

28. $-2e^{2-5Q}$

29. $-0.5e^{-2t}+2e^{t+2}+3t+c$

8.3

1. 14

2. 80

3. 47.25

4. -6.67

5. $48+4\ln 5=54.44$

6. $\ln 10+\dfrac{28}{15}-\ln 3=3.07$

7. 102.22

8. 14 666.67

9. 396.75

10. $10(\ln 15-\ln 11)=3.1$

11. 14.66 **12.** $\frac{583}{2}+12\ln 12=321.32$ **13.** $4\ln 5=6.44$

14. $50+\ln 6=51.79$ **15.** 24.67 **16.** 201.2144 **17.** 614.7809

18. Simplify to $\int_{Q=0}^{Q=5}(1+4e^{-2Q})\mathrm{d}Q=6.9999$ **19.** 6.4917

20. (a)

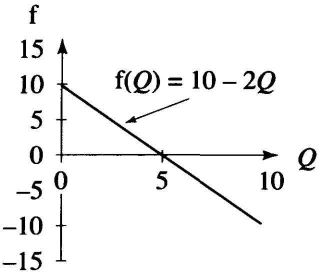


Figure PE 8.3 Q20

(b) Net area $=\int_0^{10}\mathrm{f}(Q)\mathrm{d}Q=0$

(c) $\int_0^5\mathrm{f}(Q)\mathrm{d}Q=25$ area above axis

$\int_5^{10}\mathrm{f}(Q)\mathrm{d}Q=-25$ area below axis

21. (a)

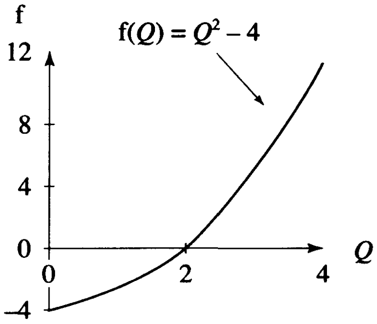


Figure PE 8.3 Q21

(b) Net area $=\int_0^4\mathrm{f}(Q)\mathrm{d}Q=\frac{16}{3}=5.33$

(c) $\int_2^4\mathrm{f}(Q)\mathrm{d}Q=\frac{32}{3}=10.67$ area above axis

$\int_0^2\mathrm{f}(Q)\mathrm{d}Q=-\frac{16}{3}=-5.33$ area below axis

22. (a)

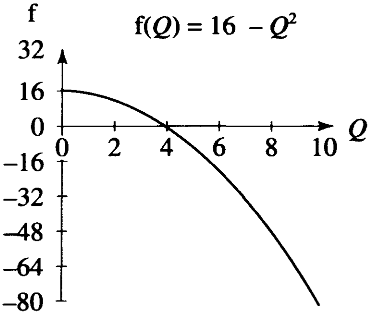


Figure PE 8.3 Q22

(b) Net area $=\int_0^{10}\mathrm{f}(Q)\mathrm{d}Q=-173.33$

(c) $\int_0^4\mathrm{f}(Q)\mathrm{d}Q=42.67$ area above axis.

$\int_4^{10}\mathrm{f}(Q)\mathrm{d}Q=-216$ area below axis

23. (a)

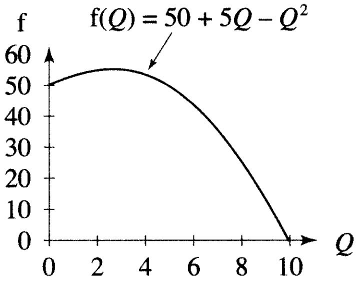


Figure PE 8.3 Q23

(b) Net area $= \int_0^{10} f(Q)dQ = 416.67$ = area above axis

24. (a)

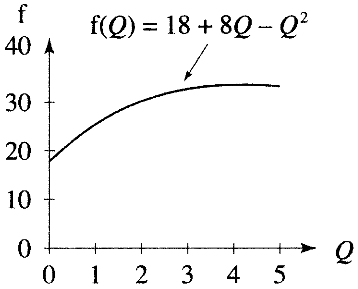


Figure PE 8.3 Q24

(b) Net area $= \int_0^{5} f(Q)dQ = 148.33$ = area above axis

25. (a)

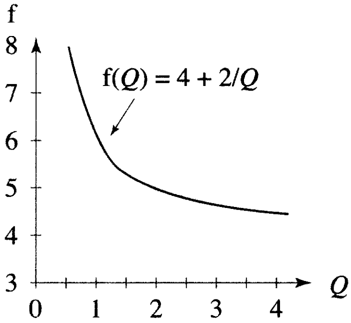


Figure PE 8.3 Q25

(b) Net area $= \int_{0.5}^{4} f(Q)dQ = 18.16$ = area above axis

8.4

1. (a)

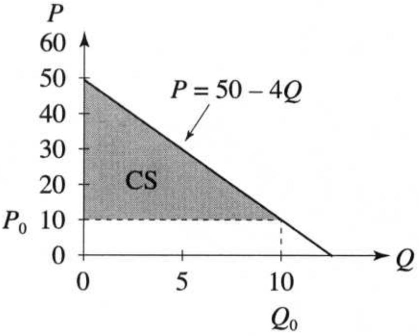


Figure PE 8.4 Q1

(b) $CS = 200$

2. (a)

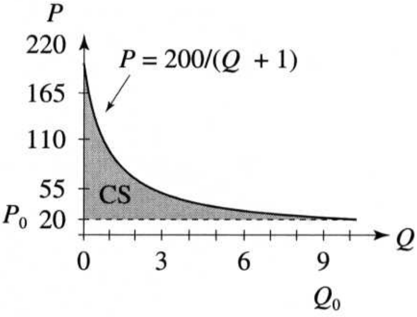


Figure PE 8.4 Q2

(b) $CS = 280.52$

3. (a)

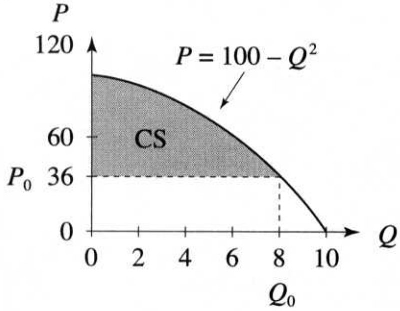


Figure PE 8.4 Q3

(b) $CS = 341$

4. (a)

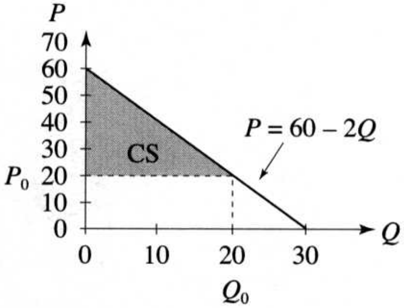


Figure PE 8.4 Q4

(b) $CS = 400$

5. (a)

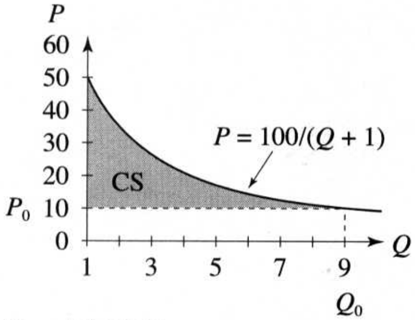


Figure PE 8.4 Q5

(b) $CS = 140.26$

6. (a)

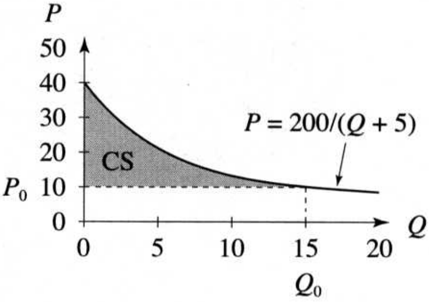


Figure PE 8.4 Q6

(b) $CS = 127.26$

7. (a)

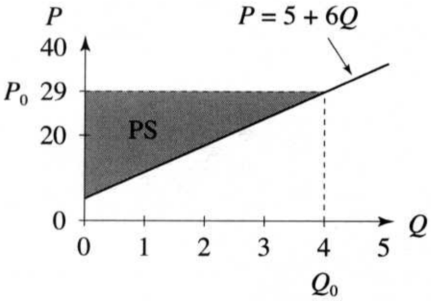


Figure PE 8.4 Q7

(b) $PS = 48$

8. (a)

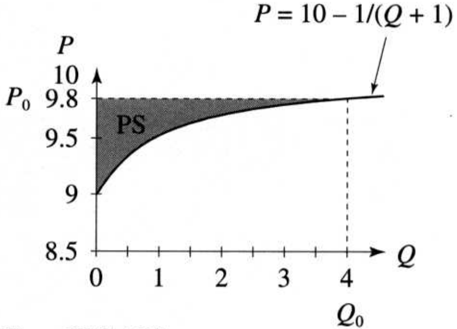


Figure PE 8.4 Q8

(b) $PS = 0.81$

9. (a)

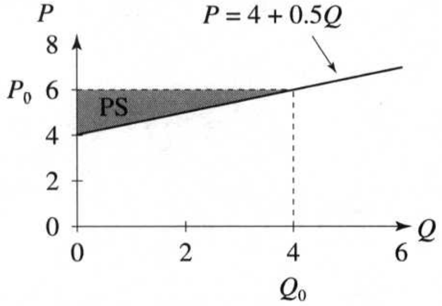


Figure PE 8.4 Q9

(b) $PS = 4$

10. (a)

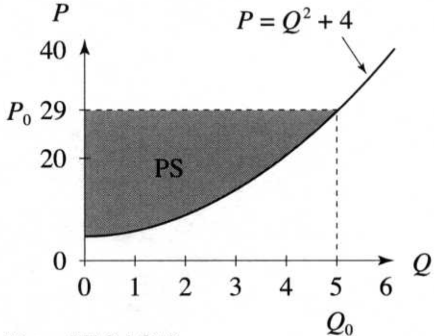


Figure PE 8.4 Q10

(b) $PS = \dfrac{250}{3} = 83.33$

11. (a) $Q = 6$, $P = 38$

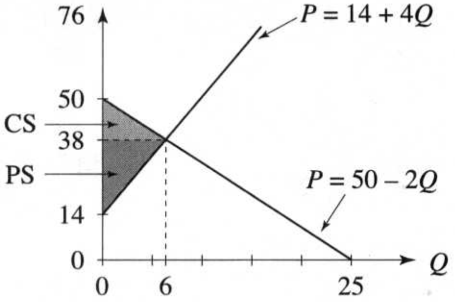


Figure PE 8.4 Q11

(b) $CS = 36$, $PS = 72$

12. (a) $Q = 90$, $P = 55$

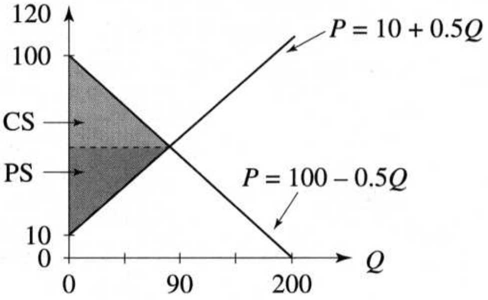


Figure PE 8.4 Q12

(b) $CS = 2025$, $PS = 2025$

13. (a) Equilibrium at $Q = 4$, $P = 34$ and $Q = -10$, $P = -50$ (not economically meaningful)

(b) $CS = 42.67$, $PS = 48$

14. (a)

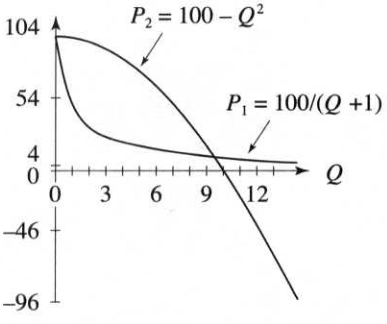


Figure PE 8.4 Q14

(b)

$$P_1 = \frac{100}{Q+1}$$

(i) $TR = 239.79$ for first ten units
(ii) $TR = 36.77$ for $Q = 8$ to $Q = 12$

$$P_2 = 100 - Q^2$$

(i) $TR = 666.67$ for first ten units
(ii) $TR = -5.33$ for $Q = 8$ to $Q = 12$

Revenue from market 2 exceeds that from market 1 until $Q \cong 9.5$. However, revenue from market 2 is zero at $Q = 10$, and negative for $Q > 10$, while revenue from market 1 is always positive but small

15.

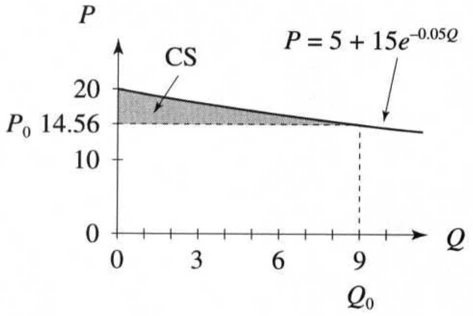


Figure PE 8.4 Q15

$CS = 22.6716$

16.

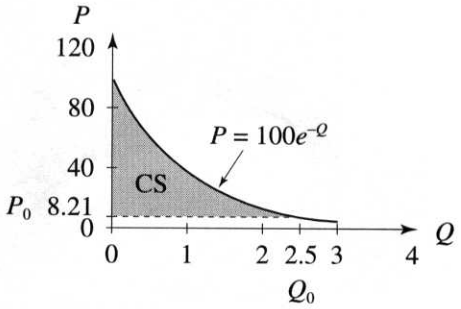


Figure PE 8.4 Q16

$CS = 71.27$

17. Rearrange: $P = e^{4-0.1Q}$

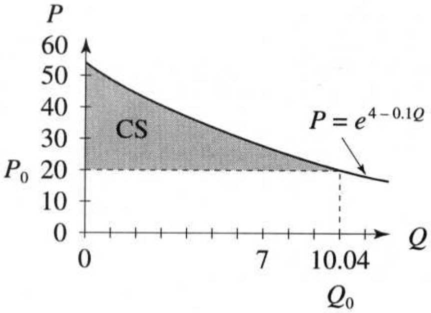


Figure PE 8.4 Q17

$CS = 144.2261$

18.

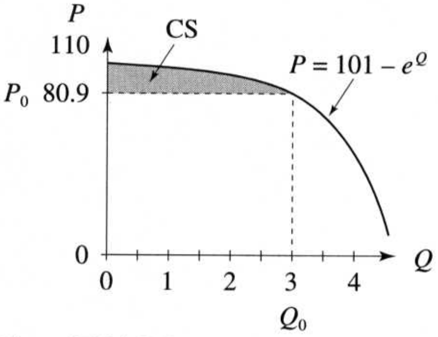
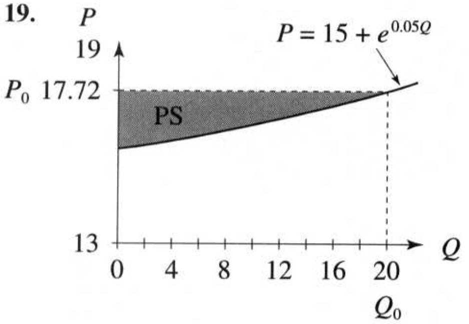


Figure PE 8.4 Q18

$CS = 41.21$

19.



FFigure PE 8.4 Q19

$PS = 20$

20.

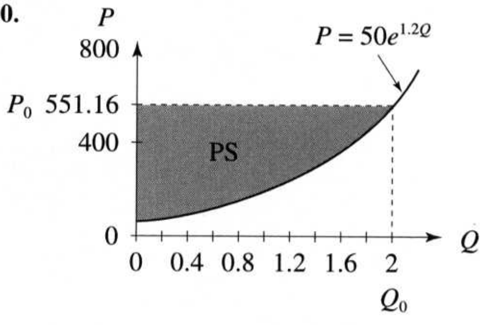


Figure PE 8.4 Q20

$PS = 684.6853$

8.5

1. $y = \frac{15x^2}{2} + c$ 2. $y = 5t^2 - t + c$ 3. $y = \ln x + c$

4. $C = 5Q^2 + c$ 5. $P = \frac{Q^3}{3} + c$ 6. $y = 2e^{0.5t} + c$

7. $y = 2.5x^2 - 6$ 8. $y = 5t^2 + 2t + 15$ 9. $y = -0.5x + 1.5\ln x + 6.5$

10. $C = 10\ln Q - 0.5Q^2 + 50.5$ 11. $P = \frac{Q^3}{3} + 2.5Q^2 + 10$

12. $y = 20e^{0.6t} + 60$

8.6

1. (a) $MC = \frac{\mathrm{d}}{\mathrm{d}Q}(TC)$ (b) $TC = -\frac{Q^3}{3} + 40Q^2 + 500$ (c) 4833 $\left(\int_{Q=3}^{Q=12} MC\,\mathrm{d}Q\right)$

2. (a) $MPC = \frac{\mathrm{d}}{\mathrm{d}Y}(C)$ (b) $C = 26.25 - 1.25e^{-0.8Y}$

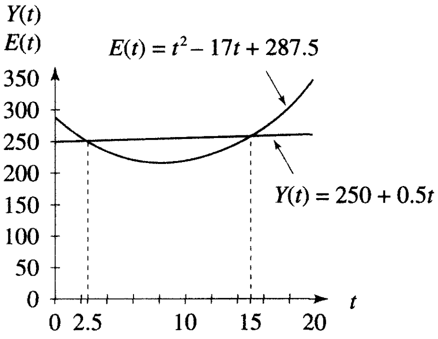
3. (a) $MR = \frac{\mathrm{d}}{\mathrm{d}Q}(TR)$ (b) $TR = 120Q - Q^2$

4. (a) $TC = \frac{Q^3}{3} - 21Q^2 + 200Q + 900$ (b) 852

5. (a) $MLC = \frac{\mathrm{d}}{\mathrm{d}L}(TLC)$ (b) $TLC = 3L + 2L^2$ (c) 114

6. (a) $MP_L = \frac{\mathrm{d}Q}{\mathrm{d}L}$ (b) $Q = 10L - L^2$ (c) 9

7. (a)



Break-even at $t = 2.5$ and $t = 15$

Figure PE 8.6 Q7(a)

(b) (i) −44.271 (loss); (ii) 325.521

8. (a)

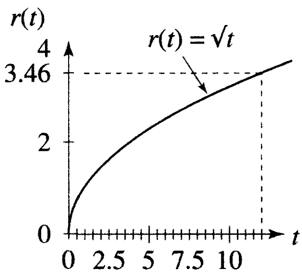


Figure PE 8.6 Q8(a)

(b) 980 gallons

9. (a)

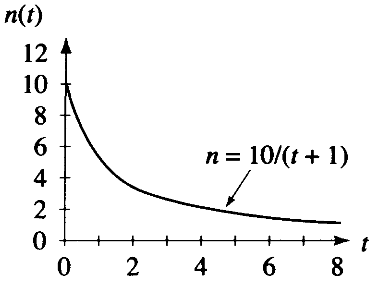


Figure PE 8.6 Q9

(b) $T = 10\ln(t+1)$

(c) (i) 24; (ii) 6.5; (iii) 3.9

10. (a)

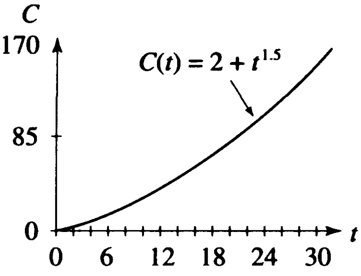


Figure PE 8.6 Q10

(b) (i) 32.36; (ii) 114.13

(c) (i) $AC = 6.472$; (ii) $AC = 22.826$

8.7

1. $y(t) = 50e^{2t}$

2. $y = 800e^{-0.1t}$

3. $N = 1807e^{-0.4t}$

4. $y^2 = 2x^2 - 100$

5. $y = 50e^{t^3}$

6. $P = 21 + 20t - \dfrac{t^2}{12}$

7. $y = 20e^{x^2/2}$

8. $I = 900e^{-0.05t}$

9. $Q = \dfrac{1}{P}$

10. $P = 120e^{-Q}$

8.8

1. (a) $A = Ce^{0.02t}$. The amount of ore mined increases at a constant proportional rate

(b) $t = 0$ at the beginning of 1989, then $A = 400e^{0.02t}$

(c) $t = 1995 - 1989 = 6$ at the beginning of 1995; $t = 2000 - 1989 = 11$ at the beginning of 2000

$$\text{Therefore, ore mined} = \int_{t=6}^{t=11} 400e^{0.02t}\,\mathrm{d}t = 2372 \text{ million tons}$$

2. (a) $P = Ce^{0.01t}$ (b) $P = 58.6e^{0.01t}$; $P = 441.75$ million at the beginning of 2200

(c) $P = 70$ million when $t = 17.78$ years

$P = 70$ million in $(1998 + 17.78) = 2015.78$, i.e. during 2015

3. $P = 50e^{-0.2t}$

(a) (i) $P = 22.47$ = amount present after 4 hours (ii) $P = 42.61$ when $t = 8$

(b) 11.5 hours

4. $I = 12e^{-0.05t}$

(a) $I = 9.11 \times 10^3$ after 5.5 years

(b) $I = 5 \times 10^3$ when $t = 17.5$ years

5. (a) $\dfrac{dQ}{dP}\dfrac{P}{Q} = -1 \rightarrow \dfrac{dQ}{dP} = -\dfrac{Q}{P}$ (b) $Q = \dfrac{576}{P}$

6. (a) $F = 80(1 - e^{-0.12t}) \times 10^3$

(b) Maximum capacity is 80 000

(c) $F = 40\,000$ at $t = 5.78$ years

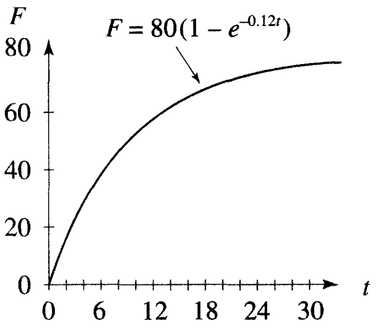


Figure PE 8.8 Q6

7. (a) $C_1 = 50e^{0.01t}$; $C_2 = 50 + 0.8t^{1.25}$

(b) At $t = 5$, $C_1 = 52.56$, $C_2 = 54.67$
C_1 is more cost effective than C_2;
the rate of increase is slower.

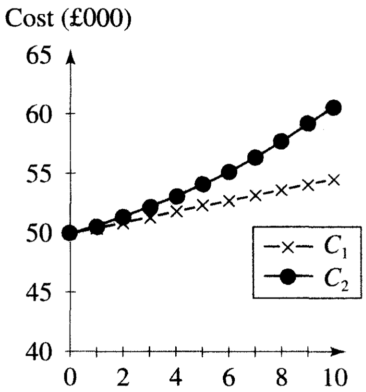


Figure PE 8.8 Q7

8. (a) $\dfrac{dQ}{dP}\dfrac{P}{Q} = -\dfrac{10}{Q}$ (b) $P = 20e^{-0.1Q}$

9. Proportional rate of growth at P_0 is $\left.\dfrac{1}{P_t}\left(\dfrac{dP_t}{dt}\right)\right|_{P_t = P_0}$

$$\frac{1}{P_t}\left(\frac{dP_t}{dt}\right) = \frac{100e^{0.2t}}{500e^{0.2t}} = \frac{1}{5} = 0.2, \text{ a constant rate for any } t$$

Proportional rate = 0.2 at $t = 5$
Proportional rate = 0.2 at $t = 10$

10. Proportional rate $= \dfrac{-68.5e^{-0.1t}}{685e^{-0.1t}} = -0.1$, a constant rate for any t

Proportional rate $= -0.1$ at $t = 5$
Proportional rate $= -0.1$ at $t = 10$

11. Proportional rate $= \dfrac{-2/(t+2)^2}{2/(t+2)} = -\dfrac{1}{t+2}$; this depends on t

Proportional rate $= -0.1429$ at $t = 5$
Proportional rate $= -0.0833$ at $t = 10$
Therefore the rate is not constant

12. Proportional rate $= \dfrac{50t^{1.5}}{20t^{2.5}} = \dfrac{2.5}{t}$; this depends on t

Proportional rate $= 0.5$ at $t = 5$
Proportional rate $= 0.25$ at $t = 10$
Therefore the rate is not constant

Chapter 9

9.1

1. (i)

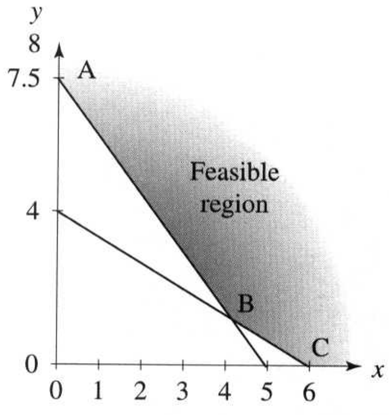


Figure PE 9.1 Q1(i)

(ii)

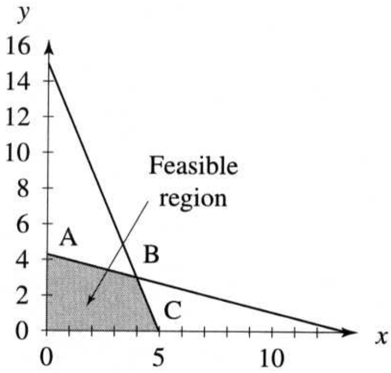


Figure PE 9.1 Q1(ii)

$A = (0, 7.5), B = (4.2, 1.2), C = (6, 0)$

$A = (0, 4.33), B = (4, 3), C = (5, 0)$

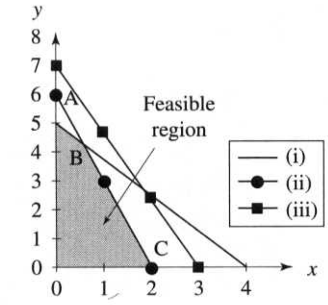


Figure PE 9.1 Q2

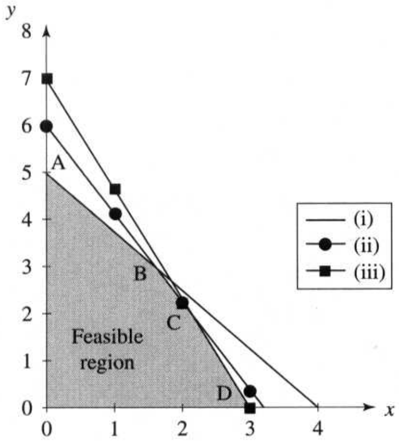


Figure PE 9.1 Q3

4. (a) and (b)

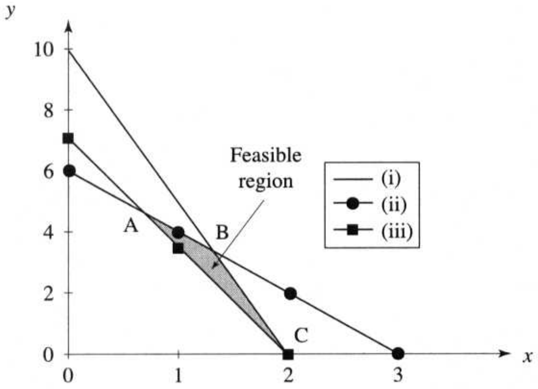


Figure PE 9.1 Q4

(c) A = (0.67, 4.67), B = (1.33, 3.33), C = (2, 0).

5. (a) $y = -0.5x + \frac{TC}{24}$

Slope $= -0.5$.

Isocost: (i) Let $TC = 24$

(ii) Let $TC = 48$

or any two values of TC

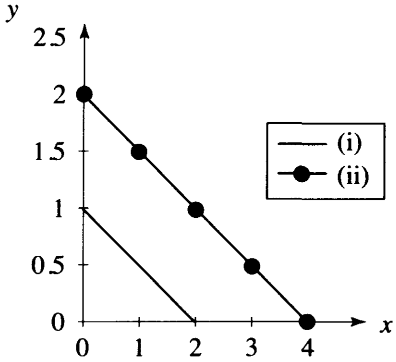


Figure PE 9.1 Q5(a)

(b) $y = -3x + \frac{TR}{3}$

Slope $= -3$.

Isorevenue: (i) Let $TR = 9$

(ii) Let $TR = 18$

or any two values of TR

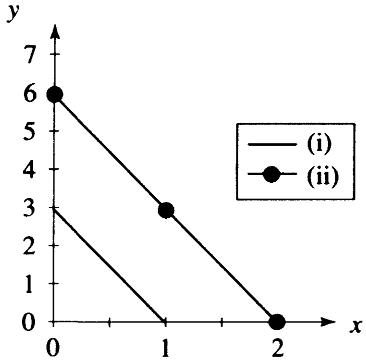


Figure PE 9.1 Q5(b)

(c) $y = -\frac{5.5}{2.85}x + \frac{\pi}{2.85}$

Slope $= -\frac{5.5}{2.85}$

Isoprofit: (i) Let $\pi = 5.5(2.85)$

(ii) Let $\pi = 10(2.85)$

or any two values of π

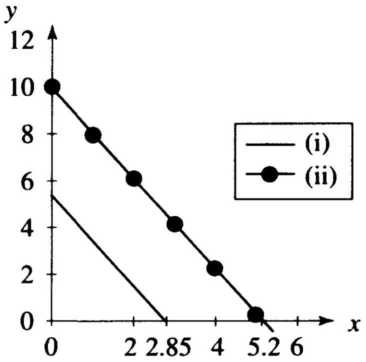


Figure PE 9.1 Q5(c)

6. (a) Algebraically: Corner points of the feasible region were determined in question 3:

A = (0, 5), B = (1.6, 3), C = (2.18, 1.91), D = (3, 0). Let W = objective function.

$W = 10$ at A; $W = 10.8$ at B; $W = 10.36$ at C; $W = 9$ at D.

Therefore W is a maximum at point B: $x = 1.6$, $y = 3$.

(b) Graphically, the function to be maximised (objective function) is drawn:

$y = -1.5x + \frac{W}{2}$. This is a maximum at B.

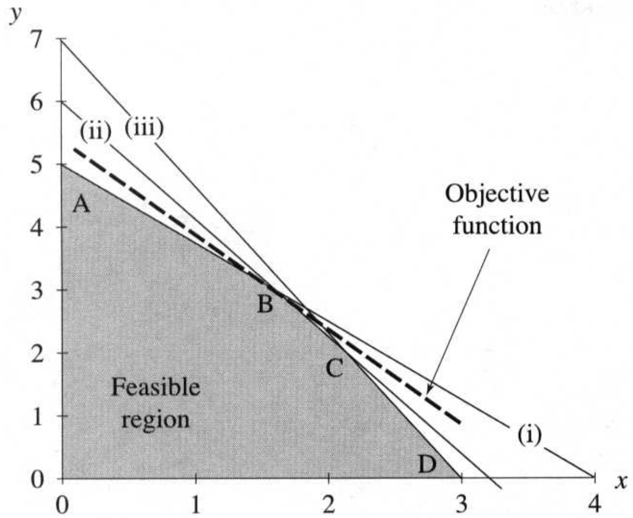


Figure PE 9.1 Q6(b)

7. (a) Algebraically, the minimum value of the objective function is at point D, when $W = 9$.

   (b) Graphically, the objective function is a minimum at D.

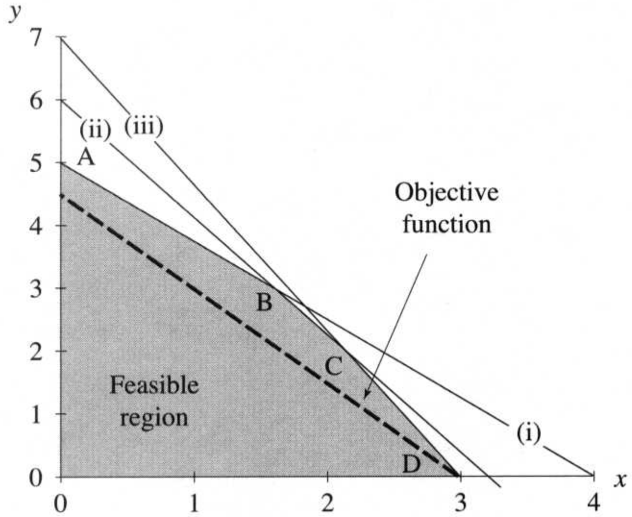


Figure PE 9.1 Q7

8. (a)

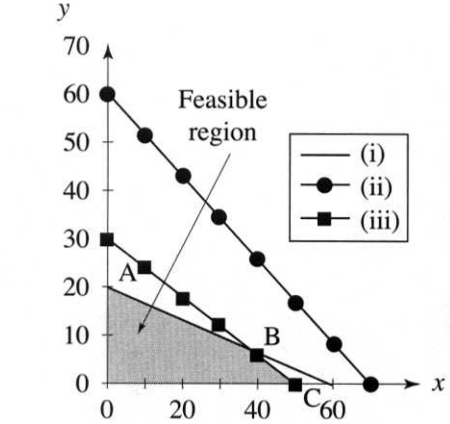


Figure PE 9.1 Q8

(b) Constraint (ii), $60x + 70y \leq 4200$, is not a limitation.

(c) Corner points:

$A = (0, 20), B = (37.5, 7.5), C = (50, 0)$

9. (a)

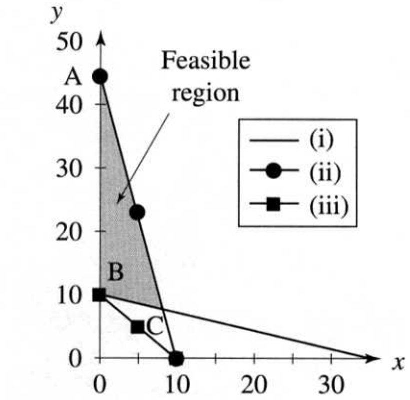


Figure PE 9.1 Q9

(b) Constraint (iii), $x + y \geq 10$, is not a limitation.

(c) Corner points:

$A = (0, 44), B = (0, 10), C = (8.32, 7.40)$

10. (a) (i) $6A + 3B \leq 360$

(ii) $2A + 4B \leq 280$

(iii) $2A + 2B \leq 160$

$\pi = 20A + 30B$

(b)

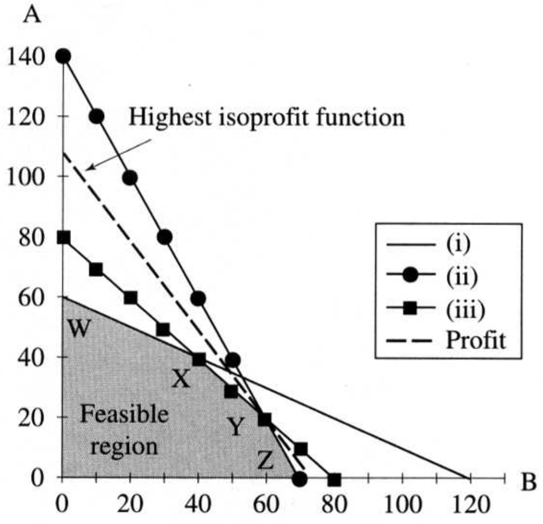


Figure PE 9.1 Q10

$W = (0, 60); \quad X = (40, 40); \quad Y = (60, 20); \quad Z = (70, 0)$

(c) Profit at $W = 20(60) + 30(0) = 1200$

Profit at $X = 20(40) + 30(40) = 2000$

Profit at $Y = 20(20) + 30(60) = 2200$

Profit at $Z = 20(0) + 30(70) = 2100$

Therefore maximum profit at Y. 20 A, 60 B machines used.

11. (a) (i) $4y + 20x \geq 80$

(ii) $y + 2x \geq 15$

(iii) $2y + x \geq 10$

$C = 18x + 6y$

(b)

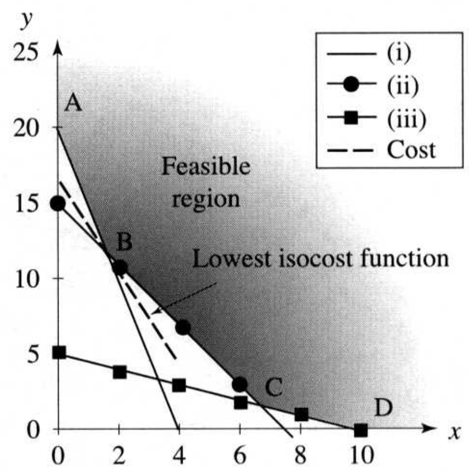


Figure PE 9.1 Q11

A = (0, 20); B = (1.67, 11.65); C = (6.67, 1.66); D = (10, 0)

(c) Cost at A = $18(0) + 6(20) = 120$

Cost at B = $18(1.67) + 6(11.65) = 100$

Cost at C = $18(6.67) + 6(1.66) = 130$

Cost at D = $18(10) + 6(0) = 180$

Therefore, minimum cost at B. The gardener should use $1\frac{2}{3}$ kg of brand X and 11.65 kg of brand Y.

9.2

1. (a) $\begin{pmatrix} 5 & -1 \\ -7 & -9 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & -7 \\ 7 & -9 \end{pmatrix}$ (c) $\begin{pmatrix} 17 & 8 \\ -28 & -9 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -4 \\ 0 & -8 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -4 \\ 0 & -9 \end{pmatrix}$ (f) A + C not possible: different dimensions.

(g) $\begin{pmatrix} 5 & -11 \\ 3 & -9 \end{pmatrix}$ (h) $\begin{pmatrix} 56 & -4 & 2 \\ -35 & 7 & 7 \end{pmatrix}$ (i) Not possible: CB = $(2 \times 3)(2 \times 2)$

(j) Not possible: $CB^T = (2 \times 3)(2 \times 2)$ (k) $\begin{pmatrix} 32 & 63 \\ 3 & 0 \end{pmatrix}$ (l) Not possible

(m) $\begin{pmatrix} 5 & -128 \\ -1 & 4 \\ -1 & -14 \end{pmatrix}$ (n) $\begin{pmatrix} 56 & -35 \\ -4 & 7 \\ 2 & 7 \end{pmatrix}$ (o) Not possible

2. (a) A + C not possible: different dimensions. (b) $\begin{pmatrix} 2 & 0 \\ 4 & -4 \\ -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 5 & -10 \\ 3 & -6 \end{pmatrix}$ (d) $(5 \quad 4)$ (e) Not possible

3. Let quantity sold matrix = Q and profit per unit matrix = P.

(a) π, total profit = QP^{T} (a 3 × 3 matrix) where

π_{11} = profit for burgers = £485

π_{22} = profit for chips = £1165

π_{33} = profit for drinks = £750

(b) Profit for each shop (unit profit × quantity for all three items)

= $Q^{\mathrm{T}}P$ (a 3 × 3 matrix) where

πs_{11} = profit for shop A = £785

πs_{22} = profit for shop B = £850

πs_{33} = profit for shop C = £765

4. Let R = party by area matrix and V = voters in areas 1, 2 and 3.

(a) Votes per candidate, X

$X = R^{\mathrm{T}}V$ where

X_{11} = votes for A = 79240

X_{22} = votes for B = 52400

X_{33} = votes for C = 51360

5. (a) $\mathrm{AB} = \begin{pmatrix} 1 & 0 \\ -3 & 10 \end{pmatrix}$. $\mathrm{BA} = \begin{pmatrix} 4 & 3 \\ 6 & 7 \end{pmatrix}$

(b) (i) AC not possible as (2 × 2)(3 × 1)

(ii) $\mathrm{AD} = \begin{pmatrix} 6 & 1 & 3 \\ 12 & 7 & 11 \end{pmatrix}$ (iii) $\mathrm{DC} = \begin{pmatrix} 14 \\ 8 \end{pmatrix}$ (iv) $\begin{pmatrix} 14 & 70 & 42 \\ 8 & 40 & 24 \end{pmatrix}$

9.3

1. (a, b) $x = 279/4$, $y = -87/4$, $z = -26$

2. (a) $x = 7$, $y = 5$, $z = -9.5$ (b) $x = 15.33$, $y = -3.33$, $z = -13.67$

(c) On completing the elimination, the equation O = −4 is obtained. This contradiction indicates no solution.

(d) $x = 26/3$, $y = -6$, $z = -14/3$

3. (a) $x = 1, y = 21, z = -17$

4. $P_1 = 26, P_2 = 38, P_3 = 66$

5. $Y = 500, C = 100, T = 100$

6. $x = 12/5, y = 3/5, z = 0$

7. $x = 3.8, y = 1.53, z = 1.53$

8. $P_1 = 7, Q_1 = 51; P_2 = 9, Q_2 = 103; P_3 = 10, Q_3 = 39$

9.4

1. (a) 3 (b) −2 (c) 35 (d) 0 (e) 6

(f) 0 (g) $c^2 - 2c$ (h) a (i) $1 - \frac{a}{1-a}$

2. $x = 0; y = 5$ **3.** $x = 4.6; y = -2.8$ **4.** $P_1 = 0.73; P_2 = 3.59$

5. $P = 56.92; Q = 5.38$ **6.** $P = 4.27; Q = -2.96$

7. $Y = 37.6; r = 88$ **8.** $P_1 = 15.09, Q_1 = 39.27; P_2 = 6.09, Q_2 = 69.08$

9. (a) $Y - C = I_0; -bY + C = C_0$

(b) $Y_e = \frac{I_0 + C_0}{1-b}; \quad C_e = \frac{C_0 + bI_0}{1-b}$

10. (a) $0.8Y + 0.4r = 247.8; \quad 0.1Y - 0.3r = 29.225$

(b) $Y = 307.25; \quad r = 5$

11. Write equations $Q = f(P)$, hence solve for P_1, P_2 by Cramer's rule.
$P_1 = 0.6Q_1 + 0.8Q_2 - 3; \quad P_2 = 0.4Q_1 + 0.2Q_2 - 2$

9.5

1. (a) −8 (b) −78 (c) −234

2. (a) $x = 7; y = 5; z = -9.5$ (b) $x = \frac{46}{3}; y = -\frac{10}{3}; z = -\frac{41}{3}$

(c) No unique solution, $\Delta = 0$. (d) $x = \frac{26}{3}; y = -6; z = -\frac{14}{3}$

3. $x = 1; y = 1; z = 1$ **4.** $P_1 = 2; P_2 = 4; P_3 = 1$

5. $Y = 500$; $C = 100$; $T = 100$. **6.** $x = \frac{12}{5}$; $y = \frac{3}{5}$; $z = 0$

7. $x = 3.8$; $y = 1.53$; $z = 1.53$

8. $P_1 = 7$, $Q_1 = 51$; $P_2 = 9$, $Q_2 = 103$; $P_3 = 10$, $Q_3 = 39$

9. (b) $P_1 = 10.5$; $P_2 = 3.6$; $P_3 = 11.8$

9.6

1. (a) $\begin{pmatrix} 3 & -2.5 \\ -1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -0.5 & 0.2 \\ 1 & -0.3 \end{pmatrix}$ (c) $\begin{pmatrix} -\frac{5}{11} & \frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$

(d) $\begin{pmatrix} 0 & \frac{1}{12} \\ \frac{1}{4} & -\frac{1}{16} \end{pmatrix}$ (e) No inverse since $|A| = 0$

2. (a) $A^{-1} = \begin{pmatrix} 1 & -\frac{1}{4} & \frac{1}{12} \\ 0 & \frac{1}{4} & -\frac{1}{12} \\ -1 & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$ (b) $x = 7$; $y = 5$; $z = -9.5$

3. (a) $A^{-1} = \begin{pmatrix} 1 & \frac{1}{6} & -\frac{1}{18} \\ 0 & -\frac{1}{6} & \frac{1}{18} \\ -1 & -\frac{1}{12} & \frac{7}{36} \end{pmatrix}$ (b) $x = \frac{46}{3}$; $y = -\frac{10}{3}$; $z = -\frac{41}{3}$

4. (a) $A^{-1} = \begin{pmatrix} \frac{7}{4} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{5}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$ (b) $x = 8.5$; $y = -6.5$; $z = 2.5$

5. (a) No inverse since $|A| = 0$ (b) No unique solution.

6. (i) (a) $(I - A)^{-1} = \begin{pmatrix} 3 & 2 \\ 1.25 & 2.5 \end{pmatrix}$ (b) Total output = $\begin{pmatrix} 320 \\ 300 \end{pmatrix}$

(ii) (a) $(I - A)^{-1} = \begin{pmatrix} 2.5 & 1.25 \\ 1.0 & 2.5 \end{pmatrix}$ (b) Total output = $\begin{pmatrix} 800 \\ 1120 \end{pmatrix}$

7. (a) $(I-A)^{-1}=\begin{pmatrix}2.4 & 0.8 & 0.4\\ 2.0 & 4.0 & 2.0\\ 2.6 & 4.2 & 4.6\end{pmatrix}$ (b) 728; 2440; 4372

8. (a) $(I-A)^{-1}=\begin{pmatrix}2.9 & 1.8 & 0.95\\ 1.0 & 2.0 & 0.5\\ 1.7 & 1.4 & 1.85\end{pmatrix}$ (b) 5783; 3770; 4709

9. (a) $A=\begin{pmatrix}0.5 & 0.5\\ 0.5 & 0.25\end{pmatrix}$ (b) 7000 units from X; 6000 units from Y

10. (a) $A=\begin{pmatrix}0.1 & 0.1 & 0.25\\ 0.2 & 0.4 & 0\\ 0.4 & 0 & 0.1\end{pmatrix}$

(b) 808.81 from Horticulture; 1602.9 from Dairy; 1470.6 from Poultry

11. (a) $A=\begin{pmatrix}0.2 & 0.125 & 0.25\\ 0.1 & 0 & 0.625\\ 0.1 & 0.1 & 0\end{pmatrix}$

(b) 1086.8; 1281.7; 1236.8 from sectors A, B and C respectively.

12. (a) $A=\begin{pmatrix}0.4 & 0.2 & 0\\ 0.1 & 0.4 & 0.2\\ 0.2 & 0.2 & 0.1\end{pmatrix}$

(b) 1719.0 from Dairy; 1406.9 from Beef; 2361.3 from Leather

Chapter 10

10.1

1. (a) Order 1, homogeneous. (b) Order 1, non-homogeneous.

(c) Order 2, non-homogeneous.

2. (a) 5, 14, 21.2, 26.96, 31.568, therefore $Y_5 = 31.568$

(b) 20, 18, −88, −496, −1280, therefore $P_5 = -1280$

(c) 100, 140, 164, 178, 187, therefore $P_5 = 187$

3. (a) $Y_t = A(0.8)^t$. $Y_t = 6.25(0.8)^t$: $Y_{20} = 0.072$, $Y_{50} = 0.000089$.

(b) $Y_t = A(-0.8)^t$. $Y_t = -6.25(-0.8)^t$: $Y_{20} = -0.072$, $Y_{50} = -0.000089$.

(c) $Y_t = A(0.8)^t + 50$. $Y_t = -56.25(0.8)^t + 50$: $Y_{20} = 49.35$, $Y_{50} = 49.999$

4. (a) Stable. $Y_t = A(0.8)^t$ decreases to 0 as t increases. See Y_{20} and Y_{50}.

(b) Stable: Y_t oscillates to 0 as t increases.

(c) Stable: Y_t increases to 50 as t increases.

5. (a) $Y_t = A(10)^t - 100$; $Y_t = 120(10)^t - 100$. Not stable.

(b) $P_t = A(0.9)^t + 800$; $P_t = -700(0.9)^t + 800$. Stable, P_t increases to 800.

6. (a) Price increases by 5% each season.

(b) $P_t = 6000(1.05)^t$. Price will increase indefinitely.

7. (a) (ii) is different: $P_{t+1} = \dfrac{1}{1.05} P_t$

(b) (ii) is different, the index should be $(n - 2)$.
(iii) is different, the index should be $(n - 1)$.

8. (a) Investment in any year is 90% of that in the previous year, plus £60m (an injection of £60m)

(b) (i) $I_0 = £15\text{m}$, then $I_t = -585(0.9)^t + 600$. Investment increases to £600m. Reaches £400m in 10.1 years.

(ii) $I_0 = £180\text{m}$, then $I_t = -420(0.9)^t + 600$. Investment increases to £600m, more slowly than (i). Reaches £400m in 7 years.
(iii) $I_0 = £900\text{m}$, then $I_t = 300(0.9)^t + 600$. Investment decreases to £600m, therefore never drops to £400m.

9. (a) $Y_t = A(0.5)^t - 10$

(b) $Y_t = 360(0.5)^t - 10$. Stabilises, decreases steadily to -10. $Y_t = 0$ at $t = 5.17$

10. (a) $P_n = 1.05P_{n-1} - 8000$ (b) $P_n = A(1.05)^n + 160\,000$

(c) If $P_0 = 100\,000$, $P_n = -60\,000(1.05)^n + 160\,000$

11. (a) $Y_t = A(0.6)^t + 30(2)^t$

(b) $Y_t = A(0.7)^t + 120(0.8)^t$. $Y_t = -204.71(0.7)^t + 120(0.8)^t$

12. (a) $P_t = 186(0.75)^t + 4(3)^t$. Not stable.

(b) $Y_t = 8.75(0.84)^t + 31.25$. Stable.

10.2

1. (a) $P_{t+1} = P_t + rP_t = (1 + r)P_t$

(b) Solve $P_{t+1} - (1 + r)P_t = 0$

$$P_t = A(1 + r)^t$$

When $t = 0$, $P_t = P_0$, therefore $P_0 = A(1+r)^0 = A$, therefore $A = P_0$.

$$\text{Therefore } P_t = P_0(1+r)^t$$

2. (a) $Y_t = A(0.8)^t + 3000$. Stabilises to 3000.
If $A > 0$, Y_t decreases to 3000. If $A < 0$, Y_t increases to 3000.

(b) $Y_t = A(0.75)^t + 1750(1.05)^t$. Not stable, increases indefinitely.

3. (a) $P_t + P_{t-1} = 42$. $P_t = A(-1)^t + 21$. Price oscillates about 21.

(b) $P_t = 10.5(-1)^t + 21$. Price alternates between 31.5 and 10.5.

4. (a) $P_{t+1} - 0.88P_t = 54$

(b) $P_t = A(0.88)^t + 450$

(c) Price stabilises to 450.

5. (a) $0.12Y_t = 4.5(Y_t - Y_{t-1}) \rightarrow 4.38Y_t - 4.5Y_{t-1} = 0$

(b) $Y_t = 314(1.027)^t$

(c) $I_t = 4.5(Y_t - Y_{t-1}) = 4.5(314)(1.027)^t\{1 - (1.027)^{-1}\} = 37.148(1.027)^t$

Therefore, investment will continue to grow. All units in £m.

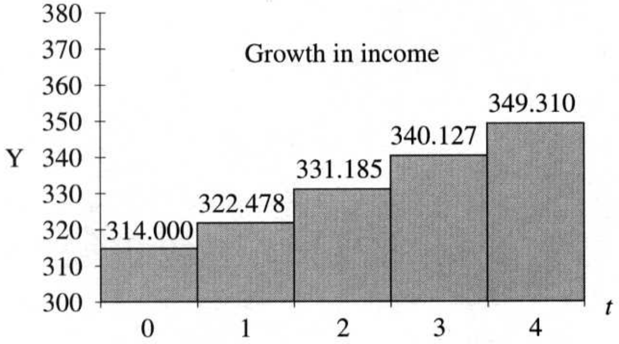


Figure PE 10.2 Q5(c)(i)

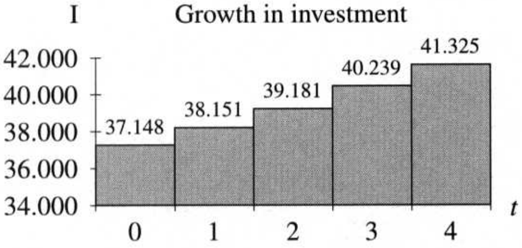


Figure PE 10.2 Q5(c)(ii)

6. (a) $-0.255Y_t = 0.9(Y_t - Y_{t-1}) \rightarrow 1.155Y_t - 0.9Y_{t-1} = 0$

(b) $Y_t = 800(0.779)^t$

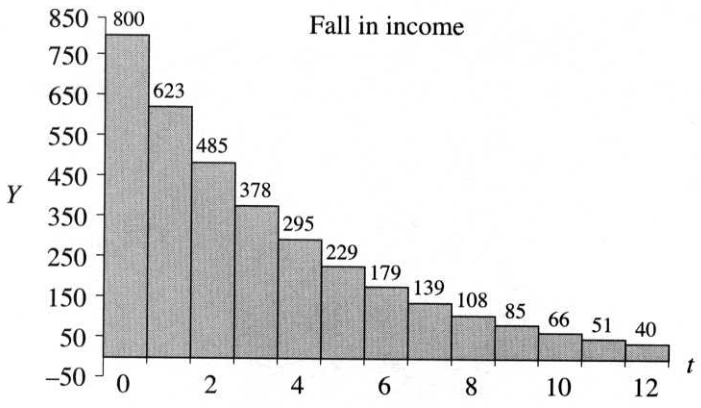


Figure PE 10.2 Q6(b)

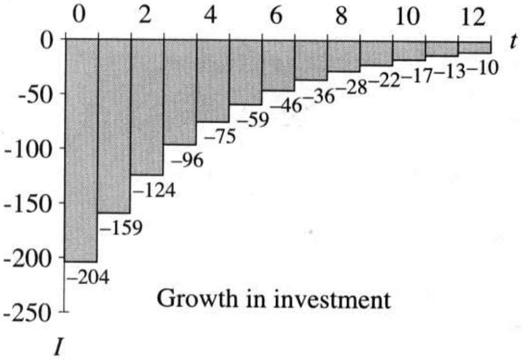


Figure PE 10.2 Q6(c)

(c) $I_t = 0.9(Y_t - Y_{t-1}) = 0.9(800)(0.779)^t\{1 - (0.779)^{-1}\} = -204.26(0.779)^t$

Income is decreasing, but investment is increasing. However, investment is negative (borrowing instead of investing). Investment eventually stabilises to zero. All units in £m.

Bibliography

Anthony, M. and Biggs, N., *Mathematics for Economics and Finance*, Cambridge University Press, Cambridge, 1996.

Binmore, K.G., *Mathematical Analysis*, 2nd edition, Cambridge University Press, Cambridge, 1982.

Binmore, K.G., *Calculus*, 2nd edition, Cambridge University Press, Cambridge, 1983.

Black, J. and Bradley, J.F., *Essential Mathematics for Economists*, 2nd edition, John Wiley, Chichester, 1980.

Chiang, Alpha C., *Fundamental Methods in Mathematical Economics*, 3rd edition, McGraw-Hill, New York, 1984.

Dowling, E.T., *Introduction to Mathematical Economics*, 2nd edition, Schaum's Outline Series, McGraw-Hill, New York, 1992.

Glass, J.C., *An Introduction to Mathematical Methods in Economics*, McGraw-Hill, New York, 1980.

Hartley, P.J. and Wynn-Evans, A., *Numerical Mathematics*, Stanley Thornes, Cheltenham, 1979.

Hatton, M.D., *Mathematical Analysis*, Hodder and Stoughton Educational, London, 1977.

Lipsey, R. and Chrystal, A., *Introduction to Positive Economics*, 8th edition, Oxford University Press, Oxford, 1995.

Ostaszewewsli, A., *Mathematics in Economics: Models and Methods*, Blackwell, Oxford, 1993.

Sloman, J., *Economics*, 2nd edition, Harvester Wheatsheaf/Prentice Hall, Hemel Hempstead, 1994.

Turley, G. and Maloney, M., *Principles of Economics, An Irish Textbook*, Gill and Macmillan, London, 1997.

Wisniewski, M., *Introductory Mathematical Methods in Economics*, 2nd edition, McGraw-Hill, Maidenhead, 1996.

Worked Examples

Items in italics are on the web site
not in the textbook

1 Mathematical preliminaries

1.1 Addition and subtraction 3
1.2 Multiplication and division 3
1.3 Add and subtract fractions 5
1.4 Multiplying fractions 6
1.5 Division with fractions 6
1.6 Solving equations 8
1.7 Currency conversions 10
1.8 Solving a variety of equation types 11
1.9 Solving simple inequalities 15
1.10 Calculations with percentages 17
1.11 Using Excel to perform calculations and plot graphs 21

2 The straight line and applications

2.1 Plotting lines given slope and intercept 33
2.2 Determine the equation of a line given slope and intercept 35
2.3 Calculation of horizontal and vertical intercepts 39
2.4 To graph a straight line from its equation $y = mx + c$ 41
2.5 Plot the line $ax + by + d = 0$ 44
2.6 Linear demand function 54
2.7 Linear supply function 58
2.8 Further analysis of the linear supply function 59
2.9 Linear total cost function 62
2.10 Linear total revenue function 64

2.11 Calculating slope given two points on the line 66
2.12 Equation of a line given slope and a point on the line 68
2.13 Equation of a line given two points on the line 69
2.14 *Vertical translations of linear functions*
2.15 *Horizontal translations of linear functions*
2.16 *Horizontal and vertical translations of linear functions*
2.17 *The effect of an excise tax on a supply function*
2.18 *Translating a linear cost function*
2.19 Determining the coefficient of point elasticity of demand 74
2.20 *Calculating the coefficient of point elasticity at different prices*
2.21 *Calculating the coefficient of income elasticity of demand*
2.22 *Graphing and interpreting the budget constraint*
2.23 *Budget constraint, changes in prices and income*
2.24 Use Excel to show the effect of price and income changes on a budget constraint 81

3 **Simultaneous equations**
3.1 Solving simultaneous equations 1 90
3.2 Solving simultaneous equations 2 92
3.3 Solving simultaneous equations 3 93
3.4 Simultaneous equations with no solution 95
3.5 Simultaneous equations with infinitely many solutions 96
3.6 Solve three equations in three unknowns 97
3.7 Goods market equilibrium 99
3.8 Labour market equilibrium 100
3.9 Goods market equilibrium and price ceilings 102
3.10 Labour market equilibrium and price floors 103
3.11 Equilibrium for two substitute goods 106
3.12 Taxes and their distribution 107
3.13 Subsidies and their distribution 109
3.14 Calculating the break-even point 111
3.15 Consumer and producer surplus at market equilibrium 116
3.16 Equilibrium national income when $E = C + I$ 119
3.17 *Effect of changes in MPC and* I_0 *on* Y_e
3.18 *Equilibrium national income and effect of taxation*
3.19 *Expenditure multiplier with imports and trade balance*
3.20 *IS-LM analysis*
3.21 Cost, revenue, break-even, per unit tax with Excel 123
3.22 Distribution of tax with Excel 124

4 **Non-linear functions and applications**
4.1 Solving less general quadratic equations 134
4.2 Solving quadratic equations 135
4.3 Sketching a quadratic function $f(x) = \pm x^2$ 137

4.4 Comparing graphs of quadratic functions

4.5 Sketching any quadratic equation 138
4.6 Non-linear demand and supply functions 141
4.7 Non-linear total revenue function 142
4.8 Calculating break-even points 144
4.9 Plotting cubic functions 146
4.10 Graphs of more general cubic functions 148
4.11 *TR*, *TC* and profit functions 150
4.12 Graphing exponential functions 152
4.13 Simplifying exponential expressions 156
4.14 Solving exponential equations 159
4.15 Unlimited growth: population growth 161
4.16 Limited growth: consumption and changes in income 163
4.17 Use logs to solve certain equations 168
4.18 Finding the time for the given population to grow to 1750 168
4.19 Graphs of logarithmic functions 170
4.20 Using log rules 172
4.21 Solving certain equations using rule 3 for logs 173
4.22 Solve equations containing logs and exponentials 175
4.23 Sketches of hyperbolic functions 179
4.24 Hyperbolic demand function 180
4.25 Total cost functions with Excel 183

5 Financial mathematics

5.1 Sum of an arithmetic series 190
5.2 Sum of a geometric series 191
5.3 Application of arithmetic and geometric series 192

5.4 Simple interest calculations

5.5 Compound interest calculations 197
5.6 Future and present values with compound interest 197
5.7 Calculating the compound interest rate and time period 198
5.8 Compounding daily, monthly and semi-annually 201
5.9 Continuous compounding 202
5.10 Annual percentage rates 204
5.11 Future value of asset and reducing-balance depreciation 206
5.12 Present value of asset and reducing-balance depreciation 206
5.13 Calculating net present value 208
5.14 *IRR* determined graphically (Excel) and by calculation 210
5.15 Compound interest for fixed periodic deposits 214
5.16 Annuities 215
5.17 Present value of annuities 217
5.18 Mortgage repayments 219
5.19 How much of the repayment is interest? 220
5.20 Sinking funds 222
5.21 The interest rate and the price of bonds 224

5.22 Growth of an investment using different methods of compounding (Excel) 227

6 Differentiation and applications

6.1 Equation for the slope of $y = x^2$ from first principles 236
6.2 Using the power rule 239
6.3 More differentiation using the power rule 242
6.4 Calculating higher derivatives 243
6.5 Determining the rate of change of the slope of a curve
6.6 Calculating marginal revenue given the demand function 246
6.7 Calculating marginal revenue over an interval 247
6.8 Derive marginal cost equation from total cost function 248
6.9 *MR*, *AR* for a perfectly competitive firm and a monopolist 250
6.10 Derive marginal cost from average cost 253
6.11 Deduce the equation for the marginal and average product of labour from a given production function 256
6.12 Find TLC, MLC, ALC given the labour supply functions
6.13 Find TLC, MLC, ALC for a perfectly competitive firm and a monopsonist
6.14 *MPC*, *MPS*, *APC*, *APS* 258
6.15 Calculating the marginal utility
6.16 Finding turning points 261
6.17 Maximum and minimum turning points 266
6.18 Intervals along which a curve is increasing or decreasing 268
6.19 Derived curves 269
6.20 Sketching functions 274
6.21 Maximum *TR* and a sketch of the *TR* function 277
6.22 Break-even, profit, loss and graphs 279
6.23 Maximum and minimum output for a firm over time 281
6.24 Profit maximisation and price discrimination 284
6.25 Profit maximisation for a perfectly competitive firm 285
6.26 Profit maximisation for a monopolist 287
6.27 Curvature of curves: convex or concave towards the origin 295
6.28 Locate the point of inflection 298
6.29 Relationship between the APL and MP_L *functions*
6.30 Point of inflection on the production function: law of diminishing returns
6.31 Relationship between *TC* and *MC* 299
6.32 Relationship between *AC*, *AVC*, *AFC* and *MC* functions 301
6.33 Derivatives of other standard functions 307
6.34 Using the chain rule for differentiation 309
6.35 Using the product rule for differentiation 311
6.36 Using the quotient rule for differentiation 314
6.37 Find *MC* given a logarithmic *TC* function 316
6.38 Demand, *TR*, *MR* expressed in terms of exponentials 317
6.39 Expressions for point elasticity of demand in terms of *P*, *Q* or both for linear and non-linear demand functions 320

6.40 Point elasticity of demand for non-linear demand functions 322
6.41 Constant elasticity demand function 323
6.42 Elasticity, marginal revenue and total revenue

7 Functions of several variables

7.1 Plot isoquants for a given production function 334
7.2 Partial differentiation: a first example 337
7.3 Determining first-order partial derivatives 338
7.4 Determining second-order partial derivatives 341
7.5 Differentials for functions of one variable 345
7.6 Differentials and incremental changes 346
7.7 MP_L and MP_K: increasing or decreasing? 351
7.8 Slope of an isoquant in terms of MP_L, MP_K 354
7.9 Constant, increasing and decreasing returns to scale 358
7.10 Indifference curves and slope 362
7.11 Partial elasticities of demand 365
7.12 Partial elasticities of labour and capital 366
7.13 Use partial derivatives to derive expressions for various multipliers 367
7.14 Optimum points for functions of two variables 371
7.15 Monopolist maximising total revenue for two goods 373
7.16 Maximise profit for a multi-product firm 374
7.17 Monopolist: price and non-price discrimination 376
7.18 Maximising total revenue subject to a budget constraint 379
7.19 Lagrange multipliers and utility maximisation 382
7.20 Use Lagrange multipliers to derive the identity $U_x/U_y = P_X/P_Y$ 384
7.21 Meaning of λ 384
7.22 Maximise output subject to a cost constraint 386
7.23 Minimise costs subject to a production constraint 387

8 Integration and applications

8.1 Using the power rule for integration 397
8.2 Integrating sums and differences, constant multiplied by variable term 399
8.3 Integrating more general functions 400
8.4 Integrating functions containing e^x 402
8.5 Integrating functions of linear functions by substitution 403
8.6 Integrating linear functions raised to a power 405
8.7 More examples on integrating functions of linear functions 406
8.8 Evaluating the definite integral 409
8.9 Definite integral and e^x 411
8.10 Definite integration and net area between curve and x-axis 411
8.11 Definite integration and logs 413
8.12 Using the definite integral to calculate consumer surplus 415
8.13 Using the definite integral to calculate producer surplus 417
8.14 Consumer and producer surplus for exponential functions 419
8.15 Solution of differential equations: $\mathrm{d}y/\mathrm{d}x = \mathrm{f}(x)$ 424

8.16 Find total cost from marginal cost 425
8.17 Differential equations and rates of change 427
8.18 Solving differential equations of the form $dy/dx = ky$ 430
8.19 Solving differential equations of the form $dy/dx = f(x)g(y)$ 432
8.20 Limited growth 434
8.21 Determining the proportional rates of growth 436

9 Linear algebra and applications

9.1 Find the minimum cost subject to constraints 442
9.2 Profit maximisation subject to constraints 446
9.3 Adding and subtracting matrices 454
9.4 Multiplication of a matrix by a scalar 455
9.5 Matrix multiplication 456
9.6 Applications of matrix arithmetic 458
9.7 Solution of a system of equations: Gaussian elimination 462
9.8 More Gaussian elimination 464
9.9 Gauss–Jordan elimination 466
9.10 Using determinants to solve simultaneous equations 469
9.11 Using Cramer's rule to solve simultaneous equations 472
9.12 Find the market equilibrium using Cramer's rule 473
9.13 Use Cramer's rule for the income-determination model 474
9.14 Evaluation of a 3 × 3 determinant 476
9.15 Solve three simultaneous equations by Cramer's rule 476
9.16 Equilibrium levels in the national income model 478
9.17 The inverse of a matrix: elimination method 481
9.18 The inverse of a 3 × 3 matrix 484
9.19 Solve a system of equations by the inverse matrix 487
9.20 Input/output analysis 490
9.21 Use Excel to solve systems of linear equations 495

10 Difference equations

10.1 Solving difference equations by iteration 505
10.2 General solution of a homogeneous first-order difference equation 506
10.3 General and particular solutions of first-order homogeneous difference equations 507
10.4 Stability of solutions of first-order difference equations 509
10.5 Solve non-homogeneous first-order difference equations 1 512
10.6 Solve non-homogeneous first-order difference equations 2 514
10.7 The lagged income model 517
10.8 The cobweb model 520
10.9 The Harrod–Domar growth model 523

Index

Absolute maximum 261
Absolute minimum 261
Abstract models (static) 47
Addition 2–3
  fractions 4–5
  matrices 453, 454
Adjoint matrix 486
Aggregate 50
Aggregate expenditure 118, 120
Amortisation 218
Annual percentage rate (*APR*) 202–4, 230
  progress exercises 204
Annuities 213–22, 230–1
  ordinary 215
  present value 217–18
Annuity factor 217
Antilogs 174
Arc price elasticity formula 78
Arc price elasticity of demand 78
Area between two curves 420–1
Area of triangle 116
Area under a curve 439
  approximation 407–9
  integration 407–8
Arithmetic progression (*AP*) 189
  applications 194–5
  progress exercises 191–2
  sum of terms 231
Arithmetic sequence 189
  characteristics 194
Arithmetic series 189–90, 229
  application 192–4
  sum of 190
Asset and reducing-balance depreciation
  future value 206
  present value 206–7
Asymptotes 178, 181, 273
Average cost (*AC*) 250, 268, 301–5
  deriving marginal cost from 253–4
Average fixed cost (*AFC*) 301–5
Average function (*AF*) 249–55, 329, 352
  applications 245–60
Average product of capital (*APK*) 352
Average product of labour (*APL*) 256–7, 352
Average propensity to consume (*APC*) 257–9
Average propensity to save (*APS*) 258–9
Average revenue (*AR*) 249–50
  and price 250
  for perfectly competitive firm and monopolist 250–3
  function and price 250
Average variable cost (*AVC*) 301–5

Black market and price ceiling 103
Bonds 224
Brackets 1–2, 4, 45
Break-even point 111–12, 128, 144–5, 150–1, 280
Budget constraint 80

  maximum utility for 381–5
  total revenue maximisation subject to 379–80

Calculator 19–20, 26–7
Calculus 236
Capital recovery factor 219
Cash flows, investment project 208
Cells 21
Chain rule for differentiation 308–10, 328
Chords, and slopes of chords 235–6
Circular flow model 50–1
Cobb–Douglas production function 349–51, 356–9, 366, 386, 391
Cobb–Douglas utility function 360
Cobweb model 520–2, 526
Coefficient for price elasticity of demand 73
Coefficient of income elasticity of demand 78
Coefficient of point elasticity of demand 74–6, 322–3
  calculating 77
  categories 77
Coefficient of price elasticity of demand, numerical scale 77
Coefficient of price elasticity of supply 78
Cofactor method 484
Cofactor of an element 483–4
Common denominator 4–5
Common difference 189
Common ratio 189
Complementary function (CF) 511, 512, 514, 518, 521, 526
Complementary goods 52
  market equilibrium 105–7, 128
Compound interest 196
  calculations 197
  compounded at various intervals 204–5
  compounded daily, monthly and semi-annually 201
  compounded several times per year 200
  continuous compounding 202
  formula 196–7, 198
  future value 197–8
  growth of investment using different methods of compounding 227–9
  present value 197–8
  progress exercises 199–200
Compound interest rate
  for fixed deposits at regular intervals of time 213–14
  and time period 198–9
Concave down 292, 293, 297, 329
Concave up 292, 293–4, 297, 329
Constant elasticity demand function 323–4
Constant proportional rates of growth 435–6
Constrained maximum or minimum 378–9
Constrained optimisation 378–90
Constraints
  profit maximisation subject to 446–50
  see also Budget constraint; Cost constraint
Consumer surplus (*CS*) 114–17, 128, 414–17, 439
  and e^x 419–20
Consumption and changes in income 163
Consumption function 120, 163, 257, 258
Consumption with limited growth 163
Continuous functions 504
Cost 61–3
Cost constraint 80
  output maximisation subject to 386–7
Cost function 284
  equation 443
  plotting 444
Cost minimisation subject to production constraint 387–8
Cramer's rule 469–75
  applications 478
  income-determination model 474
  market equilibrium 473–4
  simultaneous equations 476–8
Cross-price elasticity of demand 365, 392
Cubic equations 146
  general properties 149
  graphical solution 151
Cubic functions 133, 146–51
  graphs 147
  plotting 146–7
Cubic total cost functions 133
  graphs 148–9
Cumulative present value factor 217
Currency conversions 9–14
Curvature 291–306, 329
  convex or concave towards the origin 295–6
  definition 292
  economics 293–4
  production function 299
  total cost function 299
Curve sketching 272–6

Debt repayments 213–22, 230–1
Decay curves 154
Decimal places 20
Definite integral 407–14, 439
  consumer surplus (*CS*) 415–17

  evaluating 409–10, 412–13
  e^x 411
  producer surplus (*PS*) 417–19
Definite integration
  and logarithms 412
  applications 420–1
  net area contained between curve and x-axis 411–12
Demand and supply 51–65
Demand curve 54
Demand function 53–6, 284
  and point elasticity of demand 77
  calculating marginal revenue given 246
  equation 53
  hyperbolic 180–2
  in terms of exponentials 317–18
  linear 54–6
  non-linear 141–2
  sketching 279
  slope 53, 54
Demand schedule 55
Denominator 4–5
Dependent variable 45, 504, 505
Depreciation 205–7, 230
Derivatives 238–9, 328, 390
  first 268, 292
  higher 243
  rules for finding 307
  second 268, 292
  slope of a curve at a point 240
  standard functions 306–7
Derived curves 269–72
Determinants 468–80, 483, 499
  2×2 468
  3×3 468, 476–8, 484–6
  definitions 468
  evaluating 468, 476–9
  terminology 483–4
  use to solve equations 469–75
Deterministic model 47
Difference equations 503–27
  applications 517–25
  definition 503–4
  first-order 505–17, 526–7
  general solution 506–8, 514
  homogeneous 505, 506–8, 526
  introduction 503–5
  linear 505, 517
  macroeconomics 517–25
  microeconomics 517–25
  non-homogeneous 505, 511–15, 526
  order 505, 526
  particular solution 507–8, 509, 515
  progress exercises 515–17
  solving 505–17
  stability of solutions 508–11, 526
  terminology 504–5
  time path to stability 508–11
Differential equations
  $dy/dt = r(A - y)$ 434
  $dy/dx = f(x)$ 424
  $dy/dx = f(x)g(y)$ 432
  $dy/dx = ky$ 430–2
  $dy/dx = ry$ 433
  economics 425
  first-order 422–33, 439, 440
  general solution 423, 432
  integration 424
  limited and unlimited growth 433–8
  particular solution 423, 432
  rates of change 426–8
  solution 422–4, 430, 432
  total cost from marginal cost 425–6
Differentials 390
  and small changes 344–9
  for functions of one variable 345–6
  for functions of two variables 346–9
Differentiation 234–331
  and slope of a curve 235–45
  applications 245–60
  chain rule for 308–10, 328
  definition 238
  function of a function 308–10
  general power rule 306
  power rule for 238–41, 242, 243–5
  product rule for 311–13, 328
  quotient rule for 313–15, 328
  working rules for sums and differences of several functions 241–2
  see also Partial differentiation
Diminishing marginal rate of substitution 362
Diminishing marginal rate of technical substitution 353
Discount factors 209
Discrete functions 504
Division 3–4
  exponential functions 155
  fractions 6–7
  matrices. *See* Inverse matrix

e (number) 152, 401
Economic agents 50
Economic models 50–1
Economics 15, 29

  curvature 293–4
  differential equations 425
  functions of two variables 333–4
  maximum/minimum 276–91
  points of inflection 297–8
  quadratic functions 141–6
Effective annual rate. *See* Annual percentage rate (*APR*)
Elastic demand 77
Elasticity 319–27, 329
  concept of 72
  formula 73
  of demand, supply and income 72–80
  w.r.t. capital 392
  w.r.t. labour 392
Elimination methods 462–7
  Gauss–Jordan 466–7
  Gaussian 462–5
  inverse of a matrix 481–3
Equations
  budget constraints 80
  demand function 53
  given slope and point on line 67–8
  given two points on line 66–7
  horizontal and vertical lines 40
  matrix form 486–7
  solving 2, 7–9, 11–13
    elimination methods 462–7
    using Excel 495–8
    using inverse matrix 487–8
    using logarithms 167–8, 173–5
  straight line 35–9, 41, 53
  supply function 56–7
  translated lines 71–2
Equilibrium
  in goods and labour markets 99–101
  in income-determination model 474
  see also Market equilibrium
Equilibrium level in national income model 478–9
Equilibrium level of income, partial differentiation 367
Equilibrium national income 119–22
  with consumption and investment 120
Euro exchange rates 9
e^x
  and consumer surplus (*CS*) 419–20
  and producer surplus (*PS*) 419–20
  definite integral 411
  integration of functions containing 402
Excel 20–6, 129, 186, 231
  cost, revenue, break-even per unit tax 123–4
  effect of price and income changes on budget constraint 81–4
  financial mathematics 227–9
  internal rate of return (*IRR*) 210–12
  for linear algebra 495–8
  linear functions 80–6
  non-linear functions 183–5
  simultaneous equations 123–7
  tax distribution 124–7
  total cost function 183–5
Exchange rates 9
Expenditure multiplier 121
  and *MPC* 121
  with imports and trade balance 121
  with taxes 121, 121
Exponential equations, solving 159–60
Exponential functions 151–65, 401–2
  applications 161–5
  consumer and producer surplus 419–20
  definition 151–2
  division 155
  graphs 152–4, 164
  multiplication 155
  properties 154–5
  rules 155–6
  solving 156–8
Exponential indices 186
Exponentials 317–18
  solving equations containing 175–6
Extreme point theorem 445

Feasible region 443
Financial mathematics 188–233
  Excel 227–9
Finite series 189
First-order difference equations 505–17, 526–7
First-order differential equations 422–33, 439, 440
First-order partial derivatives 336–40, 390
Fixed costs 62
Fractions 4–7
  addition 4–5
  division 6–7
  multiplication 5–6
  subtraction 4–5
  terminology 4
Functions 45, 328
  of linear functions 438
  of one variable 390
    optimisation 260–76, 369–70
  of two variables 390
    economics 333–4

    optimisation of 370–2
  of two or more variables 332–93
    general form 333
  sketching 274–6
Future value 188
  asset and reducing-balance depreciation 206
  compound interest 197–8

Gauss–Jordan elimination 466–7
Gaussian elimination 462–5
Geometric progression (GP) 190
  applications 194–5
  infinite number of terms 191
  progress exercises 191–2
  sum of terms 191
Geometric sequences 189, 190, 229
  characteristics 194
Geometric series 190
  application 192–4
  sum of 191
Goods market equilibrium 99–100, 128
  and price ceilings 102–3
  and subsidies 110–11
  and taxes 109
Government expenditure and taxation 121
Government expenditure multiplier 368
Government role in economy 51
Graphs
  background 31
  cubic functions 146–7
  cubic total cost functions 148
  exponential functions 152–4, 164
  from equation $y = mx + c$ 41–3
  functions of two variables, economics 333–4
  hyperbolic functions 179–80
  inequalities 443
  of line $ax + by + d = 0$ 43–5
  logarithmic functions 170–1
  minimum cost 445
  plotting 31, 33–4
  straight line 41–6
  y, y', y'' 269–72, 292
Greater than (symbol) 14
Growth curves 154
Growth laws 161–5
Growth problems 165

Harrod–Domar growth model 522–4, 527
Homogeneous functions of degree r 358
Horizontal asymptote 178
Horizontal intercepts 39–40, 54, 55
Horizontal translations 71
Households' consumption expenditure 118, 120
Hyperbolic curve 302
Hyperbolic demand function 180–2
Hyperbolic functions 177–83, 186
  defining 177–8
  equations and applications based on 182–3
  graphs 179–80
  sketching 177–8, 179–80

Identity matrix 453
Imports, expenditure multiplier 121
Income, consumption and changes in 163
Income constraint, utility maximisation subject to 381–5
Income-determination model
  Cramer's rule 474
  equilibrium in 474
Income elasticity of demand 365, 392
Income tax rate multiplier 368
Incremental changes 344–9, 359, 363
Independent variable 45, 504
Indices 155, 156
  exponential 186
  rules for 158–9, 160–1
Indifference curves 360
  slope 360–3
Inelastic demand 77
Inequalities 14–17
  constraints 442–3
  graphs 443
  manipulating 15
  solving 15–16
  symbols 14
Inequality statements 15
  intervals defined by 17
Infinite series 189
Injections into circular flow 51
Input/output analysis 488–93, 500
  progress exercises 493–5
Integral 396
Integral of constant term 398
Integration 394–440
  area under the curve 407–14
  as reverse of differentiation 394–5
  by algebraic substitution 402–7
  constant multiplied by variable term 399
  differential equations 424
  functions containing e^x 402
  general functions 400
  linear functions 406

  linear functions raised to power 405–6
  of linear functions using substitution 402–5
  natural exponential function 401–2
  power rule for 396–401
  rules 438
  sums and differences 399
  working rules 399
Intercepts 31–5
Interest, proportion of loan repayment 220–1
Interest rates
  and price of bonds 224–9
Internal rate of return (*IRR*) 209–13, 230
  calculation 209–12
  comparison of appraisal techniques 212
  Excel 210–12
  graphical method 209, 210–12
  progress exercises 212–13
Intersection with axis 276
Intervals
  along which a curve increases or decreases 272
  along which a function increases or decreases 268–9
  defined by inequality statements 17
Inverse function 46
Inverse matrix 481–95, 500
  2 × 2 484
  3 × 3 484–6
  cofactor method 483–98
  elimination method 481–3
  progress exercises 493–5
  solving set of equations using 487–8
  terminology 481
Investment multiplier 368
Investment project cash flows 208
  net present value (*NPV*) 208
Investments, growth using different methods of compounding (Excel) 227–9
IS schedule 122
IS-LM model 118–22, 129
Isocost function 444
Isocost line 444, 445
Isoquants 334, 340
  production function 334–6
  slope 353–6
  three-dimensional view 335
  two-dimensional view 336
Iteration method, solving difference equations by 505–6

Labour demand 128
Labour market equilibrium 100–1, 128
  and price floors 103–4
Labour supply 128
Lagged income model 517–19, 526
Lagrange multiplier 379–87, 391
  interpretation 383–4
  use to derive identity $U_x/U_y = P_X/P_Y$ 382–3
  utility maximisation 381, 382
Lagrangian function 379
Laplace expansion 484
Law of demand 52
Law of limited growth 434
Law of supply 57
Law of unlimited growth 433
Less than (symbol) 14
Limited growth 163
  differential equations 433–5
  graphs 164
Limiting value 236
Linear algebra 441–502
  Excel for 495–8
Linear demand function 54–6
Linear functions
  Excel 80–6
  functions of 438
  integration 406
  integration using substitution 402–5
  raised to power, integrating 405–6
  translation 71–2
Linear national income model, multipliers 367–9
Linear programming 441–52, 498–9
  progress exercises 451–2
Linear supply function 58–63
Linear total cost function 62–3
Linear total revenue function 64–5, 132
LM schedule 122
Local maximum 261
Local minimum 261
Logarithm of a number 165–7
Logarithmic functions 162, 165–77
  graphs 170–1
  properties 170
  rules for combining and manipulating 171–3
Logarithms 186
  and definite integration 413
  rules for 186
  solving equations containing 173–5
  to base 10, 167
  to base *e* 167
  use in solving equations 167–8, 173–5
Logistic growth 164

Macroeconomics 50
  difference equations 517–25
Marginal cost (*MC*) 245, 248, 299–306
  deriving from average cost 253–4
  equation 248–9
  given logarithmic *TC* function 316
  total cost from 425–6
Marginal functions 245–9, 329, 352, 425
  applications 245–60
  partial derivatives 351
  total function from 439
Marginal product of capital (MP_K) 351–2, 354–6
Marginal product of labour (MP_L) 256–7, 351–2, 354–6
Marginal products 353
  ratio 353–4
Marginal propensity to consume (*MPC*) 119, 121, 257
Marginal propensity to save (MP_S) 257
Marginal rate of substitution (*MRS*) 360–3
Marginal rate of technical substitution (*MRTS*) 353–6, 391
Marginal revenue (*MR*) 245–7, 277–9, 324–6
  calculating
    given demand function 246
    over an interval 247–8
  for perfectly competitive firm and monopolist 250–3
  in terms of exponentials 317–18
  sketching 279
Marginal utility (*MU*) 259, 360, 362
Market equilibrium 102, 103, 182
  Cramer's rule 473–4
  for substitute and complementary goods 105–7, 128
Mathematical functions 19–20
Mathematical modelling 47–50
  construction 48–50
  steps in 48–50
Matrices 452–62
  addition 453, 454
  definition 452–3
  division. *See* Inverse matrix
  equal 453
  multiplication 455–8
  subtraction 454
  system of equations 486–7
  terminology 452
  transpose 453
  see also Inverse matrix
Matrix algebra 499
  progress exercises 461–2
Matrix arithmetic, applications 458–60
Matrix of technical coefficients 500
Maximisation 446
Maximum points 263–7, 329, 370–1, 391
Maximum utility for budget constraint 381–5
Maximum/minimum method 265, 280, 288, 329
  economics 276–91
Microeconomics 50
  difference equations 517–25
Midpoints elasticity formula 78
Minimum cost graph 445
  mathematical methods 445–6
  subject to constraints 442–5
Minimum points 263–7, 329, 370–1, 391
Minor of an element 483
Minus *b* formula 133–5, 267
Minus one exception to power rule 398
Mixed second-order partial derivatives 341–2, 344
Monopolist
  maximising total revenue for two goods 373
  price and non-price discrimination 376
  price discrimination for single good 375–6
Monopoly, profit maximisation 287–9
Mortgage repayments 219
Multiplication 3
  exponential functions 155
  fractions 5–6
  matrices 455–8

National income model 118–22, 128
  equilibrium levels in 119–21, 478–9
  multipliers 368–9, 392
Natural exponential function, integration 401–2
Negative numbers 14–15
Negative second derivatives 351
Negative slope 54
Net present value (*NPV*) 207–9, 230
  comparison of appraisal techniques 212
  investment project 208
  progress exercises 212–13
Non-linear functions 131–87
  Excel 183–5
Non-linear supply and demand functions 141–2
Non-linear total revenue function 132, 142–4
Null matrix 453
Number line 14
Numerator 4

Objective function 443
Optimisation 329
  constrained 378–90, 391
  functions of one variable 260–76, 369–70
  functions of two variables 370–2
  terminology 261
  unconstrained 369–78, 391
Ordinary annuity 215
Output for a firm over time 281–3
Output maximisation subject to cost constraint 386–7
Own-price elasticity 73

Partial derivatives
  expressions for multipliers 367–8
  first-order 336–40, 348, 390
  marginal functions 351
  second-order 340–4, 348, 390
Partial differentiation 332–49, 356–7
  applications 349–69
  equilibrium level of income 367–8
  production functions 349–57
Partial elasticity 368–9, 302
  w.r.t. capital 366
  w.r.t. labour 366–7
Partial elasticity of demand 364–5
Particular integral (PI) 511, 512, 514, 518, 521, 526
Percentages 17–20
Perfectly competitive firm
  profit maximisation 285–7
Physical models
  dynamic 47
  static 47
pi (Π) 152
Point elasticity of demand 74, 319–23, 329
  and demand function 77
  and price and vertical intercept 78
  in terms of P, Q or both for non-linear demand functions 320–2
Point of inflection (PoI) 292, 296–306, 370, 371, 391
  economics 297–8
  total cost function 299
Polynomials 149
Population growth 161–2, 168–9
Power rule
  differentiation 238–41, 242, 243–5
  integration 396–401 405–6
  minus one exception to 398
Powers 155
Present value 188, 230
  asset and reducing-balance depreciation 206–7
  compound interest 197–8
  formula 196
Price
  and average revenue (AR) function 250
  and quantity 15, 29–30
Price ceilings 102–3
  and black market 103
Price changes 324–6
Price discrimination 283–5
  for monopolist producing a single good 375–6
Price elasticity of demand 73–80, 319, 322–3, 365, 376, 392
Price elasticity of supply 78
Price floors 103–4
Price of bonds and interest rates 224–9
Price–quantity relationship, graphs 30
Producer surplus (PS) 115–16, 128, 417–19, 439
  and e^x 419–20
  definite integral 417–19
Product rule for differentiation 311–13, 328
Production conditions 352
Production constraint, cost minimisation subject to 387–8
Production functions 255–7, 356–7, 363–4, 385–7, 392
  first and second derivatives 351
  general form 255
  graphical representation 352–3
  isoquants 334–6
  partial differentiation 349–57
Profit function 150–1, 288
  sketching 280
Profit maximisation 283–9, 373
  for multi-product firm 374–5
  monopoly 287–9
  perfectly competitive firm 285–7
  subject to constraints 446–50
Proportional rates of growth 435–7

Q function, sketching 282–3
Quadratic equations 185
  roots 133–4
  sketching 138–9
  solving 133–6
Quadratic functions 132
  applications 145–6
  economics 141–6
  graphs 136–9

  properties 136–40
  sketching 137–8
Quotient rule for differentiation 313–15, 328

Rate of change, differential equations 426–8
Reducing-balance depreciation 206–7
Repayment, loan, proportion of interest 220–1
Returns to scale 357–9, 363–4
  constant, increasing and decreasing 358–9
Revenue 64–5
Rounding numbers 20

Saddle point 370, 371, 391
Scalar multiplication 455
Second-order partial derivatives 340–4, 390
Sequence 189
Series 189, 229
Short-run production function 330
Simple interest 195–6
  progress exercises 199–200
Simultaneous equations 89–130
  applications 98–114
  Cramer's rule 476–8
  Excel 123–7
  infinitely many solutions 96–7
  no solution 94, 95
  solving 89–98
    by determinants 469–75
  three equations in three unknowns 97–8
  two equations in two unknowns 90–7
  two and three unknowns 98
  unique solution 91, 93, 94
Sinking funds 213–22, 230–1
Sketching
  curves 272–6
  functions 274–6
Slope of a curve 235–45, 260–1
  at a point 236, 240
  indifference curves 360–3
  turning points and 236
  $y = x^2$ 236–8
Slope of a line 31–5
  calculating 66–7
  measuring 32–3
Slope of chords 236
Slope of demand function 53, 54
Small changes 344–9
  formula 359, 391
Spreadsheet 20–7
Square roots 2
Standard functions 438
Stationary points 261
Stationary points of inflection 297
Stochastic model 48
Straight line 29–88
  defining 31
  depreciation 205
  equation 35–9, 53
  given slope and point on line 67–8
  given two points on line 66–7
  graphs 41–6
  translation 71–2
Straight second-order partial derivatives 341–3
Subsidies and their distribution 109–11
Substitute goods 52
  market equilibrium 105–7, 128
Subtraction 2–3
  fractions 4–5
  matrices 454
Supply function 56–60
  equation 57–8
  linear 58–60
  non-linear 141–2
Symbols 2, 14

Tax distribution 128
  Excel 124–7
  paid by consumer and producer 129–30
Taxation
  and equilibrium national income 121
  and government expenditure 121
Taxes and their distribution 107–9, 113
Total amount accumulated over given time interval 439
Total cost (*TC*) 62, 150–1, 248–9, 299–300, 316
  cubic 148–9
  curvature 299
  Excel 183–5
  from marginal functions 439
  point of inflection (PoI) 299
  sketching 279
Total differential 390
Total fixed cost (*TFC*) 301
Total revenue (*TR*) 63–4, 150–1, 246, 276–9, 324–6
  in terms of exponentials 317–18
  linear 132
  maximisation 372
    subject to budget constraint 379–80
  maximum 276–9
  non-linear 142–4
  sketching 277–9

Total surplus 116
Total variable cost (TVC) 301
Translated lines, equation 72
Translation, linear functions 71–2
Turning point(s) 236, 260–3, 269–70, 271, 273–6, 282, 288, 290
  maximum and minimum 263–7
  slope of a curve and 236–8

Unconstrained optimisation 369–78, 391
Unique solution 91, 93, 94
Unit elastic demand 77
Unit matrix 453
Unlimited growth 161–2
  differential equations 433–5
  graphs 164
Utility functions 359–64
  graphical representation 360
Utility maximisation
  graphical analysis 382–3
  Lagrange multiplier 381–2
  subject to income constraint 381–5

Variable costs 62
Variables and letters 2
Vertical asymptote 178
Vertical intercepts 31, 53, 54, 62
Vertical translations 71

Withdrawals from circular flow 51